Degree of Approximation of Functions in Lipschitz Class with Muckenhoupt Weights by Matrix Means

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***Abstract*—In this paper, we investigate the approximation properties of the matrix means of trigonometric Fourier series of f belonging to weighted Lipschitz class $Lip(\alpha, p, w)$ with Muckenhoupt weights generated by $T \equiv (a_{n,k})$ under relaxed conditions. Our theorem extends some of the previous results pertaining to the degree of approximation of functions in weighted Lipschitz class $Lip(\alpha, p, w)$ and the ordinary Lipschitz class $Lip(\alpha, p)$.**

***Index Terms*—Fourier series, Matrix means, Muckenhoupt class, Class $Lip(\alpha, p, w)$.**

I. Introduction

A MEASURABLE 2π-periodic function $w : [0, 2\pi] \to [0, \infty]$ is said to be a weight function if the set $w^{-1}(\{0, \infty\})$ has the Lebesgue measure zero. We say that $f \in L^p_w[0, 2\pi](= L^p_w)$, the weighted Lebesgue space of all measurable 2π-periodic functions if

$$\|f\|_{p,w} = \left(\int_0^{2\pi} |f(x)|^p \, w(x) dx\right)^{1/p} < \infty, \; 1 \le p < \infty.$$

Let $1 < p < \infty$. A weight function w belongs to the Muckenhoupt class A_p if

$$\sup_I \left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I [w(x)]^{-1/(p-1)}dx\right)^{p-1} < \infty,$$

where the supremum is taken over all intervals I with length $|I| \le 2\pi$. The weight functions belonging to A_p class, introduced by Hunt et al. [1], play an important role in different fields of mathematical analysis.

Let $w \in A_p$ and $f \in L^p_w$. The modulus of continuity of the function f is defined by

$$\Omega(f, \delta)_{p,w} = \sup_{|h| \le \delta} \|\Delta_h(f)\|_{p,w}, \;\; \delta > 0,$$

where $\quad (\Delta_h f)(x) = \frac{1}{h}\int_0^h |f(x+t) - f(x)| \, dt.$

The existence of the modulus of continuity of $f \in L^p_w$ follows from the boundedness of the Hardy-Littlewood maximal function in the space L^p_w [2]. The modulus of continuity $\Omega(f, \cdot)_{p,w}$ defined by Ky [3] is non-decreasing, non-negative, continuous function such that

$$\lim_{\delta \to 0} \Omega(f, \delta)_{p,w} = 0$$

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and $\quad \Omega(f_1 + f_2, \cdot)_{p,w} \le \Omega(f_1, \cdot)_{p,w} + \Omega(f_2, \cdot)_{p,w}.$

The modulus of continuity $\Omega(f, \cdot)_{p,w}$ is defined in this way, since the space L^p_w is non-invariant, in general, under the usual shift $f(x) \to f(x+h)$. We note that in the case $w \equiv 1$, the modulus of continuity $\Omega(f, \cdot)_{p,w}$ and the classical integral modulus of continuity $w_p(f, \cdot)$ are equivalent [3]. The weighted Lipschitz class $Lip(\alpha, p, w)$ for $0 < \alpha \le 1$ is defined by

$$Lip(\alpha, p, w) = \{f \in L^p_w : \Omega(f, \delta)_{p,w} = O(\delta^\alpha), \delta > 0\}.$$

For $w(x) = 1 \, \forall x \in [0, 2\pi]$ the weighted $Lip\,(\alpha, p, w)$ class reduces to well known Lipschitz class $Lip\,(\alpha, p)$.

Let $f \in L_p[0, 2\pi]$ $(p \ge 1)$ be a 2π-periodic function. Then, for $n \in \mathbb{N} \cup \{0\}$ we write

$$\begin{aligned} s_n(f;x) &= \; a_0/2 + \sum\nolimits_{k=1}^n (a_k \cos kx + b_k \sin kx) \\ &= \; \sum\nolimits_{k=0}^n u_k(f;x), \; u_0(f;x) = s_0(f;x) = a_0/2, \end{aligned}$$

the $(n+1)^{th}$ partial sum of Fourier series of f at point x, which is a trigonometric polynomial of order (or degree) n. The matrix means of the Fourier series of f are defined by the sequence to sequence transformation

$$\tau_n(f;x) = \tau_n(x) = \sum\nolimits_{k=0}^n a_{n,k} s_k(f;x), \;\; n \in \mathbb{N} \cup \{0\}, \tag{1}$$

