An Application of Best L_1 Piecewise Monotonic Data Approximation to Signal Restoration

I. C. Demetriou

***Abstract*—We consider an application of the best L_1 piecewise monotonic data approximation method to univariate signal restoration. We extend numerical examples concerned with the L_2 analogous method to the L_1 case and we show the efficacy of a relevant software package that implements the method in data fitting and in denoising data from a medical image. The piecewise monotonic approximation method makes the smallest change to the data such that the first differences of the smoothed data change sign a prescribed number of times. Our results exhibit some strengths and certain advantages of the method. Therefore, they may be helpful to the development of new algorithms that are suitable to signal restoration calculations.**

***Index Terms*—absolute value, data smoothing, divided differences, L_1-norm, piecewise monotonic approximation, signal restoration**

I. Introduction

Let $\{\phi(x_i) : i = 1, 2, \ldots, n\}$ be a sequence of values of a signal $\phi(x)$ measured at the abscissae $x_1 < x_2 < \cdots < x_n$, but these measurements include errors and the data are to be used to provide a restoration to $\phi(x)$. Most methods of data approximation assume that measurements of function values can be approximated by a form that depends on relatively few parameters. Spline functions, for instance, are candidates for approximation, but sometimes it is difficult to choose a suitable one. Therefore, Demetriou and Powell [10] take the view that some useful smoothing should be possible if the data fail to possess a property that is usually obtained by the underlying function. They assume that if the signal has turning points, then the number of measurements is substantially greater than the number of turning points. Therefore they propose algorithms that modify the measurements if their first differences $\{y_{i+1}-y_i : i = 1, \ldots, n-1\}$ include more than $k-1$ sign changes, a condition which allows k monotonic sections to the smoothed data, where k is a prescribed positive integer. In [10] (best L_2 approximation), the $k-1$ optimal turning points and the least sum of squares change to the data are computed in $O(n^2 + kn \log n)$ computer operations. The important result in this work is the substantial reduction of the number of data that need be considered in finding the optimal turning points, among $O(n^k)$ combinations of possible combinations of turning points. The special cases $k = 1, 2$ are solved in only $O(n)$ operations. Applications of the method in spectroscopy, signal restoration and image processing are presented by [4], [16], [26]) and references therein. For general references in signal and image processing see [21] and [13].

In [7], piecewise monotonic data approximation is studied in the sense of the least sum of the moduli of the errors (best L_1 approximation). In general, a best L_1 approximation has the remarkable property, which makes it particularly suitable to data smoothing when there are few gross errors in the data, that the magnitudes of the errors make no difference to the best fit (see, for example, [20]). The main result of [7] is that a best L_1 calculation, as the corresponding L_2 one, can be decomposed into optimal monotonic calculations between adjacent turning points. L1PMA is the Fortran software package that the author has developed [8] to implement the method of [7] with certain extensions that improve efficiency. L1PMA calculates a best L_1 fit with at most k monotonic sections of the data in $O(n^3 + kn^2)$ computer operations. This complexity reduces to $O(n^2)$ when $k = 1$ or $k = 2$. The software package has been tested on a variety of data sets showing in practice quadratic performance with respect to n. The package employs techniques for calculating the median and the best L_1 monotonic approximation ($k = 1$), which is an integral part of the package. The monotonic problem during the last 60 years has received considerable attention in various fields, including engineering, economics, operations research and statistics.

Some advantages that are gained by employing the piecewise monotonic constraints to data approximation are as follows. We avoid the assumption that $\phi(x)$ has a form that depends of a few parameters, which occurs in many other approximation techniques, as for example, in splines [3] and wavelets [14]; the smoothing process is a projection because, if it is applied to the smoothed data, then no changes are made to; piecewise monotonicity is a property that occurs in a wide range of underlying functions; any degree of undulation of the data can be accommodated. Moreover, the piecewise monotonic approximation method is particularly suitable when the data errors are large and uncorrelated.

This paper is concerned with an application of the best L_1 piecewise monotonic data approximation method [8] to univariate signal restoration. It presents a survey of the method and extends some numerical examples from least squares piecewise monotonic data approximation [9] to the L_1 case. There are many similarities as well as considerable differences between these methods.

The paper is organized as follows. In Section II we outline the method for best L_1 piecewise monotonic data approximation. In Section III we consider numerical examples that illustrate the method on data from a periodic function with simulated errors and data from a noisy signal. The results are analyzed, the smoothing capability of the method is demonstrated and a direct comparison is made between the results of our method and those of the analogous least squares method. In Section IV we present some concluding remarks and discuss on the possibility of future directions of this research.

