Numerical Method for solving Volterra Integral Equations with a Convolution Kernel

Changqing Yang, Jianhua Hou

Abstract—**This paper presents a numerical method for solving the Volterra integral equation with a convolution kernel. The integral equation was first converted to an algebraic equation using the Laplace transform, after which its numerical inversion was determined by power series. The Padé approximants were effectively used to improve the convergence rate and accuracy of the computed series. The method is described and illustrated with numerical examples. The results revealed that the method is accurate and easy to implement.**

Index Terms—**Volterra integral equation, Laplace transform, Taylor expansion, series solution, Padé approximant.**

I. Introduction

VOLTERRA integral equations have many applications in various areas, including mathematical physics, chemistry, electrochemistry, semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reaction and population dynamics[1], [2].

This paper focuses on a class of Volterra integral equations with a convolution kernel given by

$$u(x) = f(x) + \int_0^x k(x-t)u(t)dt,\ x \in [0,T], \tag{1}$$

where the source function f and the kernel function k are given, and $u(x)$ is the unknown function. Several numerical methods are available for approximating the Volterra integral equation. In particular, Huang[3] used the Taylor expansion of unknown function and obtained an approximate solution. Yang[4] proposed a method for the solution of integral equation using the Chebyshev polynomials, while Yousefi[5] presented a numerical method for the Abel integral equation by Legendre wavelets. Khodabin [6] numerically solved the stochastic Volterra integral equations using triangular functions and their operational matrix of integration. Kamyad [7] proposed a new algorithm based on the calculus of variations and discretisation method, in order to solve linear and nonlinear Volterra integral equations. The Adomian decomposition [8], [9], [10], Homotopy perturbation [10], [11] and the Laplace decomposition methods[12] were proposed for obtaining the approximate analytic solution of the integral equation.

In this paper, Volterra integral equations were first reduced to algebraic equations using the Laplace transform. We obtained a series that was uniformly convergent to the exact solution after applying the Taylor expansion and the inverse Laplace transforms to the mentioned algebraic equations. The main advantage of this method is its simplicity, such that only few of the terms of the expansion are needed to obtain good convergent numerical results.

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II. Laplace transform and their properties

This section provides the definition and properties of the Laplace transform[13].

Definition II.1. The Laplace transform of a function $f(x), x > 0$ is defined as

$$\mathcal{L}[f(x)] = F(s) = \int_0^{+\infty} e^{-sx} f(x)dx,$$

where s can either be real or complex.

The Laplace transform has several properties, as explained below:

1) Linearity property

$$\mathcal{L}[af(x) + bg(x)] = a\mathcal{L}[f(x)] + b\mathcal{L}[g(x)],$$

where a, b are constants.

2) The convolution theorem
Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by

$$\mathcal{L}[f_1(x)] = F_1(s), \quad \mathcal{L}[f_2(x)] = F_2(s).$$

The Laplace convolution product of these two functions is defined by

$$\mathcal{L}\left[\int_0^x f_1(x-t)f_2(t)dt\right] = F_1(s)F_2(s), \tag{2}$$

Theorem II.2. *[13] Suppose $F(s)$ is the Laplace transform of $f(x)$, which has a Maclaurin power series expansion in the form*

$$f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}. \tag{3}$$

Taking the Laplace transform, it is possible to write formally

$$F(s) = \sum_{i=0}^{\infty} \frac{a_i}{s^{i+1}}. \tag{4}$$

Conversely, we derive (3) form a given expansion (4).

III. Padé approximant

A Padé approximant refer to the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function. The $[L/M]$ Padé approximant to a formal power series $y(t) = \sum_{i=0}^{\infty} a_i t^i$ is given by:

$$\left[\frac{L}{M}\right] = \frac{P_L(t)}{Q_M(t)} = \frac{p_0 + p_1 t + \cdots + p_L t^L}{1 + q_1 t + \cdots + q_M t^M}. \tag{5}$$

The two polynomials in the numerator and denominator of (5) have no common factor, thus indicating the formal power series expressed by

$$y(t) = \frac{P_L(t)}{Q_M(t)} + O(t^{L+M+1}).$$

In this case, the Padé approximant $[L/M]$ is uniquely determined.

