Unbiased Simultaneous Prediction Limits on Future Order Statistics with Applications

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***Abstract* — Statistical prediction is the earliest and most prevalent form of statistical inference. It is the provision of an estimate, usually in the one-sided or two-sided interval form, for future observations based on the results obtained from past observations. In particular, the minimum, maximum, mean, median of a future sample or ranges of given number of samples could also be aims of prediction. Prediction has its uses in a variety of disciplines such as medicine, engineering and business. In this paper, we consider the problems of constructing unbiased simultaneous prediction limits on the order statistics of all of *k* future samples using the results of a previous sample from the same underlying distribution belonging to invariant family. The prediction limits obtained in the paper are generalizations of the usual prediction limits on observations or functions of observations of only one future sample. Attention is restricted to invariant families of distributions. The technique used here emphasizes pivotal quantities relevant for obtaining ancillary statistics and is applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. It does not require the construction of any tables and is applicable whether the data are complete or Type II censored. Applications of the proposed procedures are given for the two-parameter exponential and Weibull distributions. The proposed technique is conceptually simple and easy to use. The exact prediction limits are found and illustrated using some practical examples.**

***Index Terms* — Future samples of observations, order statistics, simultaneous prediction limits**

I. Introduction

Many statistical applications involve the prediction of future values of some random variables, based on previously observed data. We consider here a general parametric framework in which prediction is based on a family of models specified up to unknown parameter θ (in general, vector). Statistical intervals used by engineers and others include confidence intervals on a population parameter, such as the mean, and tolerance intervals. Confidence intervals give information about parameter of the population or a function of population parameters such as a percentile; tolerance intervals give information about a region which contains a specified proportion of a population.

Often one desires to construct from the results of a previous sample an interval which will have a high probability of containing the values of all of k future observations. For example, such an interval would be required in establishing limits on the values of some performance variable for a small shipment of equipment when the satisfactory performance of all units is to be guaranteed, or in setting acceptance limits on a specific lot of material, when acceptance requires the values of all items in a future sample to fall within the limits. An interval which contains the values of a specified number of future observations with a specified probability is known as a prediction interval. Such an interval need be distinguished both from a confidence interval on an unknown distribution parameter, and from a tolerance interval to contain the values of a specified proportion of the population. Research works on prediction intervals related to a single future statistic are abundant (see Hahn and Meeker [1], Patel [2], and references therein).

In many situations of interest, it is desirable to construct lower simultaneous prediction limits that are exceeded with probability γ by observations or functions of observations of all of k future samples, each consisting of m units. The prediction limits depend upon a previously available complete or type II censored sample from the same distribution. For instance, two situations where such limits are required are:

1. A customer has placed an order for a product which has an underlying time-to-failure distribution. The terms of his purchase call for k monthly shipments. From each shipment the customer will select a random sample of m units and accept the shipment only if the smallest time to failure for this sample exceeds a specified lower limit. The manufacturer wishes to use the results of a previous sample of n units to calculate this limit so that the probability is γ that all k shipments will be accepted. It is assumed that the n past units and the km future units are random samples from the same population. This situation is considered in [3].

2. A system consists of n identical components whose times to failure follow an underlying distribution. Initially one component is operating and the remaining n−1 components are in a standby mode; a new component goes into operation as soon as the preceding component has failed. The system is said to fail when all n components have failed. Thus, the system time to failure is the total of the failure times for the n components. A simultaneous lower prediction limit to be exceeded with probability γ by the system time to failure of all of k future systems is desired. This limit is to be calculated from the times to failure of n previously tested components. Similar problems also arise in

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various product maintenance and servicing problems.

Prediction limits can be of several forms. Hahn [4] dealt with simultaneous prediction limits on the standard deviations of all of the k future samples from a normal population. Hahn [5] considered the problem of obtaining simultaneous prediction limits on the means of all of k future samples from an exponential distribution. In addition, Hahn and Nelson [6] discussed such limits and their applications. Mann, Schafer, and Singpurwalla [7] gave an interval that contains, with probability γ, all m observations of a single future sample from the same population. Fertig and Mann [8] constructed prediction intervals to contain at least $m - k + 1$ out of m future observations from a normal distribution with probability $1-\beta$. They considered life-test data, and the performance variate of interest is the failure time of an item. Their lower prediction limit constitutes a "warranty period".

In this paper we give an expression for obtaining unbiased simultaneous prediction limits on order statistics of all of k future samples. In order to obtain the unbiased simultaneous prediction limits, attention is restricted to invariant families of distributions. In particular, the case is considered where a previously available complete or type II censored sample is from a continuous distribution with cumulative distribution function (cdf) $F((x-\mu)/\sigma)$ and probability density function (pdf) $1/\sigma f((x-\mu)/\sigma)$, where $F(\cdot)$ is known but both the location (μ) and scale (σ) parameters are unknown. For such family of distributions the decision problem remains invariant under a group of transformations (a subgroup of the full affine group) which takes μ (the location parameter) and σ (the scale) into $c\mu + b$ and $c\sigma$, respectively, where b lies in the range of μ, $c > 0$. This group acts transitively on the parameter space and, consequently, the risk of any equivariant estimator is a constant. Among the class of such estimators there is therefore a "best" one. The effect of imposing the principle of invariance, in this case, is to reduce the class of all possible estimators to one. In the present paper we investigate this question for the problem of constructing the unbiased simultaneous prediction limits on order statistics in future samples.

The results have direct application in reliability theory, where the time until the first failure in a group of several items in service provides a measure of assurance regarding the operation of the items. The simultaneous prediction limits are required as specifications on future life for components, as warranty limits for the future performance of a specified number of systems with standby units, and in various other applications. Prediction limit is an important statistical tool in the area of quality control. The lower simultaneous prediction limits are often used as warranty criteria by manufacturers. The initial sample and k future samples are available, and the manufacturer wants to have a high assurance that all of the k future orders will be accepted. It is assumed throughout that $k + 1$ samples are obtained by taking random samples from the same population. In other words, the manufacturing process remains constant. The results in this paper are generalizations of the usual prediction limits on observations or functions of observations of only one future sample. In the paper, attention is restricted to invariant families of distributions.

The technique used here emphasizes pivotal quantities relevant for obtaining ancillary statistics. It is a special case of the method of invariant embedding of sample statistics into a performance index [9-12] applicable whenever the statistical problem is invariant under a group of transformations which acts transitively on the parameter space (i.e., in problems where there is a unique best invariant procedure). The exact unbiased simultaneous prediction limits on order statistics of all of k future samples are obtained via the technique of invariant embedding and illustrated with some numerical examples.

II. Mathematical Preliminaries

The main theorem, which shows how to construct lower (upper) simultaneous prediction limit for the order statistics in all of k future samples when prediction limit for a single future sample is available, is given below.