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II. Best L_1 Piecewise Monotonic Data Approximation

We regard the measurements as components $\{\phi_i = \phi(x_i) : i = 1, 2, \ldots, n\}$ of a $n-$vector $\underline{\phi}$. The user provides k and specifies whether the first monotonic section is increasing or decreasing. Then the method of [7] automatically derives the optimal turning points and the best L_1 fit. Specifically, the method calculates a vector $\underline{y}$ that minimizes the sum of the moduli of the errors

$$\Phi(\underline{y}) = \sum_{i=1}^{n} |y_i - \phi_i| \quad (1)$$

subject to the piecewise monotonicity constraints

$$\left.\begin{array}{ll} y_{t_{j-1}} \leq y_{t_{j-1}+1} \leq \cdots \leq y_{t_j}, & \text{if } j \text{ is odd} \\ y_{t_{j-1}} \geq y_{t_{j-1}+1} \geq \cdots \geq y_{t_j}, & \text{if } j \text{ is even} \end{array}\right\}, \quad (2)$$

where the integers $\{t_j : j = 0, 1, \ldots, k\}$, namely the positions of the turning points or extrema of the fit, satisfy the conditions

$$1 = t_0 \leq t_1 \leq \cdots \leq t_k = n. \quad (3)$$

Since the integers $\{t_j : j = 1, 2, \ldots, k-1\}$ are variables in the optimization calculation that gives a best L_1 fit, the number of combinations of integer variables is raised to the order $O(n^k)$, which makes non practicable to investigate all these combinations individually for optimality. Fortunately the piecewise monotonic approximation problem has a rich structure that allows an efficient calculation of an optimal fit.

To begin with, the constraints prevent the equation $\underline{y} = \underline{\phi}$, because if $\underline{\phi}$ does not satisfy the piecewise monotonicity constraints, then $\{t_j : j = 1, 2, \ldots, k-1\}$ are all different. Moreover, the optimal value of y_{t_j} is independent of the components $\{y_i : i \neq t_j\}$, which gives the interpolation conditions

$$y_{t_j} = \phi_{t_j}, j = 1, 2, \ldots, k-1. \quad (4)$$

The most important property, however, is that each monotonic section in a best L_1 piecewise monotonic fit is an optimal fit itself to the corresponding data, so it can be obtained by a separate calculation. Indeed, the components $\{y_i : i = t_{j-1}, t_{j-1}+1, \ldots, t_j\}$ on $[x_{t_{j-1}}, x_{t_j}]$ minimize the sum of the moduli

$$\sum_{i=t_{j-1}}^{t_j} |y_i - \phi_i| \quad (5)$$

subject to the constraints

$$y_i \leq y_{i+1}, i = t_{j-1}, \ldots, t_j - 1, \text{ if } j \text{ is odd} \quad (6)$$

or subject to the constraints

$$y_i \geq y_{i+1}, i = t_{j-1}, \ldots, t_j - 1, \text{ if } j \text{ is even.} \quad (7)$$

In the former case the sequence $\{y_i : i = t_{j-1}, t_{j-1} + 1, \ldots, t_j\}$ is a best L_1 monotonic increasing fit to $\{\phi_i : i = t_{j-1}, t_{j-1}+1, \ldots, t_j\}$ and on the latter case a best L_1 monotonic decreasing one. Therefore, provided that $\{t_i : i = 1, 2, \ldots, k-1\}$ are known, the components of $\underline{y}$ can be generated by solving a separate monotonic problem on each section $[x_{t_{j-1}}, x_{t_j}]$. The problem with the decreasing monotonicity constraints may be treated computationally as the problem with the increasing monotonicity constraints after reversing the order of the data. We introduce the notation $\alpha(t_{j-1}, t_j)$ and $\beta(t_{j-1}, t_j)$ for the least value of (5) subject to the constraints (6) and (7) respectively. We denote by $\delta(k, n)$ the least value of (1) at the required minimum and, if k is odd, we obtain the expression $\delta(k, n) = \alpha(t_0, t_1) + \beta(t_1, t_2) + \alpha(t_2, t_3) + \cdots + \alpha(t_{k-1}, t_k)$ and analogously if k is even, where we replace the last term in this sum by $\beta(t_{k-1}, t_k)$.

