A Series Solution to a Partial Integro-Differential Equation Arising in Viscoelasticity

J.-M. Yoon, S. Xie, and V. Hrynkiv

***Abstract*—A linear partial integro-differential equation is solved both numerically and analytically using variational iteration method. This equation typically arises in viscoelasticity and other areas. The analytic solution is represented by an infinite series.**

***Index Terms*—Analytical solution, partial integro-differential equations, series solution, singular kernel, variational iteration method**

I. INTRODUCTION

THE increasing attempts in applied mathematics to model real world phenomena often lead to integral and integro-differential equations [1], [3], [4], [7], [9], [10]. This explains a growing interest in the applied mathematics community to integro-differential equations, and in particular, to partial integro-differential equations. They frequently arise and play an important role in many areas of mathematics, physics, engineering, biology, and other sciences. Main challenges in solving these kinds of problems, both numerically and analytically, are due to different factors, such as large range of variables, nonlinearity and non-local phenomena, multi-dimensionality, etc.

This paper deals with the following linear partial integro-differential equation

$$u_t = \mu u_{xx} + \int_0^t K(t-s) u_{xx}(x,s)\, ds, \tag{1}$$

where $\mu > 0$, $K(t-s) := (t-s)^{-1/2}$ is the kernel function, and the unknown real function $u(x,t)$ is sought for $0 \le t \le T,\ 0 \le x \le 1$, with the initial condition

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1, \tag{2}$$

(see [2], [13]) and the boundary conditions

$$u(0,t) = u(1,t) = 0, \ \ 0 \le t \le T. \tag{3}$$

Equations of this nature appear in many applications such as heat conduction in materials with memory, population dynamics, viscoelasticity, etc. [1], [7], [10]. In viscoelastic problems, the memory integral in (1) can be thought of as representing viscoelastic forces, whereas μu_{xx} term represents Newtonian contribution to viscosity [10].

The significance and novelty of the paper is as follows.

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(i) The most significant result of this paper is that an analytic solution has been found to the problem (1)-(3) where the Newtonian contribution to viscosity $\mu \neq 0$. There have been many numerical approaches to solve this problem [2], [8], [11], [13], [14], [16] but an exact solution obtained in some of those papers was only for the non-Newtonian problem (that is, when $\mu = 0$). To our knowledge this is the first paper that gives an exact solution to the Newtonian problem (i.e., when $\mu \neq 0$).
(ii) An analytic solution to (1) when $\mu = 0$ was derived using the variational iteration method (VIM) in [15]. This paper derives an analytic solution to the case when $\mu \neq 0$. It is a significant extension of the result from [15]. To our knowledge, this is the first successful attempt using VIM to solve partial integro-differential equation (1) with $\mu \neq 0$ that models the Newtonian problem.
(iii) A popular numerical method that has been used so far to solve these types of problems is Crank - Nicolson method. Compared with the Crank - Nicolson, the method used in this paper can achieve any desirable accuracy at a much faster speed since the solution that we found is exact in the form of an infinite series.

In this paper we develop a numerical algorithm based on the variational iteration method and then we derive an analytic solution to (1)-(3). The paper is organized as follows. In section II we derive the main results of the paper. Conclusions are in section III.

II. MAIN RESULTS

A. Numerical solution

To solve (1), we employ the VIM. The VIM, [5], [6], was proposed by J. H. He to solve differential equations using an iterative scheme. To illustrate the main idea of VIM, consider the following, in general, nonlinear equation

$$Lu(t) + Ru(t) = g(t),$$

where L is a linear operator, R is a nonlinear operator, and g is a given function. One constructs a correction functional as follows

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^t \lambda (Lu_n(s) + R\tilde{u}_n(s) - g(s))\, ds,$$

where λ is a Lagrange multiplier, and $\tilde{u}_n$ is considered a restricted variation (see [5], [6]). This gives the desired iterative scheme.

Applying this method to (1), we find $\lambda = -1$ (see [12]), and as a result we obtain the following iteration formula (with $u_0 = \sin(\pi x)$).

