Explicit Multi-Soliton Solutions for the KdV Equation by Darboux Transformation

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Abstract—**With the aid of the known Darboux transformation, starting from an arbitrary constant solution, a series of explicit two-soliton and three-soliton solutions to the Korteweg-de Vries (KdV) equation are constructed.**

***Index Terms*—KdV equation, two-soliton solution, three-soliton solution, Darboux transformation.**

I. Introduction

As a prototype example for the exactly integrable nonlinear equations, we consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

which plays an outstanding role in physical problems, for example, stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics and so on [1]. We know that the most remarkable property of exactly integrable equations is the presence of exact solitonic solutions, and the existence of one-soliton solution is not itself a specific property of integrable partial differential equations, many non-integrable equations also possess simple localized solutions that may be called one-solitonic. However, there are integrable equations only, which posses exact multi-soliton solutions which describe purely elastic interactions between individual solitons [2], and the KdV equation is one of these integrable equations.

Although the inverse scattering method [3], the B$\ddot{a}$cklund transformation method [4,5,6] and the Hirota method [7] pave the way to generation of multi-soliton solutions to the nonlinear evolution equation, the explicit multi-soliton solution cannot be obtained by pure intuition or by elementary calculations because of itself complications [8,9]. The known multi-wave solutions to the KdV equation are scarce [10,11,12,13,14,15], it has been known for a long time that equation (1) possesses explicit multi-soliton solutions described in [1].

II. Explicit two-soliton solutions

As mentioned in [16], the Lax pair for equation (1) is given by

$$\begin{cases} \Phi_x = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix} \Phi, \\ \Phi_t = \begin{pmatrix} u_x & -(4\lambda + 2u) \\ A & -u_x \end{pmatrix} \Phi, \end{cases} \tag{2}$$

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where $A = -(4\lambda + 2u)(\lambda - u) + u_{xx}$, with the Darboux matrix

$$D(x, t, \lambda) = \begin{pmatrix} -\sigma_i & 1 \\ \lambda - \lambda_i + \sigma_i^2 & -\sigma_i \end{pmatrix}, \tag{3}$$

where $i = 0, 1, 2,\ \lambda, \lambda_i$ are the spectral parameters, in particular, when $\Phi_i(x, t\lambda) = (a_{jk}^{(i)}(x, t, \lambda))_{2\times 2}$ is the fundamental solution matrix to the lax pair on u_i, σ_i is defined as

$$\sigma_i = \frac{a_{21}^{(i)}(x, t, \lambda_i)\mu_i + a_{22}^{(i)}(x, t, \lambda_i)\gamma_i}{a_{11}^{(i)}(x, t, \lambda_i)\mu_i + a_{12}^{(i)}(x, t, \lambda_i)\gamma_i}, \tag{4}$$

here, μ_i and γ_i are arbitrary constants, but $\mu_i^2 + \gamma_i^2 \neq 0$. A theorem borrowed from [16] points out, if u_i is a given solution to equation (1), then

$$u_{i+1} = 2\lambda_i - u_i - 2\sigma_i^2 \tag{5}$$

becomes new solution based on u_i.

The starting point for constructing two-soliton solution is to solve the fundamental solution matrix of the lax pair on constant solution u_0. Substituting u_0 into the system (2) yields

$$\begin{cases} \Phi_x = \begin{pmatrix} 0 & 1 \\ \lambda - u_0 & 0 \end{pmatrix} \Phi, \\ \Phi_t = -(4\lambda + 2u_0) \begin{pmatrix} 0 & 1 \\ \lambda - u_0 & 0 \end{pmatrix} \Phi. \end{cases} \tag{6}$$

By the eigenvalue method, we obtain the fundamental solution matrix to the system (6)

$$\Phi_0(x, t, \lambda) = \begin{pmatrix} e^{\eta} & e^{-\eta} \\ \omega\ e^{\eta} & -\omega\ e^{-\eta} \end{pmatrix}, \tag{7}$$

where $\eta = \eta(\lambda) = \omega\ [x - (4\lambda + 2u_0)t],\ \omega = \omega(\lambda) = \sqrt{\lambda - u_0},\ \lambda > u_0$.