The problem of minimizing (5) subject to (6) is a linear programming problem that need not have a unique solution (see [11], [12], [18] for general references and methods on L_1 approximation and linear programming). Although it is usual to calculate the solution of a L_1 approximation problem by applying linear programming techniques (see, [1], [2]), we have developed a method that is faster than a general linear programming calculation. Specifically, the calculation of a best monotonically increasing approximation to $\underline{\phi}$ seeks intervals where its components have different constant values. In the L_1 case these values are equal to the median of the corresponding data points, while in the L_2 case they are equal to their mean value. The intervals are formed by using the remarkable property that any constraints which are satisfied as equalities by a best L_1 approximation subject to a subset of the monotonicity constraints are also satisfied as equalities by a best L_1 approximation subject to all monotonicity constraints (6). The consideration of [5] that the specific value of the median should be chosen carefully in order that the final $\underline{y}$ satisfy the constraints (6) has been taken into account in the development of Algorithm 2 of [8] that performs the calculation of a best L_1 monotonic increasing fit on $[x_{t_{j-1}}, x_{t_j}]$ together with all the numbers

$$\alpha(t_{j-1}, t) = \sum_{i=t_{j-1}}^{t} |y_i - \phi_i|, t = t_{j-1}, \ldots, t_j \quad (8)$$

in $O((t_j - t_{j-1})^2)$ computer operations. General algorithms for the best L_1 monotonic increasing approximation problem are given by [17], [22] and [24].

Further, it is proved that an optimal fit $\underline{y}$ associated with the integer variables $\{t_j : j = 1, 2, \ldots, k-1\}$ can split at t_{k-1} into two optimal sections. One section that provides an optimal fit on $[x_1, x_{t_{k-1}}]$, which in fact is similar to $\underline{y}$ with one monotonic section less, and one section on $[x_{t_{k-1}}, x_n]$ that is a single best L_1 monotonic fit to the remaining data. Therefore with the initial values

$$\delta(1, t) = \alpha(t_0, t), \text{ for } t = 1, 2, \ldots, n, \quad (9)$$

the optimization problem can be expressed in terms of the dynamic programming formula

$$\delta(r, t) = \min_{1 \leq s \leq t} [\delta(r-1, s) + \alpha(s, t)], \text{ if } r \text{ is odd} \quad (10)$$

or

$$\delta(r, t) = \min_{1 \leq s \leq t} [\delta(r-1, s) + \alpha(s, t)], \text{ if } r \text{ is even,} \quad (11)$$

where $1 \leq t \leq n$. The implementation of these formulae includes several options that are considered by [10] and [7]. Demetriou [8], especially, has implemented this method in Fortran and provided a software package that derives a solution in $O(n^2 m + km^2)$ computer operations, where m

is the number of local extrema of the data. For example, an integer p is the index of a local maximum of the sequence $\phi_i, i = 2, \dots, n-1$ if $\phi_{p-1} \leq \phi_p$ and $\phi_p > \phi_{p+1}$, and similarly for a local minimum. Since m is a fraction of n, the complexity of the formulae (10) and (11) is reduced at least be a factor of 4. Further, it is stated in [8] that as more data are inserted into the calculation, that is as n increases for a fixed k, or as the number of monotonic sections increases by 2 for a fixed n, the rightmost extremum of the associated optimal approximation increases as well. The reported numerical results [8] show that increasing k beyond 3 either has a small effect in the computation times or no effect at all. Indeed, as k increases, the ranges where the monotonic algorithm is applied are decreased due to the increasing property of the rightmost extremum. We see that the complexity of the piecewise monotonic approximation method is dominated by the term n^2m, but in practice the mentioned properties restrict considerably the range of s in the minimization formulae (10) and (11) and make the calculation very efficient.

The method that gives a piecewise monotonic approximation may also be applied to the problem where inequalities (2) are replaced by the reversed ones, in which case the first section of the fit is decreasing. The latter problem may be treated computationally as the former one after an overall change of sign of $\underline{\phi}$.

III. Numerical Examples in Signal Restoration with Piecewise Monotonicity Constraints and L_1 Optimality

To illustrate the efficacy of the method in signal restoration we present calculations from two numerical examples that are taken from [9]. In addition, the reader will be able to compare the results of the least squares case of [9] with those of this paper. We shall see that the results are very similar.