where $T \equiv (a_{n,k})$ is a lower triangular matrix with non-negative entries such that $a_{n,-1} = 0$, $A_{n,k} = \sum_{r=k}^n a_{n,r}$ and $A_{n,0} = 1 \; \forall n \ge 0$. The Fourier series of f is said to be T-summable to $s(x)$, if $\tau_n(f;x) \to s(x)$ as $n \to \infty$. If for every convergent sequence $\{s_n\}$, $\lim_{n\to\infty} s_n = s$ implies $\lim_{n\to\infty} \tau_n = s$, then matrix T is said to be regular. In particular, if

$$a_{n,k} = \begin{cases} p_{n-k}/P_n, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

$$\text{and} \quad a_{n,k} = \begin{cases} p_k/P_n, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{2}$$

where $P_n(= \sum_{k=0}^n p_k \neq 0) \to \infty$ as $n \to \infty$ and $P_{-1} = 0 = p_{-1}$, then the summability matrix T reduces to Nörlund and Riesz matrices, respectively, and $\tau_n(f;x)$ in (1) defines corresponding means $N_n(f;x)$ and $R_n(f;x)$. In case of $a_{n,k} = 1/(n+1)$ for $0 \le k \le n$ and $a_{n,k} = 0$ for $k > n$, the $\tau_n(f;x)$ reduces to Cesàro means of order one, denoted by $\sigma_n(f;x)$.

A positive sequence $a = \{a_{n,k}\}$ is called almost monotonically decreasing with respect to k, if there exists a constant $K = K(a)$, depending on the sequence a only,

such that $a_{n,p} \leq Ka_{n,m}$ for all $p \geq m$ and we write that $a \in AMDS$. Similarly $a = \{a_{n,k}\}$ is called almost monotonically increasing with respect to k, if $a_{n,p} \leq Ka_{n,m}$ for all $p \leq m$ and we write that $a \in AMIS$ [4, 5]. We note that every monotone sequence is an almost monotone sequence.
We also write $\Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}$ and $[x]$, the greatest integer contained in x.

The Fourier series and trigonometric polynomials play an important role in various scientific and engineering fields, e.g., Lo and Hui [6] use the Fourier series expansion in a very nice way. Based upon the Fourier series expansion, they propose a simple and easy-to-use approach for computing accurate estimates of Black-Scholes double barrier option prices with time-dependent parameters.

Chandra [7] has studied the approximation properties of the means $N_n(f;x)$ and $R_n(f;x)$ in $Lip(\alpha,p)$, $1 \leq p < \infty$, $0 < \alpha \leq 1$ with monotonicity conditions on the means generating sequence $\{p_n\}$ and proved $\|N_n(f;x) - f(x)\|_p = O(n^{-\alpha}) = \|R_n(f;x) - f(x)\|_p$, $n = 1,2,3....$ Mittal et al. [8] generalized the paper of Chandra [7] partially, and extended its Theorem 1 and Theorem 2 (ii) to matrix means with $|\sum_{k=0}^{n} a_{n,k} - 1| = O(n^{-\alpha})$. On the other hand, Leindler [4] has relaxed the condition of monotonicity on $\{p_n\}$ and proved some of the results of Chandra [7] for almost monotone weights $\{p_n\}$. Mittal et al. [5] extended the results of Leindler [4] to matrix means with almost monotone sequence $\{a_{n,k}\}$ and row sums 1. Following [9], [10] and [11], recently Guven [12] has extended some of the results of Chandra [7] in another direction. He has extended Lipschitz class $Lip(\alpha,p)$ to $Lip(\alpha,p,w)$, and proved the weighted version of the Theorem 1 and Theorem 2 of Chandra [7] for $1 < p < \infty$, $0 < \alpha \leq 1$ i.e., for $f \in Lip(\alpha,p,w)$ and monotone $\{p_n\}$ he proved

$$\begin{aligned}&\|N_n(f;x) - f(x)\|_{p,w}\\ &= O(n^{-\alpha}) = \|R_n(f;x) - f(x)\|_{p,w}\,,\ n = 1,2,3,...\ .\end{aligned}$$

Following Mittal et al. [5], very recently, Singh and Sonker [13] have studied the degree of approximation of periodic functions in generalized Hölder metric space through matrix means of Fourier series, where matrix $T \equiv (a_{n,k})$ has almost monotone rows. On the other hand, Guven [14, Theorem 1 and Theorem 2] has given the weighted version of [4] and some of the results of [8] by assuming $T \equiv (a_{n,k})$ almost monotone with $|\sum_{k=0}^{n} a_{n,k} - 1| = O(n^{-\alpha})$. We note that condition $|\sum_{k=0}^{n} a_{n,k} - 1| = O(n^{-\alpha})$ was not used by Leindler [4] as $\sum_{k=0}^{n} a_{n,k} = 1$ for Nörlund matrix. It also appears that the author in [14] have followed many conditions and calculations as given in [5] and [13] without citing them.