For simplicity, we set $\omega_i = \sqrt{\lambda_i - u_0},\ \eta_i = \eta(\lambda_i),\ \theta_i = \eta_i + c_i$, where c_i is an arbitrary constant, and $i = 0, 1, 2$.

From (4), we get

$$\sigma_0 = \omega_0\ \frac{e^{\eta_0}\mu_0 - e^{-\eta_0}\gamma_0}{e^{\eta_0}\mu_0 + e^{-\eta_0}\gamma_0}. \tag{8}$$

Choosing $\mu_0 = e^{c_0}, \gamma_0 = e^{-c_0}$ in (8), we have

$$\sigma_{0t} = \omega_0 \tanh \theta_0, \tag{9}$$

then substituting (9) into (5), we obtain the solitary wave solution

$$u_{11} = u_0 + 2\omega_0^2 \mathrm{sech}^2 \theta_0.$$

Similarly, choosing $\mu_0 = e^{c_0}, \gamma_0 = -e^{-c_0}$ in (8), we have

$$\sigma_{0c} = \omega_0 \coth \theta_0, \tag{10}$$

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which further leads to

$$u_{12} = u_0 - 2\omega_0^2 \text{csch}^2\theta_0.$$

Now we construct the two-soliton solutions generated from u_1. For convenience, we first give the new solution which is expressed in terms of σ_0 rather than u_1, then substitute (9) and (10) into the relative solution, respectively. According to [16], we can obtain the fundamental solution matrix to the lax pair associated with the known solitary wave solution u_1 in the following manner

$$\Phi_1(x,t,\lambda) = \begin{pmatrix} -\sigma_0 & 1 \\ \lambda - \lambda_0 + \sigma_0^2 & -\sigma_0 \end{pmatrix} \Phi_0(x,t,\lambda) = \begin{pmatrix} (-\sigma_0+\omega)e^{\eta} & -(\sigma_0+\omega)e^{-\eta} \\ Be^{\eta} & De^{-\eta} \end{pmatrix}, \quad (11)$$

where $B = \lambda - \lambda_0 + \sigma_0^2 - \sigma_0\omega,\ D = \lambda - \lambda_0 + \sigma_0^2 + \sigma_0\omega.$ From (4) and (11), we have

$$\sigma_1 = \frac{(\lambda_1 - \lambda_0 + \sigma_0^2) - \sigma_0\omega_1 \frac{e^{\eta_1}\mu_1 - e^{-\eta_1}\gamma_1}{e^{\eta_1}\mu_1 + e^{-\eta_1}\gamma_1}}{-\sigma_0 + \omega_1 \frac{e^{\eta_1}\mu_1 - e^{-\eta_1}\gamma_1}{e^{\eta_1}\mu_1 + e^{-\eta_1}\gamma_1}}. \quad (12)$$

By analogy with μ_0, γ_0 in (8), there are two special cases to consider in (12).

1) Choosing $\mu_1 = e^{c_1}, \gamma_1 = e^{-c_1}$ in (12), we get

$$\sigma_{1t} = \frac{(\lambda_1 - \lambda_0 + \sigma_0^2) - \sigma_0\omega_1 \tanh\theta_1}{-\sigma_0 + \omega_1 \tanh\theta_1}, \quad (13)$$

combining (5) and (13), we see that

$$u_2 = u_0 + \frac{2(\lambda_1 - \lambda_0)(\omega_0^2 - \sigma_0^2 - \omega_1^2 \text{sech}^2\theta_1)}{(\sigma_0 - \omega_1 \tanh\theta_1)^2}. \quad (14)$$

Substituting (9) and (10) into (14), respectively, we obtain explicit two-soliton solutions

$$u_{21} = u_0 + \frac{2(\lambda_1 - \lambda_0)(\omega_0^2 \text{sech}^2\theta_0 - \omega_1^2 \text{sech}^2\theta_1)}{(\omega_0 \tanh\theta_0 - \omega_1 \tanh\theta_1)^2} \quad (15)$$

and

$$u_{22} = u_0 - \frac{2(\lambda_1 - \lambda_0)(\omega_0^2 \text{csch}^2\theta_0 + \omega_1^2 \text{sech}^2\theta_1)}{(\omega_0 \coth\theta_0 - \omega_1 \tanh\theta_1)^2}, \quad (16)$$

respectively.