Theorem 1 (Lower (upper) simultaneous prediction limit under complete information). Let m_j "future" observations $(Y_{1_j},\ldots,Y_{m_j})$ represent the jth random sample from the cdf $F_\theta(.)$, where θ is the parameter (in general, vector), $j\in\{1, \ldots, k\}$, and let $Y_{(r_j,m_j)}$ denote the r_jth order statistic in the jth sample of size m_j.. Assume that all of k samples from the same cdf are independent. Then a lower simultaneous $(1-\alpha)$ prediction limit h on the r_jth order statistics $Y_{(r_j,m_j)}$, $j=1, \ldots, k$, of all of k future samples may be obtained from

$$P_\theta\left\{Y_{(r_1,m_1)}>h,\ldots,Y_{(r_j,m_j)}>h,\ldots,Y_{(r_k,m_k)}>h\right\}$$

$$=\sum_{i_1=0}^{r_1-1}\cdots\sum_{i_j=0}^{r_j-1}\cdots\sum_{i_k=0}^{r_k-1}\binom{m_1}{i_1}\cdots\binom{m_j}{i_j}\cdots\binom{m_k}{i_k}$$

$$\times\frac{P_\theta\left\{Y_{(i_\Sigma+1,m_\Sigma)}>h\right\}-P_\theta\left\{Y_{(i_\Sigma,m_\Sigma)}>h\right\}}{\binom{m_\Sigma}{i_\Sigma}}=1-\alpha, \quad (1)$$

where

$$i_\Sigma=\sum_{j=1}^{k}i_j, \quad m_\Sigma=\sum_{j=1}^{k}m_j. \quad (2)$$

(Observe that an upper simultaneous α prediction limit h may be obtained from a lower simultaneous prediction limit by replacing $1-\alpha$ by α.)

Proof.

$$P_\theta\left\{Y_{(r_1,m_1)}>h,\ldots,Y_{(r_j,m_j)}>h,\ldots,Y_{(r_k,m_k)}>h\right\}$$

$$=\prod_{j=1}^{k}P_\theta\left\{Y_{(r_j,m_j)}>h\right\}$$

$$=\prod_{j=1}^{k}\sum_{i_j=0}^{r_j-1}\binom{m_j}{i_j}[F_\theta(h)]^{i_j}[1-F_\theta(h)]^{m_j-i_j}$$

$$=\sum_{i_1=0}^{r_1-1}\cdots\sum_{i_j=0}^{r_j-1}\cdots\sum_{i_k=0}^{r_k-1}\binom{m_1}{i_1}\cdots\binom{m_j}{i_j}\cdots\binom{m_k}{i_k}[F_\theta(h)]^{i_\Sigma}[1-F_\theta(h)]^{m_\Sigma-i_\Sigma}. \tag{3}$$

Since

$$[F_\theta(h)]^{i_\Sigma}[1-F_\theta(h)]^{m_\Sigma-i_\Sigma}$$

$$=\binom{m_\Sigma}{i_\Sigma}^{-1}\left[\sum_{i=0}^{i_\Sigma}\binom{m_\Sigma}{i}[F_\theta(h)]^{i}[1-F_\theta(h)]^{m_\Sigma-i}-\sum_{i=0}^{i_\Sigma-1}\binom{m_\Sigma}{i}[F_\theta(h)]^{i}[1-F_\theta(h)]^{m_\Sigma-i}\right]$$

$$=\frac{P_\theta\left\{Y_{(i_\Sigma+1,m_\Sigma)}>h\right\}-P_\theta\left\{Y_{(i_\Sigma,m_\Sigma)}>h\right\}}{\binom{m_\Sigma}{i_\Sigma}}, \tag{4}$$

the joint probability can be written as

$$P_\theta\left\{Y_{(r_1,m_1)}>h,\ldots,Y_{(r_j,m_j)}>h,\ldots,Y_{(r_k,m_k)}>h\right\}$$

$$=\sum_{i_1=0}^{r_1-1}\cdots\sum_{i_j=0}^{r_j-1}\cdots\sum_{i_k=0}^{r_k-1}\binom{m_1}{i_1}\cdots\binom{m_j}{i_j}\cdots\binom{m_k}{i_k}$$

$$\times\frac{P_\theta\left\{Y_{(i_\Sigma+1,m_\Sigma)}>h\right\}-P_\theta\left\{Y_{(i_\Sigma,m_\Sigma)}>h\right\}}{\binom{m_\Sigma}{i_\Sigma}}. \tag{5}$$

This ends the proof. □

Corollary 1.1. If $r_j=1$, $\forall j=1(1)k$, then a lower simultaneous $(1-\alpha)$ prediction limit h on the smallest order statistics $Y_{(1,m_j)}$, $j=1, \ldots, k$, of all of k future samples may be obtained from

$$P_\theta\left\{Y_{(1,m_1)}>h,\ldots,Y_{(1,m_j)}>h,\ldots,Y_{(1,m_k)}>h\right\}$$

$$=P_\theta\left\{Y_{(1,m_\Sigma)}>h\right\}=1-\alpha. \tag{6}$$

Proof.

$$P_\theta\left\{Y_{(1,m_1)}>h,\ldots,Y_{(1,m_j)}>h,\ldots,Y_{(1,m_k)}>h\right\}$$

$$=\prod_{j=1}^{k}P_\theta\left\{Y_{(1,m_j)}>h\right\}=\prod_{j=1}^{k}[1-F_\theta(h)]^{m_j}$$

$$=[1-F_\theta(h)]^{m_\Sigma}=P_\theta\left\{Y_{(1,m_\Sigma)}>h\right\}. \tag{7}$$

This ends the proof. □

Corollary 1.2. If $r_j=m_j$, $\forall j=1(1)k$, then an upper simultaneous $(1-\alpha)$ prediction limit h on the largest order statistics $Y_{(m_j,m_j)}$, $j=1, \ldots, k$, of all of k future samples may be obtained from

$$P_\theta\left\{Y_{(m_1,m_1)}\le h,\ldots,Y_{(m_j,m_j)}\le h,\ldots,Y_{(m_k,m_k)}\le h\right\}$$

$$=P_\theta\left\{Y_{(m_\Sigma,m_\Sigma)}\le h\right\}=1-\alpha. \tag{8}$$

Proof.

$$P_\theta\left\{Y_{(m_1,m_1)}\le h,\ldots,Y_{(m_j,m_j)}\le h,\ldots,Y_{(m_k,m_k)}\le h\right\}$$

$$=\prod_{j=1}^{k}P_\theta\left\{Y_{(m_j,m_j)}\le h\right\}=\prod_{j=1}^{k}[F_\theta(h)]^{m_j}$$

$$=[F_\theta(h)]^{m_\Sigma}=P_\theta\left\{Y_{(m_\Sigma,m_\Sigma)}\le h\right\}. \tag{9}$$

This ends the proof. □

Theorem 2 (Lower (upper) unbiased simultaneous prediction limit under parametric uncertainty). Let $(X_1 \le \ldots \le X_r)$ be the r smallest observations in a random sample of size n from the cdf $F_\theta(.)$, where the θ is the parameter (in general, vector), and let $(Y_{1_j},\ldots,Y_{m_j})$ be the jth random sample of m_j "future" observations from the same cdf, $j\in\{1, \ldots, k\}$. Assume that $(k+1)$ samples are independent and the parameter θ is unknown. Let $H=H(X_1, \ldots, X_r)$ be any statistic based on the preliminary sample and let $Y_{(r_j,m_j)}$ denote the r_jth order statistic in the jth sample of size m_j. Then an unbiased lower simultaneous $(1-\alpha)$ prediction limit H on the r_jth order statistics $Y_{(r_j,m_j)}$, $j=1, \ldots, k$, of all of k future samples may be obtained from

$$E_\theta\left\{P_\theta\left\{Y_{(r_1,m_1)}>H,\ldots,Y_{(r_j,m_j)}>H,\ldots,Y_{(r_k,m_k)}>H\right\}\right\}$$

$$=\sum_{i_1=0}^{r_1-1}\cdots\sum_{i_j=0}^{r_j-1}\cdots\sum_{i_k=0}^{r_k-1}\binom{m_1}{i_1}\cdots\binom{m_j}{i_j}\cdots\binom{m_k}{i_k}$$

$$\times\frac{E_\theta\left\{P_\theta\left\{Y_{(i_\Sigma+1,m_\Sigma)}>H\right\}\right\}-E_\theta\left\{P_\theta\left\{Y_{(i_\Sigma,m_\Sigma)}>H\right\}\right\}}{\binom{m_\Sigma}{i_\Sigma}}=1-\alpha. \tag{10}$$