Apart from this Nigam and Sharma [15] applied the concept of $(C,1)(E,q)$ summability method and establish a new theorem on degree of approximation of a function $f \in Lip(\xi(t),r)$ class. Very recently, the authors in [16] obtained the degree of approximation of functions conjugate to the function belonging to weighted Lipschitz class $W(L^p,\xi(t))$ and find the error of approximation free from p and sharper that the earlier obtained results in this direction.

II. Main Result

In the present paper, we continue the work of [5] and [13] and prove weighted version of the theorem of [5] for $p > 1$ which extend the result of Leindler [4] to weighted version as well as matrix version for $p > 1$. Our theorem also extends theorems of Guven [12] to matrix means $\tau_n(f;x)$ under the relaxed conditions of monotonicity and replaces the two theorems of Guven [14] by a single theorem for $\sum_{k=0}^{n} a_{n,k} = 1$. More precisely, we prove:

Theorem 1. *Let $f \in Lip(\alpha,p,w)$, $p > 1$, $w \in A_p$ and let $T \equiv (a_{n,k})$ be an infinite lower triangular regular matrix and satisfies one of the following conditions:*

(i) $0 < \alpha < 1$, $\{a_{n,k}\} \in AMIS$ in k,
(ii) $0 < \alpha < 1$, $\{a_{n,k}\} \in AMDS$ in k and $(n+1)a_{n,0} = O(1)$,
(iii) $\alpha = 1$ and $\sum_{k=0}^{n-1} (n-k)\,|\Delta_k a_{n,k}| = O(1)$,
(iv) $\alpha = 1$, $\sum_{k=0}^{n} |\Delta_k a_{n,k}| = O(a_{n,0})$ with $(n+1)a_{n,0} = O(1)$,
(v) $0 < \alpha \leq 1$, $\sum_{k=0}^{n-1} \left|\Delta_k \left(\frac{A_{n,0} - A_{n,k+1}}{k+1}\right)\right| = O\left(\frac{1}{n+1}\right)$.

Then

$$\|f(x) - \tau_n(f;x)\|_{p,w} = O((n+1)^{-\alpha}), n = 0,1,2... \quad (3)$$

III. Lemmas

To prove our theorem, we need the following lemmas.

Lemma 1 ([12]). *Let $1 < p < \infty$, $w \in A_p$ and $0 < \alpha \leq 1$. Then the estimate*

$$\|f(x) - s_n(f;x)\|_{p,w} = O((n+1)^{-\alpha}),\ n = 0,1,2,..., \quad (4)$$

holds for every $f \in Lip(\alpha,p,w)$.

Lemma 2 ([12]). *Let $1 < p < \infty$ and $w \in A_p$. Then, for $f \in Lip(1,p,w)$ the estimate*

$$\|s_n(f;x) - \sigma_n(f;x)\|_{p,w} = O((n+1)^{-1}),\ n = 0,1,2,...\,, \quad (5)$$

holds.

Lemma 3 ([5]). *Let either $\{a_{n,k}\} \in AMIS$ or $\{a_{n,k}\} \in AMDS$ with $(n+1)a_{n,0}=O(1)$. Then, for $0 < \alpha < 1$,*

$$\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} = O((n+1)^{-\alpha}).$$