The first example is a best fit with $k = 6$ monotonic sections to $n = 100$ measurements of the function

$$\phi(x) = \sin(5x) - x \tag{12}$$

at equally spaced abscissae in the interval $[-2.5, 2]$. The measurements were generated by adding uniformly distributed random numbers from the interval $(-0.5, 0.5)$ to the function values $\phi(x_i), i = 1, 2, \dots, n$. The data are presented in the second and the third column of Table I, although the abscissae are irrelevant to this calculation. Without any preliminary analysis, the data were fed to L1PMA, six monotonic sections were required and the solution was reached immediately in a common pc. The best L_1 fit is presented in the fourth column of Table I and the analogous best L_2 fit is presented in the fifth column (the L_2 fit is obtained as described in [9]); the horizontal lines indicate the turning point positions in either case. Fig. 1 shows the data and the best L_1 fit; the data are denoted by (+), the best fit by (o) and the piecewise linear interpolant to the smoothed values illustrates the fit.

The first attempt at fitting the data, on the purpose of demonstrating some features of the piecewise monotonic fit, was not entirely satisfactory. Indeed, the turning points of the fit are at the abscissae $x_8, x_{50}, x_{64}, x_{78}$ and x_{89}, satisfying the interpolation conditions (4). The components

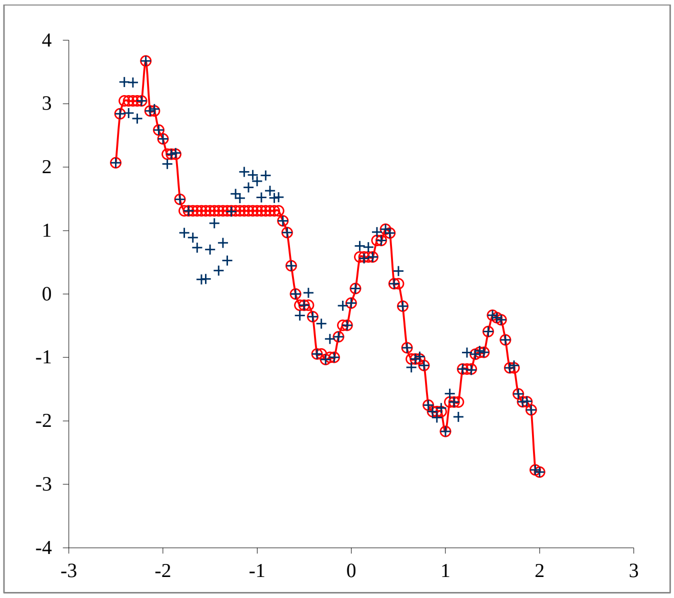

Fig. 1. Graphical representation of the data given in Table I. The data of column 2 annotate the x-axis. The data of column 3 are denoted by (+) and the best L_1 fit of column 4 by (o). The solid line illustrates the fit.

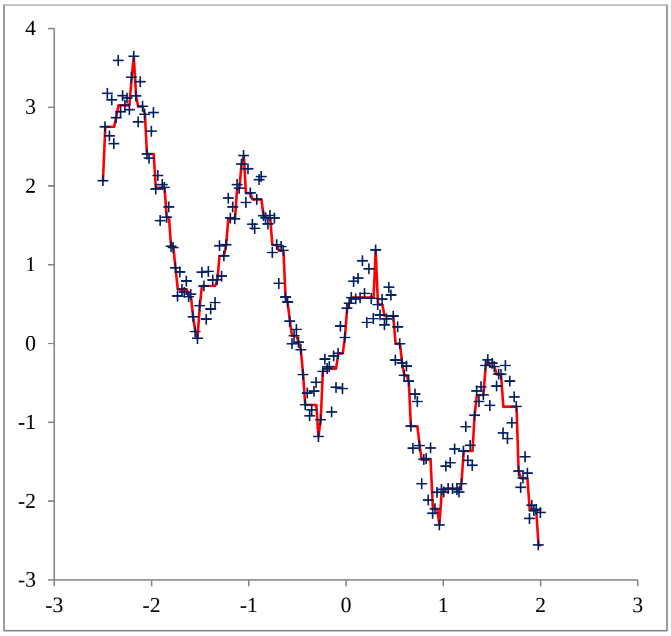

Fig. 2. Best L_1 fit (solid line) with $k = 8$ monotonic sections to $n = 200$ measurements (+) of function (12) produced as described in Section III.

of a best fit consist of ranges of constant values, each such value being the median of the corresponding data within a range. Of course, the monotonic algorithm finds the median of any data values that are needed to achieve monotonicity. For example, we see in the fourth column of Table I that the components of an optimal fit that is obtained by minimizing the function $\sum_{i=3}^{7} |y_i - \phi_i|$ subject to the constraints $y_3 \leq y_4 \leq y_5 \leq y_6 \leq y_7$ have the values $y_3 = y_4 = y_5 = y_6 = y_7 = 3.042$, where 3.042 is the (unique) median of the data $\{3.339, 2.852, 3.333, 2.765, 3.042\}$. The analogous L_2 values (fifth column) are $y_3 = y_4 = y_5 = y_6 = y_7 = (\sum_{i=3}^{7} \phi_i)/5 = 3.066$ and are obtained by minimizing the function $\sum_{i=3}^{7} (y_i - \phi_i)^2$ subject to the constraints $y_3 \leq y_4 \leq y_5 \leq y_6 \leq y_7$. Further, the median