Proof. For the proof we refer to Theorem 1. □

Corollary 2.1. If $r_j=1$, $\forall j=1(1)k$, then an unbiased lower simultaneous $(1-\alpha)$ prediction limit H on the first order statistics $Y_{(1,m_j)}$, $j=1, \ldots, k$, of all of k future samples may be obtained from

$$E_\theta\left\{P_\theta\left\{Y_{(1,m_1)}>H,\ldots,Y_{(1,m_j)}>H,\ldots,Y_{(1,m_k)}>H\right\}\right\}$$

$$=E_\theta\left\{P_\theta\left\{Y_{(1,m_\Sigma)}>H\right\}\right\}=1-\alpha. \tag{11}$$

Corollary 2.2. If $r_j= m_j$, $\forall j=1(1)k$, then an unbiased upper simultaneous $(1-\alpha)$ prediction limit H on the last order statistics $Y_{(m_j,m_j)}$, j=1, …, k, of all of k future samples may be obtained from

$$E_\theta\left\{P_\theta\left\{Y_{(m_1,m_1)}\le H,...,Y_{(m_j,m_j)}\le H,...,Y_{(m_k,m_k)}\le H\right\}\right\}$$

$$=E_\theta\left\{P_\theta\left\{Y_{(m_\Sigma,m_\Sigma)}\le H\right\}\right\}=1-\alpha. \qquad (12)$$

Remark 1. In this paper, in order to find the unbiased lower simultaneous (1−α) prediction limit H on the r_jth order statistics $Y_{(r_j,m_j)}$, j=1, …, k, of all of k future samples, the technique of invariant embedding [9-12] is used.

A. Left-Truncated Weibull Distribution

Theorem 3 ((Lower (upper) unbiased prediction limit H for the lth order statistic Y_l in a new (future) sample of m observations from the left-truncated Weibull distribution on the basis of the preliminary data sample) Let $X_1 \le ... \le X_r$ be the first r ordered observations from the preliminary sample of size n from the left-truncated Weibull distribution with the pdf

$$f_{(\mu,\sigma,\delta)}(x)=\frac{\delta}{\sigma}x^{\delta-1}\exp[-(x^\delta-\mu)/\sigma],\ \ (x^\delta\ge\mu,\sigma,\delta>0), \qquad (13)$$

where δ is the shape parameter, σ is the scale parameter, and μ is the truncation parameter. It is assumed that the parameter δ is known. Then a lower unbiased (1−α) prediction limit H on the lth order statistic Y_l from a set of m future ordered observations $Y_1 \le … \le Y_m$ also from the distribution (13) is given by

$$H=\left(X_1^\delta+w_H S\right)^{1/\delta}, \qquad (14)$$

where

$$w_H=\begin{cases}\arg\left(nl\binom{m}{l}\sum_{i=0}^{l-1}\frac{\binom{l-1}{i}(-1)^i[1+w_H(m-l+i+1)]^{-(r-1)}}{(n+m-l+i+1)(m-l+i+1)}=1-\alpha\right),\\ \text{if}\ \ \alpha>\frac{m!(n+m-l)!}{(m-l)!(n+m)!};\\ \arg\left(1-\frac{m!(m+n-l)!}{(m-l)!(m+n)!}(1-nw_H)^{-(r-1)}=1-\alpha\right),\\ \text{if}\ \ \alpha\le\frac{m!(n+m-l)!}{(m-l)!(n+m)!},\end{cases} \qquad (15)$$

$$S=\sum_{i=1}^{r}(X_i^\delta-X_1^\delta)+(n-r)(X_r^\delta-X_1^\delta). \qquad (16)$$

(Observe that an upper unbiased α prediction limit H on the lth order statistic Y_l may be obtained from a lower unbiased (1−α) prediction limit by replacing 1−α by α.)

Proof. It can be justified by using the factorization theorem that (X_1^δ,S) is a sufficient statistic for (μ,σ). We wish, on the basis of the sufficient statistic (X_1^δ,S) for (μ,σ), to construct the predictive density function of the lth order statistic Y_l from a set of m future ordered observations $Y_1 \le … \le Y_m$. By using the technique of invariant embedding [9-12] of (X_1^δ,S), if $X_1 \le Y_l$, or (Y_l^δ,S), if $X_1 > Y_l$, into a pivotal quantity $(Y_l^\delta-\mu)/\sigma$ or $(X_1^\delta-\mu)/\sigma$, respectively, we obtain an ancillary statistic

$$W_l=\left(Y_l^\delta-X_1^\delta\right)/S. \qquad (17)$$

It can be shown that the pdf of W_l is given by

$$f(w_l)=\begin{cases}n(r-1)l\binom{m}{l}\sum_{i=0}^{l-1}\frac{\binom{l-1}{i}(-1)^i[1+w_l(m-l+i+1)]^{-r}}{n+m-l+i+1}\\ \text{if}\ \ w_l>0,\\ n(r-1)\frac{m!(n+m-l)!}{(m-l)!(n+m)!}(1-nw_l)^{-r}\\ \text{if}\ \ w_l\le 0.\end{cases} \qquad (18)$$

It follows from (18) that

$$P(W_l>w_H)=\begin{cases}nl\binom{m}{l}\sum_{i=0}^{l-1}\frac{\binom{l-1}{i}(-1)^i[1+w_H(m-l+i+1)]^{-(r-1)}}{(n+m-l+i+1)(m-l+i+1)}\\ \text{if}\ \ w_H\ge 0,\\ 1-\frac{m!(m+n-l)!}{(m-l)!(m+n)!}(1-nw_H)^{-(r-1)}\\ \text{if}\ \ w_H<0.\end{cases} \qquad (19)$$

where

$$w_H=\left(H^\delta-X_1^\delta\right)/S. \qquad (20)$$

This ends the proof. □

Corollary 3.1. If l = 1, then a lower (1−α) prediction limit H on the minimum Y_1 of a set of m future ordered observations $Y_1 \le … \le Y_m$ is given by

$$H=\begin{cases}\left(X_1^\delta+\frac{S}{m}\left[\left(\frac{n}{(1-\alpha)(n+m)}\right)^{\frac{1}{r-1}}-1\right]\right)^{1/\delta} & \text{if}\ \ \alpha>\frac{m}{n+m},\\ \left(X_1^\delta-\frac{S}{n}\left[\left(\frac{m}{\alpha(n+m)}\right)^{\frac{1}{r-1}}-1\right]\right)^{1/\delta} & \text{if}\ \ \alpha\le\frac{m}{n+m}.\end{cases} \qquad (21)$$