Proof. Let $r = [n/2]$ and $\{a_{n,k}\} \in AMIS$, then

$$\begin{aligned}&\sum_{k=0}^{n}(k+1)^{-\alpha}a_{n,k}\\ &\leq Ka_{n,r}\sum_{k=0}^{r} (k+1)^{-\alpha} + (r+1)^{-\alpha}\sum_{k=r+1}^{n} a_{n,k}\\ &\leq \quad Ka_{n,r}(r+1)^{1-\alpha} + (r+1)^{-\alpha}\sum_{k=0}^{n} a_{n,k}\\ &\leq \quad K(r+1)^{-\alpha}(r+1)a_{n,r} + (r+1)^{-\alpha}A_{n,0}\\ &= \quad O(r+1)^{-\alpha} = O((n+1)^{-\alpha}),\end{aligned}$$

in view of $(r+1)a_{n,r} \leq (n-r+1)a_{n,r} \leq K(a_{n,r}+a_{n,r+1}+ ... + a_{n,n}) \leq A_{n,0}$ and $(r+1)^{-\alpha} = O((n+1)^{-\alpha})$.
If $\{a_{n,k}\} \in AMDS$ and $(n+1)a_{n,0} = O(1)$, then

$$\begin{aligned}\sum_{k=0}^{n} (k+1)^{-\alpha}a_{n,k} &\leq Ka_{n,0}\sum_{k=0}^{n} (k+1)^{-\alpha}\\ &= O((n+1)^{-\alpha}).\end{aligned}$$

This completes the proof of Lemma 3.

IV. Proof of Theorem 1

We prove the cases (i) and (ii) together by using Lemma 1 and Lemma 3. Since

$$\tau_n(f;x)-f(x)=\sum_{k=0}^{n}a_{n,k}\{s_k(f;x)-f(x)\},$$

$$\begin{aligned}\|\tau_n(f;x)-f(x)\|_{p,w} &\leq \sum_{k=0}^{n}a_{n,k}\,\|s_k(f;x)-f(x)\|_{p,w}\\ &= \sum_{k=0}^{n}a_{n,k}O(k+1)^{-\alpha}\\ &= O((n+1)^{-\alpha}).\end{aligned}$$

Next, we consider the case (iv). By Abel's transformation and $a_{n,n+1}=0$,

$$\begin{aligned}\tau_n(f;x) &= \sum_{k=0}^{n}a_{n,k}s_k(f;x)\\ &= \sum_{k=0}^{n}a_{n,k}\left(\sum_{i=0}^{k}u_i(f;x)\right)\\ &= \sum_{k=0}^{n}A_{n,k}u_k(f;x),\end{aligned}$$

and thus

$$\begin{aligned}s_n(f;x)-\tau_n(f;x) &= \sum_{k=0}^{n}(1-A_{n,k})u_k(f;x)\\ &= \sum_{k=1}^{n}k^{-1}(A_{n,0}-A_{n,k})\,ku_k(f;x).\end{aligned}$$

Hence, again by Abel's transformation and $A_{n,n+1}=0$, we get

$$s_n(f;x)-\tau_n(f;x) = \sum_{k=1}^{n}(\Delta_k k^{-1}(A_{n,0}-A_{n,k}))\times \sum_{i=1}^{k}iu_i(f;x)+(n+1)^{-1}\sum_{k=1}^{n}ku_k(f;x).$$

Therefore,

$$\begin{aligned}&\|\,s_n(f;x)-\tau_n(f;x)\|_{p,w}\\ &\leq\sum_{k=1}^{n}\left|\Delta_k k^{-1}(A_{n,0}-A_{n,k})\right|\left\|\sum_{i=1}^{k}iu_i(f;x)\right\|_{p,w}\\ &+\ (n+1)^{-1}\left\|\sum_{k=1}^{n}ku_k(f;x)\right\|_{p,w}. \qquad (6)\end{aligned}$$

Also

$$\begin{aligned}&s_n(f;x)-\sigma_n(f;x)\\ &=(n+1)^{-1}\sum_{k=0}^{n}((n+1)u_k(f;x)-s_k(f;x))\\ &=(n+1)^{-1}\sum_{k=1}^{n}ku_k(f;x),\end{aligned}$$

which implies

$$\begin{aligned}\left\|\sum_{k=1}^{n}ku_k(f;x)\right\|_{p,w} &= (n+1)\,\|\sigma_n(f;x)-s_n(f;x)\|_{p,w}\\ &= O(1), \qquad (7)\end{aligned}$$

in view of Lemma 2.
Using (7) in (6), we get

$$\|s_n(f;x)-\tau_n(f;x)\|_{p,w}\leq\sum_{k=1}^{n}\left|\Delta_k k^{-1}(A_{n,0}-A_{n,k})\right| +(n+1)^{-1}. \qquad (8)$$