TABLE I
BEST FITS WITH $k = 6$ AND $k = 8$ MONOTONIC SECTIONS TO MEASUREMENTS OF FUNCTION (12)

			Best Fit k=6		Best Fit k=8					Best Fit k=6		Best Fit k=8	
	x_i	φ_i	$y_i\,(L_1)$	$y_i\,(L_2)$	$y_i\,(L_1)$	$y_i\,(L_2)$		x_i	φ_i	$y_i\,(L_1)$	$y_i\,(L_2)$	$y_i\,(L_1)$	$y_i\,(L_2)$
1	–2.50	2.066	2.066	2.066	2.066	2.066	51	–0.23	–0.710	–0.999	–0.855	–0.999	–0.855
2	–2.45	2.839	2.839	2.839	2.839	2.839	52	–0.18	–0.999	–0.999	–0.855	–0.999	–0.855
3	–2.41	3.339	3.042	3.066	3.042	3.066	53	–0.14	–0.678	–0.678	–0.678	–0.678	–0.678
4	–2.36	2.852	3.042	3.066	3.042	3.066	54	–0.09	–0.186	–0.494	–0.340	–0.494	–0.340
5	–2.32	3.333	3.042	3.066	3.042	3.066	55	–0.05	–0.494	–0.494	–0.340	–0.494	–0.340
6	–2.27	2.765	3.042	3.066	3.042	3.066	56	0.00	–0.146	–0.146	–0.146	–0.146	–0.146
7	–2.23	3.042	3.042	3.066	3.042	3.066	57	0.05	0.086	0.086	0.086	0.086	0.086
8	–2.18	3.673	3.673	3.673	3.673	3.673	58	0.09	0.756	0.583	0.659	0.583	0.659
9	–2.14	2.884	2.884	2.899	2.884	2.899	59	0.14	0.561	0.583	0.659	0.583	0.659
10	–2.09	2.914	2.884	2.899	2.884	2.899	60	0.18	0.735	0.583	0.659	0.583	0.659
11	–2.05	2.584	2.584	2.584	2.584	2.584	61	0.23	0.583	0.583	0.659	0.583	0.659
12	–2.00	2.442	2.442	2.442	2.442	2.442	62	0.27	0.974	0.843	0.908	0.843	0.908
13	–1.95	2.046	2.203	2.157	2.203	2.157	63	0.32	0.843	0.843	0.908	0.843	0.908
14	–1.91	2.203	2.203	2.157	2.203	2.157	64	0.36	1.020	1.020	1.020	1.020	1.020
15	–1.86	2.221	2.203	2.157	2.203	2.157	65	0.41	0.960	0.960	0.960	0.960	0.960
16	–1.82	1.491	1.491	1.491	1.491	1.491	66	0.45	0.161	0.161	0.261	0.161	0.261
17	–1.77	0.962	1.311	1.198	0.962	1.136	67	0.50	0.362	0.161	0.261	0.161	0.261
18	–1.73	1.311	1.311	1.198	0.962	1.136	68	0.55	–0.195	–0.195	–0.195	–0.195	–0.195
19	–1.68	0.886	1.311	1.198	0.886	0.886	69	0.59	–0.848	–0.848	–0.848	–0.848	–0.848
20	–1.64	0.728	1.311	1.198	0.728	0.728	70	0.64	–1.157	–1.025	–1.057	–1.025	–1.057
21	–1.59	0.227	1.311	1.198	0.227	0.227	71	0.68	–1.025	–1.025	–1.057	–1.025	–1.057
22	–1.55	0.236	1.311	1.198	0.236	0.236	72	0.73	–0.989	–1.025	–1.057	–1.025	–1.057
23	–1.50	0.700	1.311	1.198	0.700	0.700	73	0.77	–1.127	–1.127	–1.127	–1.127	–1.127
24	–1.45	1.114	1.311	1.198	0.700	0.704	74	0.82	–1.753	–1.753	–1.753	–1.753	–1.753