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$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\tau)}{\partial \tau} \right. \\ &\left. -\mu \frac{\partial^2 u_n(x,\tau)}{\partial x^2} - \int_0^\tau \frac{\partial^2 u_n(x,s)/\partial x^2}{\sqrt{\tau - s}} ds \right] d\tau \\ &= u_n(x,0) + \int_0^t \mu \frac{\partial^2 u_n(x,\tau)}{\partial x^2} \, d\tau \\ &+ \int_0^t \int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2 u_n(x,s)}{\partial x^2} \, ds d\tau. \end{aligned} \quad (4)$$

Then for $n = 1, 2, \cdots$, we have

$$u_1(x,t) = \sin(\pi x) - \mu\pi^2 t \sin(\pi x) - \frac{4}{3}\pi^2 t^{3/2} \sin(\pi x),$$

$$\begin{aligned} u_2(x,t) &= \sin(\pi x) - \mu\pi^2 t \sin(\pi x) - \frac{4}{3}\pi^2 t^{3/2} \sin(\pi x) \\ &+ \frac{1}{2}\mu^2\pi^4 t^2 \sin(\pi x) + \frac{16}{15}\mu\pi^4 t^{5/2} \sin(\pi x) \\ &+ \frac{1}{6}\pi^5 t^3 \sin(\pi x), \text{ etc.} \end{aligned}$$

From this iteration process (4), it can be observed that the numerical solution of VIM shows a reasonably rapid convergence of iterates after around forty iterations. It can be shown by induction that the Mth iteration of (4) is given by

$$u_M(x,t) = \sum_{n=0}^{M}\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^n \mu^{n-k} \pi^{2n+\frac{k}{2}} t^{n+\frac{k}{2}}}{\Gamma\left(1 + n + \frac{k}{2}\right)} \sin(\pi x), \quad (5)$$

where $\Gamma(\cdot)$ denotes the gamma function.

The following table shows the comparison between the 40th iteration u_{40} of variational iteration method at $T = 1.0$ and the numerical solutions of Crank-Nicolson method with $\Delta x = 0.1$ and $\Delta t = 0.005$.

TABLE I
COMPARISON BETWEEN VIM AND CRANK - NICOLSON FOR $T = 1.0, \mu = 1$

x	VIM	Crank - Nicolson	$\mid u_{\text{VIM}} - u_{\text{CN}} \mid$
0.1	-0.00395205	-0.00388268	6.9×10^{-5}
0.2	-0.00751725	-0.00738531	1.3×10^{-4}
0.3	-0.01034661	-0.01016501	1.8×10^{-4}
0.4	-0.01216317	-0.01194968	2.1×10^{-4}
0.5	-0.01278912	-0.01256464	2.2×10^{-4}
0.6	-0.01216317	-0.01194968	2.1×10^{-4}
0.7	-0.01034661	-0.01016501	1.8×10^{-4}
0.8	-0.00751725	-0.00738531	1.3×10^{-4}
0.9	-0.00395205	-0.00388268	6.9×10^{-5}

It is important to point out that our numerical algorithm is designed in such a way that the iterations can be computed rapidly in Maple with only a few seconds to complete the 40 iterations. In comparison, the computing time in Maple for the Crank - Nicolson method is over one minute, much slower than the VIM algorithm. Furthermore, based on the theorem in the next subsection, the numerical solution (5) becomes more and more accurate than Crank-Nicolson solution as the number of iterations increases. The graph of $u(x,t)$ computed by using VIM with $n = 100$ is shown on Figure 1.

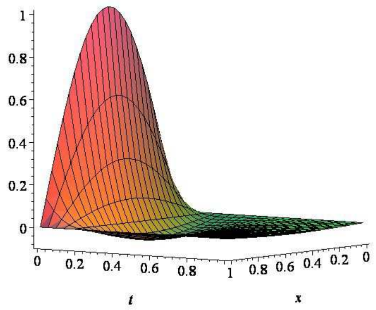

Fig. 1. $u(x,t)$ for $0 \le x \le 1$, $0 \le t \le 1, \mu = 1$

B. Analytic solution

In this subsection we derive the analytic solution to (1)-(3). The results are summarized in the following

Theorem II.1. *The series solution for the problem (1)-(3) is given by*

$$u(x,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^n \mu^{n-k} \pi^{2n+\frac{k}{2}} t^{n+\frac{k}{2}}}{\Gamma\left(1 + n + \frac{k}{2}\right)} \sin(\pi x). \quad (6)$$