B. Two-parameter Exponential Distribution

Theorem 4 (Lower (upper) unbiased prediction limit H for the lth order statistic Y_l in a new (future) sample of m observations from the two-parameter exponential distribution on the basis of the preliminary data sample) Let $X_1 \le ... \le X_r$ be the first r ordered observations from the preliminary sample of size n from the two-parameter exponential distribution with the pdf

$$f_{(\mu,\sigma)}(x)=\frac{1}{\sigma}\exp[-(x-\mu)/\sigma],\quad (x\ge\mu,\ \sigma>0), \qquad (22)$$

where σ is the scale parameter, and μ is the shift parameter. It is assumed that these parameters are unknown. Then a lower unbiased (1–α) prediction limit H on the lth order statistic Y_l from a set of m future ordered observations $Y_1 \le \dots \le Y_m$ also from the distribution (22) is given by

$$H = X_1 + w_H S, \tag{23}$$

where

$$w_H = \begin{cases} \arg\left(n\binom{m}{l}\sum_{i=0}^{l-1}\frac{\binom{l-1}{i}(-1)^i[1+w_H(m-l+i+1)]^{-(r-1)}}{(n+m-l+i+1)(m-l+i+1)} = 1-\alpha \right) \\ \text{if } \alpha > \frac{m!(n+m-l)!}{(m-l)!(n+m)!}, \\ \arg\left(1-\frac{m!(m+n-l)!}{(m-l)!(m+n)!}(1-nw_H)^{-(r-1)} = 1-\alpha\right) \\ \text{if } \alpha \le \frac{m!(n+m-l)!}{(m-l)!(n+m)!}, \end{cases} \tag{24}$$

$$S = \sum_{i=1}^{r}(X_i - X_1) + (n-r)(X_r - X_1). \tag{25}$$

(Observe that an upper unbiased α prediction limit H on the lth order statistic Y_l may be obtained from a lower unbiased (1–α) prediction limit by replacing 1–α by α.)

Proof. For the proof we refer to Theorem 3. □

Corollary 4.1. If l = 1, then a lower (1–α) prediction limit H on the minimum Y_1 of a set of m future ordered observations $Y_1 \le \dots \le Y_m$ is given by

$$H = \begin{cases} \left(X_1 + \frac{S}{m}\left[\left(\frac{n}{(1-\alpha)(n+m)}\right)^{\frac{1}{r-1}} - 1\right]\right) & \text{if } \alpha > \frac{m}{n+m}, \\ \left(X_1 - \frac{S}{n}\left[\left(\frac{m}{\alpha(n+m)}\right)^{\frac{1}{r-1}} - 1\right]\right) & \text{if } \alpha \le \frac{m}{n+m}. \end{cases} \tag{26}$$

Corollary 4.2. (Prediction a total lifetime in a future sample) A lower (1–α) prediction limit H on $\sum_{i=1}^{i=m} Y_i$ of m future observations Y_i, i=1, …, m, is given by

$$H = m\left[X_1 + \frac{S}{n}w_H\right], \tag{27}$$

where

$$w_H = \begin{cases} \arg\left(\sum_{i=0}^{m-1}\binom{r+i-2}{i}\frac{(1+mw_H/n)^{-r+1}}{[1+n/(mw_H)]^i}\left[1-\left(\frac{m}{m+n}\right)^{m-i}\right] = 1-\alpha\right) \\ \text{if } \alpha > [m/(m+n)]^m, \\ \arg\left(1-\left(\frac{m}{m+n}\right)^m (1-w_H)^{-(r-1)} = 1-\alpha\right) \\ \text{if } \alpha \le [m/(m+n)]^m. \end{cases} \tag{28}$$

Remark 2. Let us assume that the parent distributions are the two-parameter exponential

$$F_\theta(x) = 1-\exp\left(-\frac{x-\theta_2}{\theta_1}\right), \quad x \ge \theta_2, \ \theta_1 > 0, \tag{29}$$

where $\theta = (\theta_1, \theta_2)$, and the Pareto distribution

$$F_\theta(x) = 1-(\theta_2/x)^{1/\theta_1}, \quad x \ge \theta_2 > 0, \ \theta_1 > 0. \tag{30}$$

Let X be a random variable with the Pareto distribution (30), and define Y = ln$X$. Then Y becomes a random variable with the exponential distribution (29), where θ_2 is replaced by lnθ_2. Therefore it is enough to consider only the exponential distribution, because the results for the Pareto distribution are easily obtained from those for the exponential distribution.

C. Two-parameter Weibull Distribution

In this paper, the two-parameter Weibull distribution with the pdf

$$f_{(\beta,\delta)}(x) = \frac{\delta}{\beta}\left(\frac{x}{\beta}\right)^{\delta-1}\exp\left[-\left(\frac{x}{\beta}\right)^{\delta}\right], \quad x>0, \ \beta>0, \ \delta>0, \tag{31}$$

indexed by scale and shape parameters β and δ is used as the underlying distribution of a random variable X in a sample of the lifetime data.

The Weibull distribution is widely used in reliability and survival analysis due to its flexible shape and ability to model a wide range of failure rates. It can be derived theoretically as a form of extreme value distribution, governing the time to occurrence of the “weakest link” of many competing failure processes. Its special case with shape parameter δ=2 is the Rayleigh distribution which is commonly used for modeling the magnitude of radial error when x and y coordinate errors are independent normal variables with zero mean and the same standard deviation while the case δ=1 corresponds to the widely used exponential distribution. Let X follow a Weibull distribution with scale parameter β and shape parameter δ.

We consider both parameters β, δ to be unknown. Let (X_1, …, X_n) be a random sample from the two-parameter Weibull distribution (31), and let $\hat\beta$, $\hat\delta$ be maximum likelihood estimates of β, δ computed on the basis of (X_1, …, X_n). In terms of the Weibull variates, we have that

$$V_1 = \left(\frac{\hat\beta}{\beta}\right)^{\delta}, \quad V_2 = \frac{\delta}{\hat\delta}, \quad V_3 = \left(\frac{\hat\beta}{\beta}\right)^{\hat\delta} \tag{32}$$

are pivotal quantities. Furthermore, let

$$Z_i = (X_i/\hat\beta)^{\hat\delta}, \quad i=1, \dots, n. \tag{33}$$

It is readily verified that any n–2 of the Z_i’s, say Z_1, …, Z_{n-2} form a set of n–2 functionally independent ancillary statistics. The appropriate conditional approach, first suggested by Fisher [13], is to consider the distributions of V_1, V_2, V_3 conditional on the observed value of $\mathbf{Z}^{(n)}$ = (Z_1, …,

Z_n). (For purposes of symmetry of notation we include all of $Z_1, \ldots, Z_n$ in expressions stated here; it can be shown that Z_n, Z_{n-1}, can be determined as functions of $Z_1, \ldots, Z_{n-2}$ only.)