25	–1.41	0.369	1.311	1.198	0.700	0.704	75	0.86	–1.855	–1.855	–1.855	–1.855	–1.855
26	–1.36	0.805	1.311	1.198	0.700	0.704	76	0.91	–1.951	–1.855	–1.876	–1.855	–1.876
27	–1.32	0.527	1.311	1.198	0.700	0.704	77	0.95	–1.801	–1.855	–1.876	–1.855	–1.876
28	–1.27	1.297	1.311	1.198	1.297	1.297	78	1.00	–2.169	–2.169	–2.169	–2.169	–2.169
29	–1.23	1.576	1.311	1.198	1.507	1.542	79	1.05	–1.574	–1.703	–1.739	–1.703	–1.739
30	–1.18	1.507	1.311	1.198	1.507	1.542	80	1.09	–1.703	–1.703	–1.739	–1.703	–1.739
31	–1.14	1.924	1.311	1.198	1.924	1.924	81	1.14	–1.939	–1.703	–1.739	–1.703	–1.739
32	–1.09	1.680	1.311	1.198	1.777	1.777	82	1.18	–1.185	–1.185	–1.185	–1.185	–1.185
33	–1.05	1.874	1.311	1.198	1.777	1.777	83	1.23	–0.926	–1.185	–1.064	–1.185	–1.064
34	–1.00	1.777	1.311	1.198	1.777	1.777	84	1.27	–1.202	–1.185	–1.064	–1.185	–1.064
35	–0.95	1.519	1.311	1.198	1.627	1.693	85	1.32	–0.952	–0.952	–0.952	–0.952	–0.952
36	–0.91	1.867	1.311	1.198	1.627	1.693	86	1.36	–0.899	–0.922	–0.911	–0.922	–0.911
37	–0.86	1.627	1.311	1.198	1.627	1.627	87	1.41	–0.922	–0.922	–0.911	–0.922	–0.911
38	–0.82	1.514	1.311	1.198	1.514	1.520	88	1.45	–0.594	–0.594	–0.594	–0.594	–0.594
39	–0.77	1.525	1.311	1.198	1.514	1.520	89	1.50	–0.338	–0.338	–0.338	–0.338	–0.338
40	–0.73	1.150	1.150	1.150	1.150	1.150	90	1.55	–0.376	–0.376	–0.376	–0.376	–0.376
41	–0.68	0.967	0.967	0.967	0.967	0.967	91	1.59	–0.410	–0.410	–0.410	–0.410	–0.410
42	–0.64	0.442	0.442	0.442	0.442	0.442	92	1.64	–0.727	–0.727	–0.727	–0.727	–0.727
43	–0.59	–0.001	–0.001	–0.001	–0.001	–0.001	93	1.68	–1.166	–1.166	–1.147	–1.166	–1.147
44	–0.55	–0.343	–0.178	–0.168	–0.178	–0.168	94	1.73	–1.129	–1.166	–1.147	–1.166	–1.147
45	–0.50	–0.178	–0.178	–0.168	–0.178	–0.168	95	1.77	–1.578	–1.578	–1.578	–1.578	–1.578
46	–0.45	0.016	–0.178	–0.168	–0.178	–0.168	96	1.82	–1.702	–1.702	–1.698	–1.702	–1.698
47	–0.41	–0.359	–0.359	–0.359	–0.359	–0.359	97	1.86	–1.694	–1.702	–1.698	–1.702	–1.698
48	–0.36	–0.949	–0.949	–0.710	–0.949	–0.710	98	1.91	–1.830	–1.830	–1.830	–1.830	–1.830
49	–0.32	–0.470	–0.949	–0.710	–0.949	–0.710	99	1.95	–2.773	–2.773	–2.773	–2.773	–2.773
50	–0.27	–1.035	–1.035	–1.035	–1.035	–1.035	100	2.00	–2.808	–2.808	–2.808	–2.808	–2.808