Proof: First, we show that the series solution (6) converges uniformly on $[0,1] \times [0,T]$ and for $\mu > 0$. We prove it using the Weierstrass Comparison Test and the Ratio Test. Let

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^n \mu^{n-k} \pi^{2n+\frac{k}{2}} t^{n+\frac{k}{2}}}{\Gamma\left(1 + n + \frac{k}{2}\right)} \sin(\pi x),$$

and let $T_* = \max(1,T)$ and $\mu_* = \max(1,\mu)$ for $\mu > 0$ and $T > 0$. Taking into account that

$$\left| \frac{(-1)^n \mu^{n-k} \pi^{2n+\frac{k}{2}} t^{n+\frac{k}{2}}}{\Gamma(1 + n + \frac{k}{2})} \sin(\pi x) \right| \le \frac{(\mu_* \pi^3 T_*^2)^n}{\Gamma(1+n)}$$

and $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, we obtain

$$\begin{aligned} |a_n| &= \left| \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^n \mu^{n-k} \pi^{2n+\frac{k}{2}} t^{n+\frac{k}{2}}}{\Gamma(1 + n + \frac{k}{2})} \sin(\pi x) \right| \\ &\le \frac{(\mu_* \pi^3 T_*^2)^n}{\Gamma(1+n)} \sum_{k=0}^{n} \binom{n}{k} = \frac{(2\mu_* \pi^3 T_*^2)^n}{\Gamma(1+n)}. \end{aligned}$$

Now, let $Q_n = \Gamma\left(1+n\right)^{-1} (2\mu_* \pi^3 T_*^2)^n$. Then, by the ratio test, we can easily show the convergence of $\sum_{n=0}^{\infty} Q_n$. Namely,

$$\lim_{n\to\infty} \frac{Q_{n+1}}{Q_n} = \lim_{n\to\infty} \frac{2\pi^3 \mu_* T_*^2}{n+1} = 0.$$

Therefore, we conclude that the series $\sum_{n=0}^{\infty} a_n$ converges uniformly by the Weierstrass Comparison Theorem.

Next, we show that the series (6) satisfies (1)-(3). Substituting (6) into the left hand side of (1), we obtain

$$\begin{aligned} &u_t(x,t)\\ &= \sin(\pi x)\sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^n\mu^{n-k}\pi^{2n+\frac{k}{2}}(n+\frac{k}{2})t^{n+\frac{k}{2}-1}}{\Gamma\left(1+n+\frac{k}{2}\right)}\\ &= \sin(\pi x)\sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^n\mu^{n-k}\pi^{2n+\frac{k}{2}}t^{n+\frac{k}{2}-1}}{\Gamma\left(n+\frac{k}{2}\right)}. \end{aligned} \tag{7}$$

For the first term on the right hand side, we get

$$\begin{aligned} &\mu u_{xx}(x,t)\\ &= \sin(\pi x)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n+1}\mu^{n-k+1}\pi^{2n+\frac{k}{2}+2}t^{n+\frac{k}{2}}}{\Gamma\left(1+n+\frac{k}{2}\right)} \end{aligned} \tag{8}$$

$$\begin{aligned} &= \sin(\pi x)\sum_{m=1}^{\infty}\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{(-1)^m\mu^{m-k}\pi^{2m+\frac{k}{2}}t^{m+\frac{k}{2}-1}}{\Gamma\left(m+\frac{k}{2}\right)}\\ &= \sin(\pi x)\sum_{m=1}^{\infty}\left[\frac{(-1)^m\mu^m\pi^{2m}t^{m-1}}{\Gamma(m)}\right.\\ &\quad\left.+\sum_{k=1}^{m-1}\binom{m-1}{k}\frac{(-1)^m\mu^{m-k}\pi^{2m+\frac{k}{2}}t^{m+\frac{k}{2}-1}}{\Gamma\left(m+\frac{k}{2}\right)}\right]. \end{aligned}$$

where we used $m = n+1$ in (8). For the integral on the right hand side of (1), we notice that with a substitution $s = ty$,

$$\begin{aligned} \int_0^t (t-s)^{-1/2}s^{n+\frac{k}{2}}ds &= t^{n+\frac{k}{2}+\frac{1}{2}}B\left(1+n+\frac{k}{2},\frac{1}{2}\right) \\ &= t^{n+\frac{k}{2}+\frac{1}{2}}\frac{\Gamma\left(1+n+\frac{k}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+n+\frac{k}{2}\right)}, \end{aligned} \tag{9}$$