IV. Solution of Volterra integral equation and error estimate

In this section we solved Volterra integral equation (1) by the Laplace transform and Taylor series. First, the Laplace transform is applied to both sides of Equation(1)

$$\mathcal{L}[u(x)] = \mathcal{L}[f(x)] + \mathcal{L}\left[\int_0^x k(x-t)u(t)dt\right]$$

Using the Laplace transform property (2), the equation below can be obtained

$$\mathcal{L}[u] = \mathcal{L}[f] + \mathcal{L}[k]\mathcal{L}[u].$$

Thus, the given equation is equivalent to

$$\mathcal{L}[u] = \frac{\mathcal{L}[f]}{1-\mathcal{L}[k]} = F(s).$$

Applying Theorem(II.2), $F(s)$ can be expanded as an absolutely convergent series, which is given by

$$\mathcal{L}[u] = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s^3} + \cdots$$

where $c_1, c_2, c_3, \cdots$ are the known constants. Considering the inverse Laplace transform on both sides of the above equation, we can then obtain

$$u(x) = c_1 + \frac{c_2}{\Gamma(2)}x + \frac{c_3}{\Gamma(3)}x^2 + \frac{c_4}{\Gamma(4)}x^3 \cdots \tag{6}$$

which is uniformly convergent to the exact solution. We approximate the solution $u(x)$ by using

$$u_n(x) = c_1 + \frac{c_2}{\Gamma(2)}x + \frac{c_3}{\Gamma(3)}x^2 + \cdots + \frac{c_n}{\Gamma(n)}x^{n-1}.$$

Let $e_n(x) = u(x) - u_n(x)$ be the error function, where $u_n(x)$ is the estimation of the true solution $u(x)$. Using Taylor's theorem and assuming that $|u^{(n)}(x)| \le M$, the equation below can be obtained

$$|e_n| = |u(x) - u_n(x)| \le M\frac{c_{n+1}}{\Gamma(n+1)}|x^n|.$$

Padé approximants have the advantage of manipulating the polynomial approximation into a rational function in order to gain more information about $u(x)$. Consequently, the series (6) should be manipulated to construct Padé approximants, such that the performance of the approximants shows superiority over the series solutions.

V. Numerical examples

In this section, the effectiveness and simplicity of the proposed method were demonstrated using three examples. All the results were calculated using the symbolic calculus software Mathematica.

Example V.1. Consider the Abel integral equation given by

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}dt = e^{-x} - 1, \ x \in [0,1]. \tag{7}$$

We using the Laplace transform and convolution property given by

$$\mathcal{L}[u]\mathcal{L}[x^{-\frac{1}{2}}] = \mathcal{L}[e^{-x} - 1],$$

so that

$$\mathcal{L}[u] = \frac{-1}{\sqrt{\pi}\sqrt{s}(s+1)} = F(s).$$

Expanding the right hand side $F(s)$ in the power of $1/s$, we obtain

$$\mathcal{L}[u] = \frac{1}{\sqrt{\pi}}\left(-\left(\frac{1}{s}\right)^{\frac{3}{2}} + \left(\frac{1}{s}\right)^{\frac{5}{2}} - \left(\frac{1}{s}\right)^{\frac{7}{2}} + \left(\frac{1}{s}\right)^{\frac{9}{2}} \cdots\right).$$

By taking the inverse Laplace transform of both sides of the above equation, the series solution can new be expressed as:

$$\begin{aligned} u(x) = & \frac{-2\sqrt{x}}{\pi}\left(1 - \frac{2}{3}x + \frac{4}{15}x^2 - \frac{8}{105}x^3\right. \\ & + \frac{16}{945}x^4 - \frac{32}{10395}x^5 + \frac{64}{135135}x^6 \\ & \left.- \frac{128}{2027025}x^7 - \frac{256}{34459425}x^8 \cdots\right), \end{aligned} \tag{8}$$

which is consistent with the results obtained in a previous work[9].

Padé approximants also have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $u(x)$. The Padé approximants $[3/3], [4/4]$ and $[5/5]$ of $u(x)$ were constructed in order to study the structure of the obtained solution(8). For example, the Padé approximants $[4/4]$ is given by

$$u(x) \approx \frac{-2\sqrt{x}}{\pi} \cdot \frac{1 - \frac{10}{51}x + x^2 - \frac{88}{23205}x^3 + \frac{2048}{11486475}x^4}{1 + \frac{8}{17}x + \frac{8}{85}x^2 + \frac{32}{3315}x^3 + \frac{16}{36465}x^4}.$$

The numerical results show in Fig.1.