Theorem 5. (Joint pdf of the pivotal quantities V_1, V_2 from the two-parameter Weibull distribution) Let ($X_1 \le \ldots \le X_r$) be the first r ordered observations from a sample of size n from the two-parameter Weibull distribution (31). Then the joint pdf of the pivotal quantities

$$V_1 = \left(\frac{\hat{\beta}}{\beta}\right)^{\delta}, \quad V_2 = \frac{\delta}{\hat{\delta}}, \tag{34}$$

conditional on fixed

$$\mathbf{z}^{(r)} = (z_i, \ldots, z_r), \tag{35}$$

where

$$Z_i = \left(\frac{X_i}{\hat{\beta}}\right)^{\hat{\delta}}, \quad i = 1, \ldots, r, \tag{36}$$

are ancillary statistics, any r–2 of which form a functionally independent set, $\hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimates for β and δ based on the first r ordered observations ($X_1 \le \ldots \le X_r$) from a sample of size n from the two-parameter Weibull distribution (31), which can be found from solution of

$$\hat{\beta} = \left(\left[\sum_{i=1}^{r} x_i^{\hat{\delta}} + (n-r) x_r^{\hat{\delta}} \right] \Big/ r \right)^{1/\hat{\delta}}, \tag{37}$$

and

$$\hat{\delta} = \left[\left(\sum_{i=1}^{r} x_i^{\hat{\delta}} \ln x_i + (n-r) x_r^{\hat{\delta}} \ln x_r \right) \right.$$

$$\left. \times \left(\sum_{i=1}^{r} x_i^{\hat{\delta}} + (n-r) x_r^{\hat{\delta}} \right)^{-1} - \frac{1}{r} \sum_{i=1}^{r} \ln x_i \right]^{-1}, \tag{38}$$

is given by

$$f(v_1, v_2 \mid \mathbf{z}^{(r)}) = \vartheta^{\bullet}(\mathbf{z}^{(r)}) v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} v_1^{r-1}$$

$$\times \exp\left(-v_1 \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right] \right)$$

$$= f(v_2 \mid \mathbf{z}^{(r)}) f(v_1 \mid v_2, \mathbf{z}^{(r)}), \quad v_1 \in (0, \infty), \quad v_2 \in (0, \infty), \tag{39}$$

where

$$\vartheta^{\bullet}(\mathbf{z}^{(r)}) = \left[\int_0^{\infty} \Gamma(r) v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \left(\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r} dv_2 \right]^{-1}, \tag{40}$$

is the normalizing constant,

$$f(v_2 \mid \mathbf{z}^{(r)}) = \vartheta(\mathbf{z}^{(r)}) v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \left(\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r},$$

$$v_2 \in (0, \infty), \tag{41}$$

$$\vartheta(\mathbf{z}^{(r)}) = \left[\int_0^{\infty} v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \left(\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r} dv_2 \right]^{-1} \tag{42}$$

$$f(v_1 \mid v_2, \mathbf{z}^{(r)}) = \frac{\left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right]^{r}}{\Gamma(r)} v_1^{r-1}$$

$$\times \exp\left(-v_1 \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right] \right), \quad v_1 \in (0, \infty). \tag{43}$$

Proof. The joint density of $X_1 \le \ldots \le X_r$ is given by

$$f_{(\beta,\delta)}(x_1, \ldots, x_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \frac{\delta}{\beta} \left(\frac{x_i}{\beta} \right)^{\delta-1}$$

$$\times \exp\left(-\left(\frac{x_i}{\beta} \right)^{\delta} \right) \exp\left(-(n-r) \left(\frac{x_r}{\beta} \right)^{\delta} \right). \tag{44}$$

Using the invariant embedding technique [9-12], we transform (44) to

$$f_{(\beta,\delta)}(x_1, \ldots, x_r)\, d\hat{\beta}\, d\hat{\delta} = \frac{n!}{(n-r)!} \prod_{i=1}^{r} x_i^{-1} \delta^r \prod_{i=1}^{r} \left(\frac{x_i}{\beta} \right)^{\delta}$$

$$\times \exp\left(-\sum_{i=1}^{r} \left(\frac{x_i}{\beta} \right)^{\delta} - (n-r) \left(\frac{x_r}{\beta} \right)^{\delta} \right) d\hat{\beta}\, d\hat{\delta}$$

$$= -\frac{n!}{(n-r)!} \hat{\beta} \hat{\delta}^r \prod_{i=1}^{r} x_i^{-1} \left(\frac{\delta}{\hat{\delta}} \right)^{r-2} \prod_{i=1}^{r} \left(\frac{x_i}{\hat{\beta}} \right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)} \left(\frac{\hat{\beta}}{\beta} \right)^{\delta(r-1)}$$

$$\times \exp\left(-\left(\frac{\hat{\beta}}{\beta} \right)^{\delta} \left[\sum_{i=1}^{r} \left(\frac{x_i}{\hat{\beta}} \right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)} + (n-r) \left(\frac{x_r}{\hat{\beta}} \right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)} \right] \right)$$

$$\times \left(\frac{\delta}{\beta} \left(\frac{\hat{\beta}}{\beta} \right)^{\delta-1} d\hat{\beta} \right) \left(-\frac{\delta}{\hat{\delta}^2} d\hat{\delta} \right)$$

$$= -\frac{n!}{(n-r)!} \hat{\beta} \hat{\delta}^r \prod_{i=1}^{r} x_i^{-1} v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} v_1^{r-1}$$

$$\times \exp\left(-v_1 \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right] \right) dv_1 dv_2. \tag{45}$$

Normalizing (45), we obtain (39). This ends the proof. □

Theorem 6. (Joint probability density function of the pivotal quantities V_2,V_3, from the two-parameter Weibull distribution) Let $X_1 \le ... \le X_r$ be the first r ordered observations from a sample of size n from the two-parameter Weibull distribution (31). Then the joint probability density function of the pivotal quantities

$$V_2 = \frac{\delta}{\hat{\delta}}, \quad V_3 = \left(\frac{\hat{\beta}}{\beta}\right)^{\hat{\delta}}, \tag{46}$$

conditional on fixed $\mathbf{z}^{(r)}=(z_1, \ldots, z_r)$, where $Z_i = (X_i / \hat{\beta})^{\hat{\delta}}$, $i = 1, ..., r$, are ancillary statistics, any r–2 of which form a functionally independent set, $\hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimators of β and δ based on the first r ordered observations ($X_1 \le ... \le X_r$) from a sample of size n from the two-parameter Weibull distribution (31), which can be found from solution of (37) and (38), is given by

$$f(v_2, v_3 \mid \mathbf{z}^{(r)}) = \vartheta^{\bullet}(\mathbf{z}^{(r)}) v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2}$$

$$\times v_3^{v_2(r-1)} \exp\left(-v_3^{v_2}\left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2}\right]\right) v_2 v_3^{v_2-1}$$

$$= f(v_2 \mid \mathbf{z}^{(r)}) f(v_3 \mid v_2, \mathbf{z}^{(r)}), \quad v_2 \in (0, \infty), \quad v_3 \in (0, \infty), \tag{47}$$

where

$$f(v_3 \mid v_2, \mathbf{z}^{(r)}) = \frac{\left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2}\right]^r}{\Gamma(r)}$$

$$\times v_3^{v_2(r-1)} \exp\left(-v_3^{v_2}\left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2}\right]\right) v_2 v_3^{v_2-1},$$

$$v_3 \in (0, \infty). \tag{48}$$

Proof. Using the invariant embedding technique [9-12], we transform the joint density (44) of $X_1 \le ... \le X_r$ to

$$f_{(\beta,\delta)}(x_1, ..., x_r)\, d\hat{\beta}\, d\hat{\delta} = \frac{n!}{(n-r)!} \prod_{i=1}^{r} x_i^{-1} \delta^r \prod_{i=1}^{r} \left(\frac{x_i}{\beta}\right)^{\delta}$$

$$\times \exp\left(-\sum_{i=1}^{r} \left(\frac{x_i}{\beta}\right)^{\delta} - (n-r)\left(\frac{x_r}{\beta}\right)^{\delta}\right) d\hat{\beta}\, d\hat{\delta}$$