where $B(a,b)$ is the beta function, which is related to gamma function by $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Then it follows from (9) that

$$\begin{aligned} &\int_0^t (t-s)^{-1/2}u_{xx}(x,s)\,ds \\ &= \sin(\pi x)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n+1}\mu^{n-k}\pi^{2n+\frac{k}{2}+2}}{\Gamma\left(1+n+\frac{k}{2}\right)}\\ &\times\int_0^t (t-s)^{-1/2}s^{n+\frac{k}{2}}ds\\ &= \sin(\pi x)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n+1}\mu^{n-k}\pi^{2n+2+\frac{k}{2}+\frac{1}{2}}t^{n+\frac{k}{2}+\frac{1}{2}}}{\Gamma\left(\frac{3}{2}+n+\frac{k}{2}\right)}. \end{aligned} \tag{10}$$

Changing the index $p = n+1$ and $q = k+1$ in (10) gives

$$\begin{aligned} &\int_0^t \frac{u_{xx}}{\sqrt{t-s}}(x,s)ds \\ &= \sin(\pi x)\sum_{p=1}^{\infty}\sum_{q=1}^{p}\binom{p-1}{q-1}\frac{(-1)^p\mu^{p-q}\pi^{2p+\frac{q}{2}}t^{p-1+\frac{q}{2}}}{\Gamma\left(p+\frac{q}{2}\right)}\\ &= \sin(\pi x)\sum_{p=1}^{\infty}\left[\sum_{q=1}^{p-1}\binom{p-1}{q-1}\frac{(-1)^p\mu^{p-q}\pi^{2p+\frac{1}{2}q}t^{p-1+\frac{q}{2}}}{\Gamma\left(p+\frac{q}{2}\right)}\right.\\ &\left.+\frac{(-1)^p\pi^{2p+\frac{p}{2}}t^{p-1+\frac{p}{2}}}{\Gamma\left(p+\frac{1}{2}p\right)}\right]. \end{aligned} \tag{11}$$

Thus, by adding (8) and (11) we have

$$\begin{aligned} &\mu u_{xx}(x,t) + \int_0^t (t-s)^{-\frac{1}{2}}u_{xx}(x,s)ds\\ &= \sin(\pi x)\sum_{n=1}^{\infty}\left\{\frac{(-1)^n\mu^n\pi^{2n}t^{n-1}}{\Gamma(n)}\right.\\ &+\sum_{k=1}^{n-1}\left[\binom{n-1}{k}+\binom{n-1}{k-1}\right]\\ &\times\left.\frac{(-1)^n\mu^{n-k}\pi^{2n+\frac{k}{2}}t^{n+\frac{k}{2}-1}}{\Gamma\left(n+\frac{k}{2}\right)}+\frac{(-1)^n\pi^{2n+\frac{n}{2}}t^{n-1+\frac{n}{2}}}{\Gamma\left(n+\frac{n}{2}\right)}\right\}\\ &= \sin(\pi x)\sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^n\mu^{n-k}\pi^{2n+\frac{k}{2}}t^{n+\frac{k}{2}-1}}{\Gamma\left(n+\frac{k}{2}\right)}, \end{aligned} \tag{12}$$

since it is easy to verify that $\binom{n-1}{k}+\binom{n-1}{k-1} = \binom{n}{k}$. Also notice that the coefficients of the first and third terms in the curly bracket $\{\ \}$ both are 1, which can be expressed as $\binom{n}{0}$ and $\binom{n}{n}$, respectively.

Comparing (12) and (7), we see that the series (6) satisfies (1). Furthermore, one can also easily check that the series (6) satisfies the initial and boundary conditions (2) and (3), respectively. Thus, we conclude that (6) is indeed the analytic solution to the given problem. ■

III. Conclusion

We solved a partial integro-differential equation numerically and analytically using variational iteration method as an appropriate tool. We also used Maple to calculate the series solution. The numerical algorithm we developed shows a rapid convergence after a reasonable number of iterations. It is much more efficient and accurate than finite difference methods. Even more significantly, we have found an analytic solution to the problem represented by an infinite series, which makes it possible to determine the solution of the partial integro-differential problem with any desirable accuracy.

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