Example V.2. Consider the Abel integral equation of second kind expressed as[5], [14]:

$$u(x) = 2\sqrt{x} - \int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \ x \in [0,1]. \tag{9}$$

The exact solution is $u(x) = 1 - e^{\pi x}erfc(\sqrt{\pi x})$, where $erfc(\sqrt{\pi x})$ is the complementary error function and defined as

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt.$$

Applying the Laplace transform and convolution property (2), we arrive at

$$\mathcal{L}[u] = \mathcal{L}[2\sqrt{x}] - \mathcal{L}[x^{-\frac{1}{2}}]\mathcal{L}[u].$$

Hence,

$$\mathcal{L}[u] = \frac{\mathcal{L}[2\sqrt{x}]}{1 + \mathcal{L}[x^{-\frac{1}{2}}]},$$

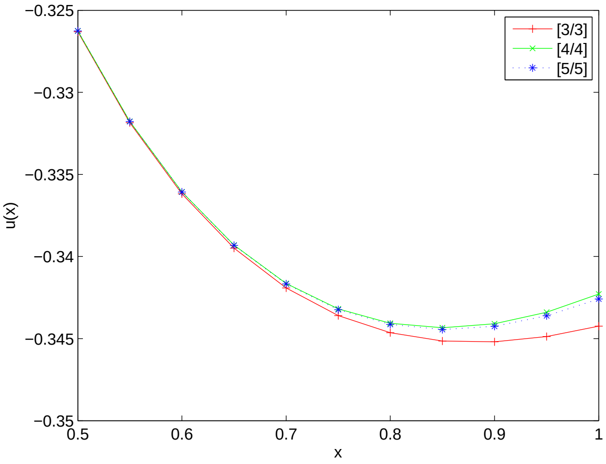

Fig. 1. Numerical results for Example1

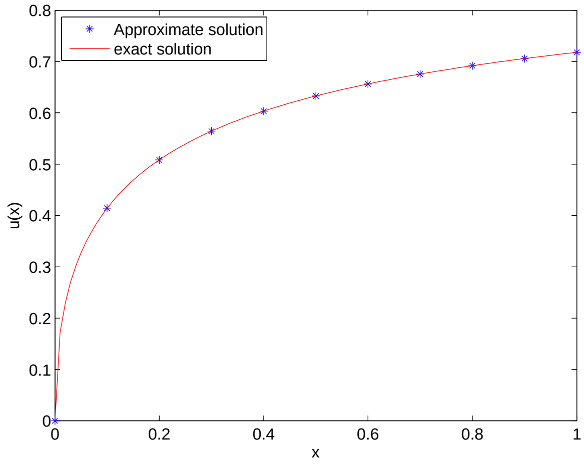

Fig. 2. Comparison of approximate solutions [4/4] and exact solution

or equivalently,

$$\mathcal{L}[u] = \frac{\sqrt{\pi}}{s(\sqrt{s}+\pi)} = F(s).$$

The right hand side of $F(s)$ expanded in the power of $1/s$ as in

$$\begin{aligned} F(s) =& \pi^{\frac{1}{2}}\left(\frac{1}{s}\right)^{\frac{3}{2}} - \pi\left(\frac{1}{s}\right)^{2} + \pi^{\frac{3}{2}}\left(\frac{1}{s}\right)^{\frac{5}{2}} \\ & - \pi^{2}\left(\frac{1}{s}\right)^{3} + \pi^{\frac{5}{2}}\left(\frac{1}{s}\right)^{\frac{7}{2}} - \pi^{2}\left(\frac{1}{s}\right)^{4} \\ & + \pi^{\frac{7}{2}}\left(\frac{1}{s}\right)^{\frac{9}{2}} - \pi^{4}\left(\frac{1}{s}\right)^{5} \cdots . \end{aligned} \tag{10}$$

By applying the inverse Laplace transform to (10), we obtain

$$\begin{aligned} u(x) =& 2x^{\frac{1}{2}} - \pi x + \frac{4\pi}{3}x^{\frac{3}{2}} - \frac{\pi^2}{2}x^2 + \frac{5\pi^2}{18}x^{\frac{5}{2}} \\ & - \frac{\pi^3}{6}x^3 + \frac{16\pi^3}{105}x^{\frac{5}{2}} - \frac{\pi^4}{24}x^4 \cdots . \end{aligned} \tag{11}$$

Similarly, $t = x^{\frac{1}{2}}$ was first substituted, the Padé approximant $[3/3]$ and $[4/4]$ of $u(t)$ was constructed, and the approximation $[4/4]$ was provided to the solution:

$$u(x) \approx \frac{2x^{\frac{1}{2}} + 3.0829x + 2.0808x^{\frac{3}{2}} + 0.5207x^2}{1 + 3.1123x^{\frac{1}{2}} + 3.8347x + 2.2230x^{\frac{3}{2}} + 0.5235x^2}.$$

The results shown in Fig. 2 demonstrate that the approximate solutions obtained using the proposed method are in good agreement with the exact solutions. When the solution $u(x)$ has a special form, the proposed method works well. Furthermore, the numerical results in Table1 reveal that higher order accuracy can be achieved by increasing some terms of the expansion. In this way, more terms would enhance the level of accuracy of the numerical approximation. Moreover, by comparing the results of the table, the results of the presented method are more accurate than those of the wavelet method presented in a previous work[14].