$$= -\frac{n!}{(n-r)!} \hat{\beta} \hat{\delta}^r \prod_{i=1}^{r} x_i^{-1} \left(\frac{\delta}{\hat{\delta}}\right)^{r-2} \prod_{i=1}^{r} \left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)} \left(\frac{\hat{\beta}}{\beta}\right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)(r-1)}$$

$$\times \exp\left(-\left(\frac{\hat{\beta}}{\beta}\right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)} \left[\sum_{i=1}^{r} \left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)} + (n-r)\left(\frac{x_r}{\hat{\beta}}\right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)}\right]\right)$$

$$\times \left(\frac{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)}{\beta} \left(\frac{\hat{\beta}}{\beta}\right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)-1} d\hat{\beta}\right) \left(-\frac{\delta}{\hat{\delta}^2} d\hat{\delta}\right)$$

$$= -\frac{n!}{(n-r)!} \hat{\beta} \hat{\delta}^r \prod_{i=1}^{r} x_i^{-1} v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} v_3^{v_2(r-1)}$$

$$\times \exp\left(-v_3^{v_2}\left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2}\right]\right) d(v_3^{v_2}) dv_2$$

$$= -\frac{n!}{(n-r)!} \hat{\beta} \hat{\delta}^r \prod_{i=1}^{r} x_i^{-1} v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} v_3^{v_2(r-1)}$$

$$\times \exp\left(-v_3^{v_2}\left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2}\right]\right) v_2 v_3^{v_2-1} dv_2 dv_3. \tag{49}$$

Normalizing (49), we obtain (47). This ends the proof. □

Theorem 7. (Lower (upper) unbiased prediction limit H for the lth order statistic Y_l in a new (future) sample of m observations from the two-parameter Weibull distribution, where the parameters β and δ are unknown, on the basis of the preliminary data sample) Let $X_1 \le ... \le X_r$ be the first r ordered observations from the preliminary sample of size n from the two-parameter Weibull distribution (31), where the parameters β and δ are unknown Then a lower unbiased (1–α) prediction limit H on the lth order statistic Y_l from a set of m future ordered observations $Y_1 \le \ldots \le Y_m$ also from the distribution (31) is given by

$$H = z_H^{1/\hat{\delta}} \hat{\beta}, \tag{50}$$

where z_H satisfies the equation

$$E_{(\beta,\delta)}\{P_{(\beta,\delta)}\{Y_l > H\}\} = E\{P\{Z_l > z_H \mid v_1, v_2\} = P\{Z_l > z_H \mid \mathbf{z}^{(r)}\}$$

$$= \frac{\int_0^\infty v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \sum_{k=0}^{l-1} \binom{m}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left((m-k+j) z_H^{v_2} + \sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r} dv_2}{\int_0^\infty v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \left(\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2}\right)^{-r} dv_2}$$

$$= 1-\alpha, \tag{51}$$

$Z_i = (X_i / \hat{\beta})^{\hat{\delta}}$, $i = 1, ..., r$; $\hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimates for β and δ based on the first r ordered observations ($X_1 \le \ldots \le X_r$) from a sample of size n from the two-parameter Weibull distribution (31).

(Observe that an upper unbiased α prediction limit H on the lth order statistic Y_l from a set of m future ordered observations $Y_1 \le \ldots \le Y_m$ may be obtained from a lower unbiased ($1-\alpha$) prediction limit by replacing $1-\alpha$ by α.)

Proof. If there is a random sample of m ordered observations $Y_1 \le \ldots \le Y_m$ from the two-parameter Weibull distribution (31) with the pdf $f_{(\beta,\delta)}(y)$ and cdf $F_{(\beta,\delta)}(y)$, then for the lth order statistic Y_l we have

$$P_{(\beta,\delta)}\{Y_l > H\} = \sum_{k=0}^{l-1}\binom{m}{k}[F_{(\beta,\delta)}(H)]^k[1-F_{(\beta,\delta)}(H)]^{m-k}$$

$$= \sum_{k=0}^{l-1}\binom{m}{k}\left[1-\exp\left(-\left(\frac{H}{\beta}\right)^{\delta}\right)\right]^k\left[\exp\left(-\left(\frac{H}{\beta}\right)^{\delta}\right)\right]^{m-k}. \quad (52)$$

Writing (52) as

$$P_{(\beta,\delta)}\{Y_l > H\} = \sum_{k=0}^{l-1}\binom{m}{k}\left[1-\exp\left(-\left(\frac{H}{\beta}\right)^{\delta}\right)\right]^k \exp\left(-(m-k)\left(\frac{H}{\beta}\right)^{\delta}\right)$$

$$= \sum_{k=0}^{l-1}\binom{m}{k}\left[1-\exp\left(-\left(\frac{H}{\widehat{\beta}}\right)^{\widehat{\delta}\left(\frac{\delta}{\widehat{\delta}}\right)}\left(\frac{\widehat{\beta}}{\beta}\right)^{\delta}\right)\right]^k$$

$$\times\exp\left(-(m-k)\left(\frac{H}{\widehat{\beta}}\right)^{\widehat{\delta}\left(\frac{\delta}{\widehat{\delta}}\right)}\left(\frac{\widehat{\beta}}{\beta}\right)^{\delta}\right)$$

$$= \sum_{k=0}^{l-1}\binom{m}{k}[1-\exp(-z_H^{v_2}v_1)]^k\exp(-(m-k)z_H^{v_2}v_1)$$

$$= \sum_{k=0}^{l-1}\binom{m}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\exp[-v_1(m-k+j)z_H^{v_2}] = P\{Z_l > z_H \mid v_1, v_2\}, \quad (53)$$

where

$$Z_l = \left(\frac{Y_l}{\widehat{\beta}}\right)^{\widehat{\delta}}, \quad z_H = \left(\frac{H}{\widehat{\beta}}\right)^{\widehat{\delta}}, \quad (54)$$

we have from (53) and (47) that

$$E_{(\beta,\delta)}\{P_{(\beta,\delta)}\{Y_l > H\}\} = E\{P\{Z_l > z_H \mid v_1, v_2\}\}$$

$$= \int_0^{\infty}\int_0^{\infty} P\{Z_l > z_H \mid v_1, v_2\} f(v_1, v_2 \mid \mathbf{z}^{(r)})dv_1 dv_2. \quad (55)$$

Now v_1 can be integrated out of (55) in a straightforward way to give (51). This completes the proof. □

Corollary 7.1. If l=1, then

$$H = \arg\left[\frac{\int_0^{\infty} v_2^{r-2}\prod_{i=1}^{r} z_i^{v_2}\left(m\left[\left(\frac{H}{\widehat{\beta}}\right)^{\widehat{\delta}}\right]^{v_2} + \sum_{i=1}^{r} z_i^{v_2} + (n-r)z_r^{v_2}\right)^{-r} dv_2}{\int_0^{\infty} v_2^{r-2}\prod_{i=1}^{r} z_i^{v_2}\left(\sum_{i=1}^{r} z_i^{v_2} + (n-r)z_r^{v_2}\right)^{-r} dv_2} = 1-\alpha\right]. \quad (56)$$

Corollary 7.2 (Lower (upper) one-sided prediction limit H on the lth order statistic Y_l in a new (future) sample of m observations from the two-parameter Weibull distribution, where the parameter β is unknown, on the basis of the previous data sample). Let $X_1 \le \ldots \le X_r$ be the first r ordered observations from a previous sample of size n from the two-parameter Weibull distribution (31), where the parameter δ is known. Thus, we deal with the exponential distribution. Then a lower one-sided conditional ($1-\alpha$) prediction limit H on the lth order statistic Y_l from a set of m future ordered observations $Y_1 \le \ldots \le Y_m$ also from the above distribution is given by

$$H = (z_H)^{1/\delta}\,\widehat{\beta}, \quad (57)$$

where z_H satisfies the equation

$$E_{\beta}\{P_{\beta}\{Y_l > H\}\} = E\{P\{Z_l > z_H \mid v_1\}\} = P\{Z_l > z_H\}$$

$$= \sum_{k=0}^{l-1}\binom{m}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left[1+(m-k+j)\frac{z_H}{r}\right]^{-r} = 1-\alpha, \quad (58)$$

$$\widehat{\beta} = \left(\left[\sum_{i=1}^{r} x_i^{\delta} + (n-r)x_r^{\delta}\right]\Big/ r\right)^{1/\delta} \quad (59)$$

is the maximum likelihood estimate for β based on the first r ordered observations ($X_1 \le \ldots \le X_r$) from a sample of size n from the two-parameter Weibull distribution (31).