Now,

$$\begin{aligned}&\Delta_k k^{-1}\ (\ A_{n,0}-A_{n,k})\\ &= \frac{A_{n,0}-A_{n,k}}{k}-\frac{A_{n,0}-A_{n,k+1}}{k+1}\\ &= k^{-1}(k+1)^{-1}(A_{n,0}-A_{n,k}-ka_{n,k})\\ &= k^{-1}(k+1)^{-1}(\sum_{i=0}^{k-1}a_{n,i}-ka_{n,k}). \qquad (9)\end{aligned}$$

Next, we shall verify by induction that,

$$\left|\sum_{i=0}^{k-1}a_{n,i}-ka_{n,k}\right|\leq\sum_{i=1}^{k}i\,|\,a_{n,i-1}-a_{n,i}|. \qquad (10)$$

For $k=1$, we have

$$\left|\sum_{i=0}^{k-1}a_{n,i}-ka_{n,k}\right|=|a_{n,0}-a_{n,1}|=1.\,|a_{n,0}-a_{n,1}|\,,$$

i.e., (10) is true for $k=1$. Let us assume that (10) is true for $k=m$, then for $k=m+1$,

$$\begin{aligned}&|\sum_{i=0}^{m}a_{n,i}-(m+1)a_{n,m+1}|\\ &= \left|\sum_{i=0}^{m-1}a_{n,i}+a_{n,m}+ma_{n,m}\right.\\ &\quad\left.-ma_{n,m}-(m+1)a_{n,m+1}\right|\\ &\leq \left|\sum_{i=0}^{m-1}a_{n,i}-ma_{n,m}\right|+(m+1)\,|a_{n,m}-a_{n,m+1}|\\ &= \sum_{i=1}^{m}i\,|a_{n,i-1}-a_{n,i}|+(m+1)\,|a_{n,m}-a_{n,m+1}|\\ &= \sum_{i=1}^{m+1}i\,|a_{n,i-1}-a_{n,i}|.\end{aligned}$$

Thus (10) is true for $k=m+1$, hence (10) is true for $1\leq k\leq n$.
Using (9) and (10), we get

$$\begin{aligned}&\sum_{k=1}^{n}\left|\Delta_k k^{-1}(A_{n,0}-A_{n,k})\right|\\ &\leq \sum_{k=1}^{n}k^{-1}(k+1)^{-1}\sum_{i=1}^{k}i\,|a_{n,i-1}-a_{n,i}|\\ &\leq \sum_{i=1}^{n}i\,|a_{n,i-1}-a_{n,i}|\sum_{k=i}^{\infty}k^{-1}(k+1)^{-1}\\ &= \sum_{i=1}^{n}|a_{n,i-1}-a_{n,i}|=\sum_{k=0}^{n-1}|\Delta_k a_{n,n-k}|\\ &= O(a_{n,0})=O((n+1)^{-1}). \qquad (11)\end{aligned}$$

Combining (8) and (11), we get

$$\|s_n(f;x)-\tau_n(f;x)\|_{p,w}=O((n+1)^{-1}). \qquad (12)$$

Using Lemma 1 and (12), we have for $\alpha=1$

$$\begin{aligned}&\|f(x)-\tau_n(f;x)\|_{p,w}\leq\|f(x)-s_n(f;x)\|_{p,w}\\ &\quad+\|s_n(f;x)-\tau_n(f;x)\|_{p,w}=O((n+1)^{-1}).\end{aligned}$$

Herewith the case (iv) is proved.
For the proof of case (iii), we first verify that the condition $\sum_{k=0}^{n-1}(n-k)\,|\Delta_k a_{n,k}|=O(1)$, implies that

$$\sum_{k=1}^{n}\left|\Delta_k k^{-1}(A_{n,0}-A_{n,k})\right|=O((n+1)^{-1}). \qquad (13)$$

From (9), we can write

$$\begin{aligned}&\sum_{k=1}^{n}|\ \Delta_k k^{-1}(A_{n,0}-A_{n,k})|\\ &\quad\leq \sum_{k=1}^{n}k^{-1}(k+1)^{-1}\sum_{i=1}^{k}i\,|a_{n,i-1}-a_{n,i}|\\ &\quad= \sum_{k=1}^{n}\Delta(k^{-1})\sum_{i=1}^{k}i\,|a_{n,i-1}-a_{n,i}|.\end{aligned}$$