continued

of an even number of data need not be unique. Indeed, the components $\{y_{58}, y_{59}, y_{60}, y_{61}\}$ of a best L_1 monotonic fit to the data $\{\phi_{58}, \phi_{59}, \phi_{60}, \phi_{61}\}$ are all set by the monotonic algorithm to 0.583, while the median of these data is in the closed interval $[0.583, 0.735]$.

In view of the preceding discussion, the first decreasing section of the fit suggests that a better approximation is possible by increasing k. Therefore, a second attempt at fitting these data with $k = 8$ resulted in the approximation values that are presented in the sixth column of Table I, having two extra turning points at x_{21} and x_{31} by enhancing the fit on the interval $[x_{17}, x_{39}]$, where the constant components provide a poor fit; all the other turning points remained unchanged. The corresponding values of the L_2 fit with $k = 8$ are presented in the seventh column. Besides that the L_1 fit and the L_2 fit have the same turning points, we see that the L_1 and L_2 piecewise monotonic approximation algorithms produce closely similar results. The piecewise monotonic fits with $k = 6$ and $k = 8$ show substantial changes to the data when the errors are large, but the monotonic constraints made no change to those data in section $[x_{89}, x_{100}]$ that satisfy the constraints, thus giving $y_i = \phi_i$, for $i = 89, 90, 91, 92, 93, 95, 96, 98, 99$ and 100. Further, the particular fit with $k = 6$ (see Fig. 1) shows that L1PMA achieves the piecewise monotonicity property it sets out to achieve and, generally, any degree of undulation in the data can be accommodated by choosing a suitable k. Hence, we repeated the experiment with $n = 200$ data points. Now the input to the program are both the data $\underline{\phi}$ and the number $k = 8$; the output is a best L_1 fit with eight monotonic sections, which is displayed in see Fig. 2.

The second example is a fit to 640 data points obtained by the 239th vertical scan line through a 640×640 gray-scale

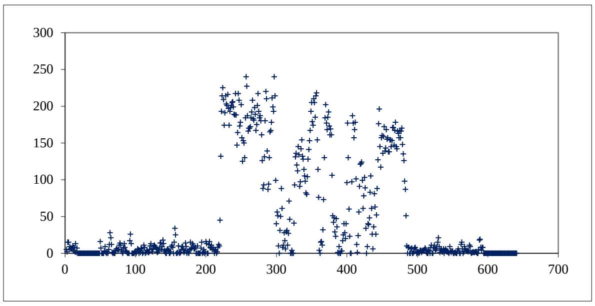

Fig. 3. Scan line of a magnetic resonance imaging axial slice from [9]. Pixel intensities are denoted by (+).

noisy image from [9]. The pixel intensities are displayed in Fig. 3. The data vary considerably and although they exhibit some turning points, reader's eye is not especially attracted. We seek turning points that might reveal major monotonic trends. We begin by noticing that the total number of local extrema of the data is 126. Next, we make use of the least squares method of [25] which is a variant of [10] that includes the trend test of [19]. The input to this method is *only* the data $\underline{\phi}$ and the output is an optimal fit where the number of monotonic sections is calculated *automatically* by the method. We fed the data to the computer program of [25] and the resultant fit gave automatically $k = 70$, thus providing too many turning points.

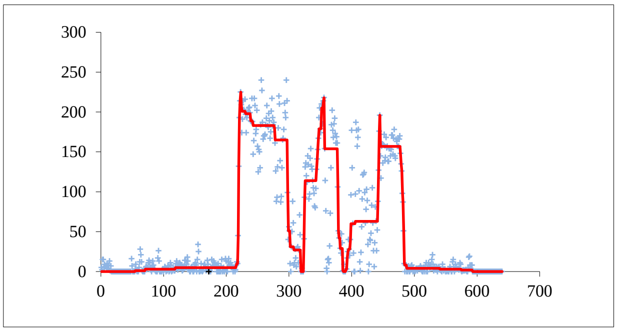

Fig. 4. Best L_1 fit with $k = 6$ (solid line) to the signal of Fig. 3.

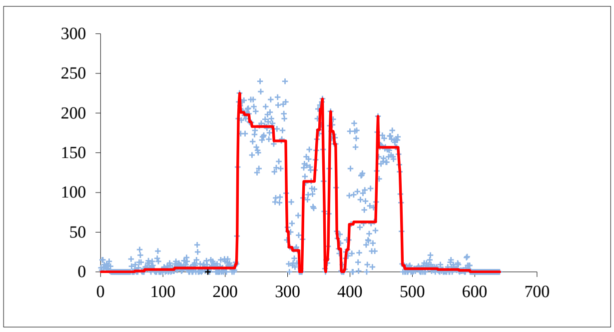

Fig. 5. Best L_1 fit with $k = 8$ (solid line) to the signal of Fig. 3.