Proof. Since the parameter δ is known, it can be shown by using (44) and the invariant embedding technique [9-12] that

$$f(v_1) = \frac{r^r}{\Gamma(r)}v_1^{r-1}\exp(-rv_1), \quad v_1 \in (0, \infty). \quad (60)$$

In this case,

$$P_{\beta}\{Y_l > H\} = \sum_{k=0}^{l-1}\binom{m}{k}\left[1-\exp\left(-\left(\frac{H}{\beta}\right)^{\delta}\right)\right]^k \exp\left(-(m-k)\left(\frac{H}{\beta}\right)^{\delta}\right)$$

$$= \sum_{k=0}^{l-1}\binom{m}{k}\left[1-\exp\left(-\left(\frac{H}{\widehat{\beta}}\right)^{\delta}\left(\frac{\widehat{\beta}}{\beta}\right)^{\delta}\right)\right]^k$$

$$\times\exp\left(-(m-k)\left(\frac{H}{\widehat{\beta}}\right)^{\delta}\left(\frac{\widehat{\beta}}{\beta}\right)^{\delta}\right)$$

$$=\sum_{k=0}^{l-1}\binom{m}{k}[1-\exp(-z_H v_1)]^k \exp(-(m-k)z_H v_1)$$

$$=\sum_{k=0}^{l-1}\binom{m}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j \exp[-v_1(m-k+j)z_H]$$

$$=P\{Z_l > z_H \mid v_1\}, \quad (61)$$

where

$$Z_l=\left(\frac{Y_l}{\beta}\right)^{\delta}, \quad z_H=\left(\frac{H}{\beta}\right)^{\delta}. \quad (62)$$

Thus, we have from (61) and (60) that

$$E_{\beta}\{P_{\beta}\{Y_l > H\}\}=E\{P\{Z_l > z_H \mid v_1\}\}$$

$$=\int_0^{\infty} P\{Z_l > z_H \mid v_1\} f(v_1)dv_1. \quad (63)$$

Now v_1 can be integrated out of (63) in a straightforward way to give

$$E\{P\{Z_l > z_H \mid v_1\}\}=P\{Z_l > z_H\}$$

$$=\sum_{k=0}^{l-1}\binom{m}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left[1+(m-k+j)\frac{z_H}{r}\right]^{-r}. \quad (64)$$

This completes the proof. □

Corollary 7.3 (Lower (upper) one-sided prediction limit H on the lth order statistic Y_l in a new (future) sample of m observations from the two-parameter Weibull distribution, where the parameter δ is unknown, on the basis of the previous data sample). Let $X_1 \le ... \le X_r$ be the first r ordered observations from a previous sample of size n from the two-parameter Weibull distribution (31), where the parameter β is known. Then a lower one-sided conditional $(1-\alpha)$ prediction limit H on the lth order statistic Y_l from a set of m future ordered observations $Y_1 \le \ldots \le Y_m$ also from the above distribution is given by

$$H=(z_H)^{1/\hat{\delta}}\beta, \quad (65)$$

where z_H satisfies the equation

$$E_{\delta}\{P_{\delta}\{Y_l > H\}\}=E\{P\{Z_l > z_H \mid v_2\}\}=P\{Z_l > z_H \mid \mathbf{z}^{(r)}\}$$

$$=\int_0^{\infty}\sum_{k=0}^{l-1}\binom{m}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j \exp[-(m-k+j)z_H^{v_2}] f(v_2 \mid \mathbf{z}^{(r)})dv_2$$

$$=1-\alpha, \quad (66)$$

$\hat{\delta}$ is the maximum likelihood estimates for δ based on the first r ordered observations ($X_1 \le ... \le X_r$) from a sample of size n from the two-parameter Weibull distribution (31)

Proof. Since the parameter β is known, it can be shown by using (44) and the invariant embedding technique [9-12] that

$$f(v_2 \mid \mathbf{z}^{(r)})=\vartheta(\mathbf{z}^{(r)})v_2^{r-1}\prod_{i=1}^{r} z_i^{v_2}\exp\left(-\left[\sum_{i=1}^{r} z_i^{v_2}+(n-r)z_r^{v_2}\right]\right),$$

$$v_2 \in (0,\infty), \quad (67)$$

where

$$\vartheta(\mathbf{z}^{(r)})=\left[\int_0^{\infty} v_2^{r-1}\prod_{i=1}^{r} z_i^{v_2}\exp\left(-\left[\sum_{i=1}^{r} z_i^{v_2}+(n-r)z_r^{v_2}\right]\right)dv_2\right]^{-1}, \quad (68)$$

$$Z_i=(X_i/\beta)^{\hat{\delta}}, \quad i=1,...,r. \quad (69)$$

In this case,

$$P_{\delta}\{Y_l > H\}=\sum_{k=0}^{l-1}\binom{m}{k}\left[1-\exp\left(-\left(\frac{H}{\beta}\right)^{\delta}\right)\right]^k \exp\left(-(m-k)\left(\frac{H}{\beta}\right)^{\delta}\right)$$

$$=\sum_{k=0}^{l-1}\binom{m}{k}\left[1-\exp\left(-\left(\frac{H}{\beta}\right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)}\right)\right]^k$$

$$\times\exp\left(-(m-k)\left(\frac{H}{\beta}\right)^{\hat{\delta}\left(\frac{\delta}{\hat{\delta}}\right)}\right)$$

$$=\sum_{k=0}^{l-1}\binom{m}{k}[1-\exp(-z_H^{v_2})]^k \exp(-(m-k)z_H^{v_2})$$

$$=\sum_{k=0}^{l-1}\binom{m}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j \exp[-(m-k+j)z_H^{v_2}]$$

$$=P\{Z_l > z_H \mid v_2\}, \quad (70)$$

where

$$Z_l=\left(\frac{Y_l}{\beta}\right)^{\hat{\delta}}. \quad z_H=\left(\frac{H}{\beta}\right)^{\hat{\delta}}. \quad (71)$$

Thus, we have from (70) and (67) that

$$E_{\delta}\{P_{\delta}\{Y_l > H\}\}=E\{P\{Z_l > z_H \mid v_2\}\}=P\{Z_l > z_H \mid \mathbf{z}^{(r)}\}$$

$$=\int_0^{\infty} P\{Z_l > z_H \mid v_2\} f(v_2 \mid \mathbf{z}^{(r)})dv_2. \quad (72)$$

This completes the proof. □

III. Numerical Examples

A. Example 1

An industrial firm has the policy to replace a certain device, used at several locations in its plant, at the end of 24-month intervals. It doesn't want too many of these items to fail before being replaced. Shipments of a lot of devices are made to each of three firms. Each firm selects a random sample of 5 items and accepts his shipment if no failures occur before a specified lifetime has accumulated. The manufacturer wishes to take a random sample and to calculate the lower prediction limit so that all shipments will be accepted with a probability of 0.95. The resulting lifetimes (rounded off to the nearest month) of an initial sample of size 15 from a population of such devices are given in Table 1.