By Abel's transformation, we have

$$\begin{aligned}
&\sum_{k=1}^{n} |\Delta_k k^{-1}(A_{n,0} - A_{n,k})| \\
&\leq \sum_{k=1}^{n+1} k^{-1}.k\,|a_{n,k-1} - a_{n,k}| \\
&\quad -\frac{1}{n+1}\sum_{k=1}^{n+1} k\,|a_{n,k-1} - a_{n,k}| \\
&= \sum_{k=1}^{n+1}\left(\frac{1}{k} - \frac{1}{n+1}\right) k\,|a_{n,k-1} - a_{n,k}| \\
&= \sum_{k=1}^{n+1}\left(\frac{n-k+1}{n+1}\right) |a_{n,k-1} - a_{n,k}| \\
&= \sum_{k=0}^{n}\left(\frac{n-k}{n+1}\right) |a_{n,k+1} - a_{n,k}| \\
&\leq \frac{1}{n+1}\sum_{k=0}^{n-1} (n-k)\,|\Delta_k a_{n,k}| = O((n+1)^{-1}),
\end{aligned}$$

which verifies (13). Combining (8), (13) and Lemma 2, we get (3) for $\alpha = 1$.

Finally, we prove the case (v). Using Lemma 1 and Abel's transform

$$\begin{aligned}
&\| \tau_n(f;x) - f(x)\|_{p,w} \leq \sum_{k=0}^{n} a_{n,k}\|s_k(f;x) - f(x)\|_{p,w} \\
&= O\{\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k}\} \\
&= O\{\sum_{k=0}^{n-1} \Delta_k (k+1)^{-\alpha}(\sum_{i=0}^{k} a_{n,i}) \\
&\quad +(n+1)^{-\alpha}\sum_{i=0}^{n} a_{n,i}\} \\
&= O\{\sum_{k=0}^{n-1} (A_{n,0} - A_{n,k+1})\{(k+1)^{-\alpha} - (k+2)^{-\alpha}\} \\
&\quad +(n+1)^{-\alpha} A_{n,0}\} \\
&= O\{\sum_{k=0}^{n} (k+1)^{-\alpha}(A_{n,0} - A_{n,k+1})/(k+1)\} \\
&\quad +O(n+1)^{-\alpha}, \qquad (14)
\end{aligned}$$

where by Abel's transformation

$$\begin{aligned}
&\sum_{k=0}^{n} (k+1)^{-\alpha}\frac{A_{n,0} - A_{n,k+1}}{k+1} \\
&= \sum_{k=0}^{n-1} \Delta_k\left\{\frac{A_{n,0} - A_{n,k+1}}{k+1}\right\}\sum_{i=0}^{k} (i+1)^{-\alpha} \\
&\quad +\frac{A_{n,0} - A_{n,n+1}}{n+1}\sum_{i=0}^{n} (i+1)^{-\alpha} \\
&\leq \sum_{k=0}^{n-1} \Delta_k\left\{\frac{A_{n,0} - A_{n,k+1}}{k+1}\right\}(k+1)^{1-\alpha} \\
&\quad +\frac{(n+1)^{1-\alpha}}{n+1} \\
&\leq (n+1)^{1-\alpha}\sum_{k=0}^{n-1} \Delta_k\left\{\frac{A_{n,0} - A_{n,k+1}}{k+1}\right\} \\
&\quad +(n+1)^{-\alpha} = O((n+1)^{-\alpha}), \qquad (15)
\end{aligned}$$

in view of $A_{n,n+1} = 0$ and condition (v) of Theorem 1.

Collecting (14) and (15), we get (3). Thus proof of Theorem 1 is complete.

V. Corollaries

In order to justify the significance of our result, we prove that the following results are the particular cases of Theorem 1 for $p > 1$. We also drive an analogous result of Theorem 1 for monotone $\{a_{n,k}\}$.

1) If we take $a_{n,k} = p_{n-k}/P_n$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$, then conditions (i) to (iv) of Theorem 1 reduce to conditions (i) to (iv) of Theorem 1 of Leindler [4, p. 131], respectively, and $\tau_n(f;x)$ means reduces to $\mathrm{N}_n(f;x)$ means. Further, we note that $Lip(\alpha,p,1) \equiv Lip(\alpha,p)$, $p > 1$. Thus our theorem generalizes Theorem 1 of [4], except for the case $p = 1$ in two directions.
2) Since $Lip(\alpha,p,1) \equiv Lip(\alpha,p)$, $p > 1$ and conditions of Theorem 1 of Mittal et al. [5, p. 4485] for $p > 1$ are included in the conditions (i) to (iv) of Theorem 1, our theorem includes weighted version of Theorem 1 of [5] for $p > 1$.
3) Since every monotone sequence is almost monotone, the conditions (i) and (ii) of Theorem 1 are satisfied in case of monotonic $\{a_{n,k}\}$. Further, every sequence $\{a_{n,k}\}$ non-decreasing with respect to k always satisfies condition (iii) of Theorem 1, e. g.,