2) Choosing $\mu_1 = e^{c_1}, \gamma_1 = -e^{-c_1}$ in (12), in a totally parallel way, we obtain

$$\sigma_{1c} = \frac{(\lambda_1 - \lambda_0 + \sigma_0^2) - \sigma_0\omega_1 \coth\theta_1}{-\sigma_0 + \omega_1 \coth\theta_1}, \quad (17)$$

which together with (5) gives

$$u_{23} = u_0 + \frac{2(\lambda_1 - \lambda_0)(\omega_0^2 \text{sech}^2\theta_0 + \omega_1^2 \text{csch}^2\theta_1)}{(\omega_0 \tanh\theta_0 - \omega_1 \coth\theta_1)^2}$$

and

$$u_{24} = u_0 - \frac{2(\lambda_1 - \lambda_0)(\omega_0^2 \text{csch}^2\theta_0 - \omega_1^2 \text{csch}^2\theta_1)}{(\omega_0 \coth\theta_0 - \omega_1 \coth\theta_1)^2}.$$

we notice that u_{23} is just a given solution in [1], when $u_0 = 0$.

III. Explicit three-soliton solutions

As shown in [16], the fundamental solution matrix $\Phi_2(x,t,\lambda)$ to the lax pair associated with u_2 is given by

$$\Phi_2(x,t,\lambda) = \begin{pmatrix} -\sigma_1 & 1 \\ \lambda - \lambda_1 + \sigma_1^2 & -\sigma_1 \end{pmatrix} \Phi_1(x,t,\lambda) = \begin{pmatrix} Pe^{\eta} & Qe^{-\eta} \\ Re^{\eta} & Se^{-\eta} \end{pmatrix}, \quad (18)$$

where $P = \lambda - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 - \omega)$,
$Q = \lambda - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 + \omega)$,
$R = (\lambda - \lambda_1 + \sigma_1^2)(-\sigma_0 + \omega) + \sigma_1(-\lambda + \lambda_0 - \sigma_0^2 + \sigma_0\omega)$,
$S = (\lambda - \lambda_1 + \sigma_1^2)(-\sigma_0 - \omega) + \sigma_1(-\lambda + \lambda_0 - \sigma_0^2 - \sigma_0\omega)$.
From (4) and (18), we further see that

$$\sigma_2 = -\frac{\sigma_0(\lambda - \lambda_1 + \sigma_1^2) + \sigma_1(\lambda - \lambda_0 + \sigma_0^2)}{\lambda - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 - \omega)K}\Big|_{\lambda=\lambda_2} + \frac{\omega(\lambda - \lambda_1 + \sigma_1^2 + \sigma_1\sigma_0)K}{\lambda - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 - \omega)K}\Big|_{\lambda=\lambda_2}, \quad (19)$$

with $K = \frac{(e^{\eta}\mu_2 - e^{-\eta}\gamma_2)}{(e^{\eta}\mu_2 + e^{-\eta}\gamma_2)}$. Because

$$u_3 = 2\lambda_2 - u_2 - 2\sigma_2^2 = u_0 + 2(\lambda_0 - u_0 - \sigma_0^2) + 2[(\lambda_2 - \lambda_1 + \sigma_1^2) - \sigma_2^2], \quad (20)$$

we first give u_3 which depends upon σ_0 and σ_1 in order to avoid tedious calculation, then consider the expressions for σ_0 and σ_1 in the relative solution.

For the special cases of σ_2 in (19), we have two groups of three-soliton solutions for equation (1).