TABLE I
The Resulting Lifetimes (in number of month intervals)

Observations							
X_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Lifetime	8	9	10	12	14	17	20
x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
25	29	30	35	40	47	54	62

Goodness-of-fit testing. It is assumed that

$$X_i \sim f_{(\mu,\sigma,\delta)}(x) = \frac{\delta}{\sigma} x^{\delta-1} \exp[-(x^{\delta} - \mu)/\sigma],$$

$$(x \ge \mu, \sigma, \delta > 0), \quad i = 1(1)15, \tag{73}$$

where the parameters μ and σ are unknown; (δ=0.87). Thus, for this example, r = n = 15, k = 3, m = 5, 1−α = 0.95, $X_1^{\delta} = 6.1$, and S = 170.8. It can be shown that the

$$U_j = 1 - \left(\frac{\sum_{i=2}^{j+1} (n-i+1)(X_i^{\delta} - X_{i-1}^{\delta})}{\sum_{i=2}^{j+2} (n-i+1)(X_i^{\delta} - X_{i-1}^{\delta})} \right)^j, \quad j = 1(1)n-2, \tag{74}$$

are i.i.d. $U(0,1)$ rv's (Nechval *et al*. [13]). We assess the statistical significance of departures from the left-truncated Weibull model by performing the Kolmogorov-Smirnov goodness-of-fit test. We use the K statistic (Muller *et al*. [14]). The rejection region for the α level of significance is $\{K \ge K_{n;\alpha}\}$. The percentage points for $K_{n;\alpha}$ were given by Muller *et al*. [14]. For this example,

$$K = 0.220 < K_{n=13;\alpha=0.05} = 0.361. \tag{75}$$

Thus, there is not evidence to rule out the left-truncated Weibull model. It follows from (6) and (21), for

$$\alpha = 0.05 < \frac{km}{n+km} = 0.5, \tag{76}$$

that

$$H = \left(X_1^{\delta} - \frac{S}{n}\left[\left(\frac{km}{\alpha(n+km)} \right)^{\frac{1}{n-1}} - 1 \right] \right)^{1/\delta}$$

$$= \left(6.1 - \frac{170.8}{15}\left[\left(\frac{15}{0.05(15+15)} \right)^{\frac{1}{14}} - 1 \right] \right)^{1/0.87} = 5. \tag{77}$$

Thus, the manufacturer has 95% assurance that no failures will occur in each shipment before H = 5 month intervals.

B. Example 2

A system consists of m(=5) identical capacitors whose times to failure follow the two-parameter exponential distribution (22). Initially one capacitor is operating and the remaining m−1 capacitors are in a standby mode; a new capacitor goes into operation as soon as the preceding capacitor has failed. The system is said to fail when all m capacitors have failed. Thus, the system time to failure is the total of the failure times for the m capacitors. A simultaneous lower prediction limit to be exceeded with probability 1−α = 0.99 by the system time to failure is desired. This limit is to be calculated from the times to failure of n(=5) previously tested capacitors. It is assumed that the first r(=3) times to failure of n previously tested capacitors were observed. Taking into account (25) and supposing that

$$S = \sum_{i=1}^{r} (X_i - X_1) + (n-r)(X_r - X_1) = 570 \text{ hours}, \tag{78}$$

where

$$X_1 = 300 \text{ hours}, \tag{79}$$

we have from (27) and (28) that

$$H = m\left[X_1 + \frac{S}{n}\left(1 - \left[\frac{1}{\alpha}\left(\frac{m}{m+n} \right)^m \right]^{1/(r-1)} \right) \right] = 1062 \text{ hours}, \tag{80}$$

where

$$\alpha = 0.01 < \left(\frac{m}{m+n} \right)^m = 0.03125. \tag{81}$$

Thus, the manufacturer has 99% assurance that no failures will occur in the system before H = 1062 hours.

C. Example 3

A manufacturer has the data on the mileages at which nineteen military carriers failed [16]. These were 162, 200, 271, 302, 393, 508, 539, 629, 706, 777, 884, 1008, 1101, 1182, 1463, 1603, 1984, 2355, 2880, and thus constitute a complete sample of observations $X_1 \le ... \le X_n$ with n = 19 from the two-parameter exponential distribution (22). It follows from (25) that

$$S = \sum_{i=1}^{19} (X_i - X_1) = 15{,}869, \qquad (82)$$

where

$$X_1 = 162. \qquad (83)$$

A buyer tells the carrier manufacturer that he wants to place two orders for the same type of military carriers to be shipped to two different destinations. The buyer wants to select a random sample of m=5 military carriers from each shipment to be tested. An order is accepted only if all of 5 military carriers in the sample meet the warranty period. What warranty should the manufacturer offer so that all of 5 military carriers in both samples meet the warranty with probability of 0.9? That is, what lower simultaneous prediction limit should the manufacturer guarantee to assure acceptance of both shipments with a probability of 0.9? To answer this question, we take into account (6) and consider a lower (1−α) prediction limit H on the minimum Y_1 of a set of km future ordered observations $Y_1 \le \ldots \le Y_m$, where k=2.

It follows from (26), for

$$\alpha = 0.01 < \frac{km}{n+km} = 0.345, \qquad (84)$$

that

$$H = X_1 - \frac{S}{n}\left[\left(\frac{km}{\alpha(n+km)}\right)^{\frac{1}{n-1}} - 1\right]$$

$$= 162 - \frac{15{,}869}{19}\left[\left(\frac{10}{0.1(19+10)}\right)^{\frac{1}{18}} - 1\right] = 102.54. \qquad (85)$$

Thus, the manufacturer has 90% assurance that no failures will occur in each shipment before H = 102.54 mileages.

IV. Conclusion and Future Work

In this paper, explicit formulae, based on previous independent observations, have been developed for computing conditional quantiles of some useful pivotal statistics. The quantiles are used to construct unbiased simultaneous prediction limits on the order statistics of all of k future samples using the results of a previous sample from the same underlying distribution belonging to invariant family. The results can be used to predict the total duration time in a Type II censoring life testing experiments, and to predict the lifetime of a k-out-of-n: F systems. Such results are required, for example, when a manufacturer wishes to assure with a high probability the acceptance of all k future shipments of a product. The computation procedures can be easily programmed and implemented for practical use. Although the quantiles of the pivotal statistics considered in this paper can be obtained through simulation, but it will be noted that simulation results are unstable; they vary from one to another. From theoretical as well as practical points of view, analytical solutions should be used if they are available. The results of this paper provide such analytical solutions. The exact prediction limits are found and illustrated with the numerical examples. The methodology described here can be extended in several different directions to handle various problems that arise in practice. We have illustrated the proposed methodology for the two-parameter exponential and Weibull distributions. Furthermore, the techniques used in this paper can be applied to obtain explicit formulae for computing conditional quantiles relating to unbiased simultaneous prediction limits for any other log-location-scale distributions.

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