$$\begin{aligned}
\sum_{k=0}^{n-1} (n-k)\,|\Delta_k a_{n,k}| &= \sum_{k=0}^{n-1} (n-k)(a_{n,k+1} - a_{n,k}) \\
&= A_{n,0} - (n+1)a_{n,0} = O(1).
\end{aligned}$$

If $\{a_{n,k}\}$ is non-increasing with respect to k, then (iv) of Theorem 1 is also true, e. g.,

$$\begin{aligned}
\sum_{k=0}^{n-1} |\Delta_k a_{n,k}| &= \sum_{k=0}^{n-1} (a_{n,k} - a_{n,k+1}) \\
&= a_{n,0} - a_{n,n} \leq a_{n,0}.
\end{aligned}$$

Thus, we have the following analogous result of Theorem 1 for monotone $\{a_{n,k}\}$:

Theorem 2. *Let $f \in Lip(\alpha,p,w)$, $p > 1$, $w \in A_p$ and let $T \equiv (a_{n,k})$ be an infinite regular triangular matrix and satisfies one of the following conditions:*

(i) $\{a_{n,k}\}$ is non-decreasing in k,
(ii) $\{a_{n,k}\}$ is non-increasing in k and $(n+1)a_{n,0} = O(1)$.

Then (3) holds.

4) If we take $a_{n,k} = p_{n-k}/P_n$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$, then Theorem 2 reduces to Theorem 1 of Guven [12, p. 101].
5) Finally, if we take $a_{n,k} = p_k/P_n$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$, then $\tau_n(f;x)$ means reduces to $R_n(f;x)$ means; and

$$\begin{aligned}
A_{n,0} - A_{n,k+1} &= \sum_{i=0}^{n} a_{n,i} - \sum_{i=k+1}^{n} a_{n,i} \\
&= \left(\sum_{i=0}^{n} p_i - \sum_{i=k+1}^{n} p_i\right)/P_n \\
&= \sum_{i=0}^{k} p_i/P_n = P_k/P_n,
\end{aligned}$$

so that

$$\begin{aligned}
\Delta_k\left(\frac{A_{n,0} - A_{n,k+1}}{k+1}\right) &= \frac{A_{n,0} - A_{n,k+2}}{k+2} \\
&\quad -\frac{A_{n,0} - A_{n,k+1}}{k+1} \\
&= \frac{1}{P_n}\left(\frac{P_{k+1}}{k+2} - \frac{P_k}{k+1}\right),
\end{aligned}$$

i. e., condition (v) of Theorem 1 reduces to condition (3) of Guven [12, Theorem 2]. Thus Theorem 1 under condition (v) extends Theorem 2 of Guven [12] to matrix means.
6) If we take $a_{n,k} = A_{n-k}^{\beta-1}\ /A_n^{\beta}$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n(\beta > 0)$, where

$$A_0^\beta = 1,\ A_k^\beta = \frac{\beta(\beta+1)...(\beta+k)}{k!},\ k \geq 1,$$

then matrix means $\tau_n(f;x)$ reduces to Cesàro means of order $\beta > 0$ denoted by $\sigma_n^\beta(f;x)$ and defined as

$$\sigma_n^\beta(f;x) = \frac{1}{A_n^\beta}\sum\nolimits_{k=0}^{n} A_{n-k}^{\beta-1} s_k(f;x).$$

Hence, Corollary 3 of Guven [12, p. 102] can also be derived from Theorem 1.

7) In the light of remark of Guven [12, p. 102], we note that Theorems 1 and 2 also hold in reflexive weighted Orlicz spaces L_w^M, which are discussed in [10] in detail.

VI. Conclusion

Theorems of this paper are an attempt to formulate the problem of approximation of $f \in Lip(\alpha, p, w)$, $p > 1$ through trigonometric polynomials generated by the summability means of the Fourier series of f in a simpler manner. The case for $p = 1$ is still an open problem which can be addressed by making certain modifications in the definition of Muckenhoupt class A_p.

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