1) Choosing $\mu_2 = e^{c_2}, \gamma_2 = e^{-c_2}$ in (19), we obtain

$$\sigma_{2t} = -\frac{(\lambda_2 - \lambda_1)(\sigma_0 - \omega_2 \tanh\theta_2)}{\lambda_2 - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 - \omega_2 \tanh\theta_2)} + \frac{\sigma_1[\lambda_2 - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 - \omega_2 \tanh\theta_2)]}{\lambda_2 - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 - \omega_2 \tanh\theta_2)}. \quad (21)$$

Substituting (21) into (20) yields

$$u_3 = u_0 + 2(\lambda_0 - u_0 - \sigma_0^2) + \frac{2(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_0 + \sigma_0^2 - \sigma_1^2)(\sigma_0 - \omega_2 \tanh\theta_2)^2}{[\lambda_2 - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 - \omega_2 \tanh\theta_2)]^2} + \frac{2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_0)(\sigma_0^2 - \omega_0^2 + \omega_2^2 \text{sech}^2\theta_2)}{[\lambda_2 - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 - \omega_2 \tanh\theta_2)]^2}. \quad (22)$$

Seeing that the structure of the three-soliton solution is overlong, we set $\omega_{21} = \omega_2^2 - \omega_1^2$, $\omega_{20} = \omega_2^2 - \omega_0^2$, $\omega_{10} = \omega_1^2 - \omega_0^2$, $T_i = \omega_i \tanh\theta_i$ and $C_i = \omega_i \coth\theta_i$, where $i = 0, 1, 2$.

Substituting (9) and (13), (10) and (13) into (22), respectively, we get

$$u_{31} = u_0 + 2\omega_0^2 \text{sech}^2\theta_0 + \frac{2\omega_{21}\omega_{10}(\omega_0^2 \text{sech}^2\theta_0 - \omega_1^2 \text{sech}^2\theta_1)(T_0 - T_2)^2}{[\omega_{20}(T_1 - T_0) + \omega_{10}(T_0 - T_2)]^2} + \frac{2\omega_{21}\omega_{20}(\omega_2^2 \text{sech}^2\theta_2 - \omega_0^2 \text{sech}^2\theta_0)(T_1 - T_0)^2}{[\omega_{20}(T_1 - T_0) + \omega_{10}(T_0 - T_2)]^2}$$

and

$$u_{32} = u_0 - 2\omega_0^2 \text{csch}^2\theta_0 - \frac{2\omega_{21}\omega_{10}(\omega_0^2 \text{csch}^2\theta_0 + \omega_1^2 \text{sech}^2\theta_1)(C_0 - T_2)^2}{[\omega_{20}(T_1 - C_0) + \omega_{10}(C_0 - T_2)]^2} + \frac{2\omega_{21}\omega_{20}(\omega_2^2 \text{sech}^2\theta_2 + \omega_0^2 \text{csch}^2\theta_0)(T_1 - C_0)^2}{[\omega_{20}(T_1 - C_0) + \omega_{10}(C_0 - T_2)]^2}$$

respectively. Similarly, substituting (9) and (17), (10) and (17) into (22), respectively, we have

$$u_{33} = u_0 + 2\omega_0^2\mathrm{sech}^2\theta_0 + \frac{2\omega_{21}\omega_{10}(\omega_0^2\mathrm{sech}^2\theta_0 + \omega_1^2\mathrm{csch}^2\theta_1)(T_0 - T_2)^2}{[\omega_{20}(C_1 - T_0) + \omega_{10}(T_0 - T_2)]^2} + \frac{2\omega_{21}\omega_{20}(\omega_2^2\mathrm{sech}^2\theta_2 - \omega_0^2\mathrm{sech}^2\theta_0)(C_1 - T_0)^2}{[\omega_{20}(C_1 - T_0) + \omega_{10}(T_0 - T_2)]^2}$$

and

$$u_{34} = u_0 - 2\omega_0^2\mathrm{csch}^2\theta_0 - \frac{2\omega_{21}\omega_{10}(\omega_0^2\mathrm{csch}^2\theta_0 - \omega_1^2\mathrm{csch}^2\theta_1)(C_0 - T_2)^2}{[\omega_{20}(C_1 - C_0) + \omega_{10}(C_0 - T_2)]^2} + \frac{2\omega_{21}\omega_{20}(\omega_2^2\mathrm{sech}^2\theta_2 + \omega_0^2\mathrm{csch}^2\theta_0)(C_1 - C_0)^2}{[\omega_{20}(C_1 - C_0) + \omega_{10}(C_0 - T_2)]^2}.$$