Example V.3. Consider the Volterra integral with a convolution kernel given by[15]

$$u(x) + \int_0^x \cos(x-t)u(t)dt = \sin(x). \tag{12}$$

TABLE I
COMPUTED ABSOLUTE ERRORS FOR EXAMPLEV.2

x	Presented method [3/3]	Presented method [3/3]	Wavelets method ($k=0, M=16$)
0	0	0	0
0.1	7.26464×10^{-7}	4.33846×10^{-9}	1.15872×10^{-2}
0.2	4.50941×10^{-6}	4.87786×10^{-8}	1.13995×10^{-2}
0.3	1.21047×10^{-5}	1.82276×10^{-7}	9.55367×10^{-3}
0.4	2.34327×10^{-5}	4.42272×10^{-7}	1.68378×10^{-3}
0.5	3.81788×10^{-5}	8.53771×10^{-7}	7.61903×10^{-3}
0.6	5.59739×10^{-5}	1.43214×10^{-6}	1.53846×10^{-3}
0.7	7.64562×10^{-5}	2.18575×10^{-6}	3.09894×10^{-3}
0.8	9.92936×10^{-5}	3.11805×10^{-6}	2.98197×10^{-3}
0.9	1.24188×10^{-4}	4.22898×10^{-6}	7.08482×10^{-4}

The exact solution is $u(x) = \frac{2\sqrt{3}}{3}sin(\sqrt{3}x/2)e^{-\frac{x}{2}}$. Taking the Laplace transforms of both sides of (12) can yield

$$\mathcal{L}[u] + \mathcal{L}\left[\int_0^x \cos(x-t)u(t)dt\right] = \mathcal{L}[\sin(x)]$$

Applying (2) we obtain

$$\mathcal{L}[u] + \mathcal{L}[\cos(x)]\,\mathcal{L}[u] = \mathcal{L}[\sin(x)],$$

which provides

$$\mathcal{L}[u] = \frac{\mathcal{L}[\sin(x)]}{1+\mathcal{L}[\cos(x)]} = \frac{1}{s^2+s+1}.$$

Expanding the right hand side of the above equation in power of $1/s$, we obtain

$$\mathcal{L}[u] = \frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s^5} - \frac{1}{s^6} + \frac{1}{s^8} - \frac{1}{s^9} + \frac{1}{s^{11}} - \frac{1}{s^{12}} \cdots .$$

Applying the inverse Laplace transform to the above equation, we obtain

$$u(x) = x - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^8}{8!} + \frac{x^{10}}{10!} - \frac{x^{11}}{11!} \cdots .$$

We construct the Padé approximants $[3/3]$, $[4/4]$ and $[5/5]$

TABLE II
ABSOLUTE ERRORS OF [3/3], [4/4] AND [5/5]FOR EXAMPLEV.3

x	[3/3]	[4/4]	[5/5]
0	0	0	0
0.1	2.47996×10^{-13}	2.12330×10^{-15}	4.16334×10^{-17}
0.2	6.34643×10^{-11}	1.04025×10^{-12}	3.05311×10^{-16}
0.3	1.62565×10^{-9}	3.81095×10^{-11}	2.86993×10^{-14}
0.4	1.62261×10^{-8}	4.83148×10^{-10}	6.49258×10^{-13}
0.5	9.66243×10^{-8}	3.42285×10^{-9}	7.20879×10^{-12}
0.6	4.14996×10^{-7}	1.67752×10^{-8}	5.10483×10^{-11}
0.7	1.42247×10^{-6}	6.37358×10^{-8}	2.64983×10^{-10}
0.8	4.13357×10^{-6}	2.00938×10^{-7}	1.09561×10^{-9}
0.9	1.05880×10^{-5}	5.49231×10^{-7}	3.80688×10^{-9}
1	2.45509×10^{-5}	1.34114×10^{-6}	1.15307×10^{-8}

of $u(x)$:

$$\left[\frac{3}{3}\right] = \frac{x + x^2 - \frac{9x^3}{20}}{1 + \frac{3x}{2} + \frac{3x^2}{10} + \frac{13x^3}{120}},$$

$$\left[\frac{4}{4}\right] = \frac{x - \frac{48x^2}{223} - \frac{113x^3}{1561} + \frac{149x^4}{9366}}{1 + \frac{127x}{446} + \frac{437x^2}{6244} + \frac{173x^3}{18732} + \frac{470x^4}{374640}},$$

$$\left[\frac{5}{5}\right] = \frac{x - \frac{4205x^2}{12627} - \frac{775x^3}{16836} + \frac{1285x^4}{50508} - \frac{49363x^5}{21213360}}{1 + \frac{4217x}{25254} + \frac{473x^2}{12627} + \frac{11x^3}{4392} + \frac{1277x^4}{4242672} - \frac{797x^5}{42426720}}.$$

The numerical results shown in Fig. 3 and Table 2 reveal that higher order accuracy can be achieved by increasing some terms of the expansion, that is, using more terms can enhance the level of accuracy of the numerical approximation. This fact ensures that the method is convergent.

VI. CONCLUSION

In this paper, we applied Laplace transform and Taylor series to solve the Volterra integral equation with a convolution kernel. The properties of the Laplace transform, together with Taylor series, are used to reduce the integral equations to the algebraic equations. The method requires much less computational work compared with traditional methods. Although we only considered a model problem in this paper, the main ideas and techniques used here are also applicable to many other problems.

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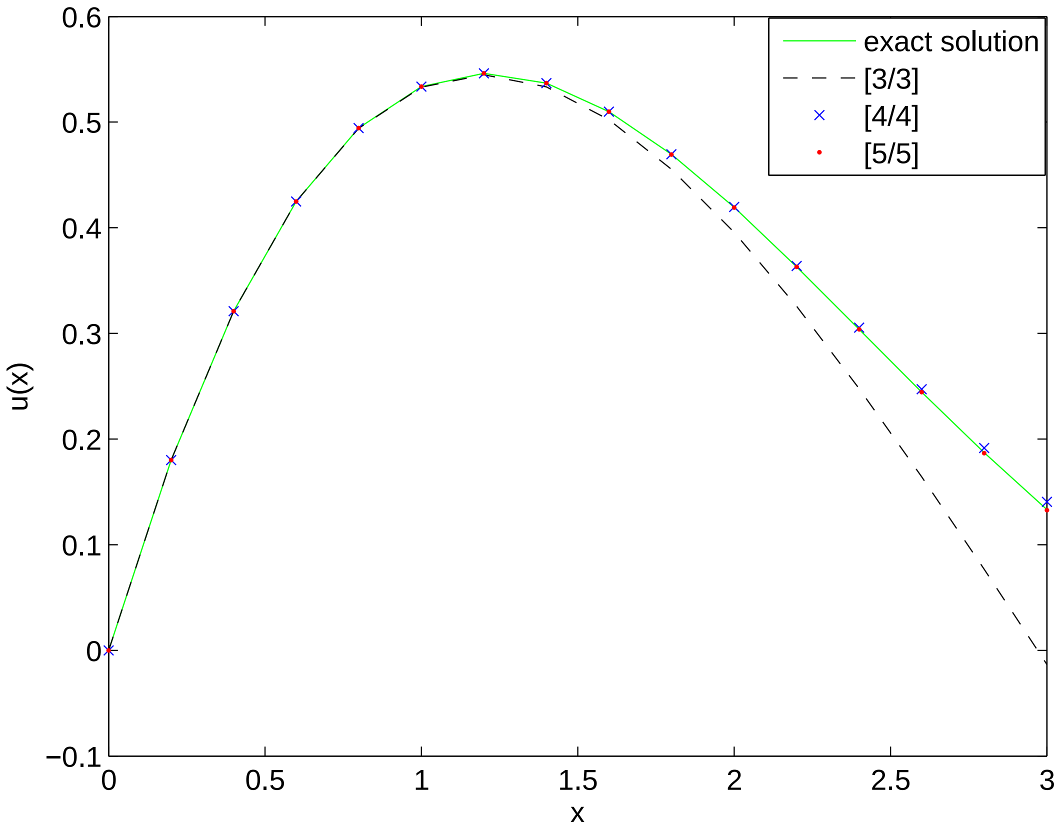

Fig. 3. Comparison of approximate solutions [3/3],[4/4],[5/5] and exact solution