2) Choosing $\mu_2 = e^{c_2}, \gamma_2 = -e^{-c_2}$ in (19), in a similar manner, we obtain

$$u_{35} = u_0 + 2\omega_0^2\mathrm{sech}^2\theta_0 + \frac{2\omega_{21}\omega_{10}(\omega_0^2\mathrm{sech}^2\theta_0 - \omega_1^2\mathrm{sech}^2\theta_1)(T_0 - C_2)^2}{[\omega_{20}(T_1 - T_0) + \omega_{10}(T_0 - C_2)]^2} - \frac{2\omega_{21}\omega_{20}(\omega_2^2\mathrm{csch}^2\theta_2 + \omega_0^2\mathrm{sech}^2\theta_0)(T_1 - T_0)^2}{[\omega_{20}(T_1 - T_0) + \omega_{10}(T_0 - C_2)]^2},$$

$$u_{36} = u_0 - 2\omega_0^2\mathrm{csch}^2\theta_0 - \frac{2\omega_{21}\omega_{10}(\omega_0^2\mathrm{csch}^2\theta_0 + \omega_1^2\mathrm{sech}^2\theta_1)(C_0 - C_2)^2}{[\omega_{20}(T_1 - C_0) + \omega_{10}(C_0 - C_2)]^2} - \frac{2\omega_{21}\omega_{20}(\omega_2^2\mathrm{csch}^2\theta_2 - \omega_0^2\mathrm{csch}^2\theta_0)(T_1 - C_0)^2}{[\omega_{20}(T_1 - C_0) + \omega_{10}(C_0 - C_2)]^2},$$

$$u_{37} = u_0 + 2\omega_0^2\mathrm{sech}^2\theta_0 + \frac{2\omega_{21}\omega_{10}(\omega_0^2\mathrm{sech}^2\theta_0 + \omega_1^2\mathrm{csch}^2\theta_1)(T_0 - C_2)^2}{[\omega_{20}(C_1 - T_0) + \omega_{10}(T_0 - C_2)]^2} - \frac{2\omega_{21}\omega_{20}(\omega_2^2\mathrm{csch}^2\theta_2 + \omega_0^2\mathrm{sech}^2\theta_0)(C_1 - T_0)^2}{[\omega_{20}(C_1 - T_0) + \omega_{10}(T_0 - C_2)]^2},$$

and

$$u_{38} = u_0 - 2\omega_0^2\mathrm{csch}^2\theta_0 - \frac{2\omega_{21}\omega_{10}(\omega_0^2\mathrm{csch}^2\theta_0 - \omega_1^2\mathrm{csch}^2\theta_1)(C_0 - C_2)^2}{[\omega_{20}(C_1 - C_0) + \omega_{10}(C_0 - C_2)]^2} - \frac{2\omega_{21}\omega_{20}(\omega_2^2\mathrm{csch}^2\theta_2 - \omega_0^2\mathrm{csch}^2\theta_0)(C_1 - C_0)^2}{[\omega_{20}(C_1 - C_0) + \omega_{10}(C_0 - C_2)]^2}.$$

IV. Conclusion

As a soliton equation which is widely used in various fields, the soliton solutions to the KdV equation have been investigated extensively in the papers and literatures, however, most of the multi-soliton solutions have been obtained in numerical form, and its explicit exact three-soliton solutions are very few, the main reason is that the calculation is too tedious to obtain succinct expression, rather than the lack of methods. Overcoming the difficulties of calculations by some techniques, we finally construct some new explicit two-soliton and three-soliton solutions for the KdV equation.

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