Hence we carried out some runs with smaller numbers of turning points, which gives more emphasis to major monotonic trends. The data were fed to L1PMA and a best L_1 fit subject to the piecewise monotonicity constraints with $k = 6$ was calculated immediately. The value of (1) equals 7175. Fig. 4 displays the data and the fit. The turning points

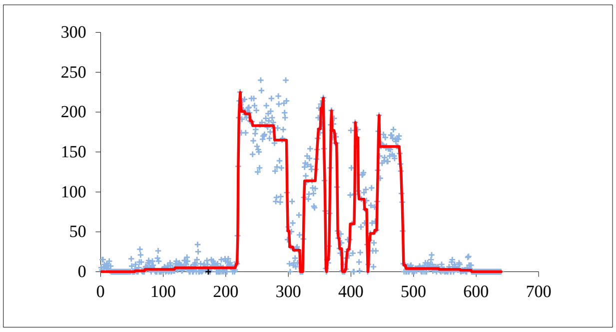

Fig. 6. Best L_1 fit with $k = 10$ (solid line) to the signal of Fig. 3.

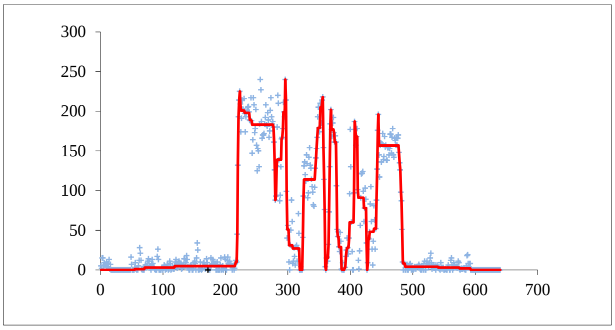

Fig. 7. Best L_1 fit with $k = 12$ (solid line) to the signal of Fig. 3.

TABLE II
THE TURNING POINT INDICES OF THE BEST L_1 AND THE BEST L_2 FIT WHEN $k = 6$

No	L_1	L_2
0	1	1
1	223	223
2	319	322
3	356	356
4	387	389
5	445	445
6	640	640

indices of the fit are presented in the second column of Table II and they are at the abscissae $x_{223}, x_{319}, x_{356}, x_{387}$ and x_{445}. Further, the turning points of the corresponding L_2 fit are at $x_{223}, x_{322}, x_{356}, x_{389}$ and x_{445}, as it is displayed in the third column of Table II, two of them being at slightly different positions than those in the L_1 fit. We see that the L_1 fit to the data is much smoother than are the data values themselves, but we should not forget that the method has revealed the k major monotonic sections. Thus, in case that the L_1 fit might be considered unsatisfactory, we carried out a second run with $k = 8$, which gave automatically two extra turning points at x_{361} and x_{369} by enhancing the fit of Fig. 4 in the interval of adjacent turning points $[x_{356}, x_{387}]$, where the fit seems rather poor. The turning point indices are presented in Table III, the L_1 fit is presented in Fig. 5 and the sum of moduli of residuals is 5919. One more run with $k = 10$, gave two extra turning points at x_{407} and x_{427} by enhancing the fit of Fig. 5 in the interval of adjacent turning points $[x_{387}, x_{445}]$. The turning point indices are presented in Table IV, the fit is presented in Fig. 6 and the sum of moduli of residuals is 5047. We see in Fig. 6 that few noticeable peaks within the range are ignored. The choices $k = 8$ or $k = 10$ may be considered satisfactory, in that the associated fit may be an adequate approximation in revealing turning points and in-between trends that seem to have real significance. Still, there is room for improvement. Therefore we proceeded to the calculation of a best L_1 fit with $k = 12$, which gave two extra turning points at x_{280} and x_{296} that are inside the interval of adjacent turning points $[x_{223}, x_{319}]$ of the fit of Fig. 6. The $k = 12$ fit is presented in Fig. 7, while the sum of moduli of residuals is 4566.

TABLE III
THE TURNING POINT INDICES OF THE BEST L_1 AND THE BEST L_2 FIT WHEN $k = 8$

No	L_1	L_2
0	1	1
1	223	223
2	319	322
3	356	356
4	361	361
5	369	369
6	387	389
7	445	445
8	640	640

In the attempt to provide structure in data when there is no underlying mathematical function, the option of the automatic calculation of turning points seems to be quite important. Furthermore, it is not inefficient to use the trend test algorithm of [25] for an initial estimation of the turning points and then apply L1PMA with specific values of k. Of course, the development of a test similar to [25] for the L_1 case would be valuable.

TABLE IV
THE TURNING POINT INDICES OF THE BEST L_1 AND THE BEST L_2 FIT WHEN $k = 10$

No	L_1	L_2
0	1	1
1	223	223
2	319	322
3	356	356
4	361	361
5	369	369
6	387	389
7	407	407
7	427	427
7	445	445
8	640	640

In order not to be misled by the results in usual practices with piecewise monotonic approximation, we mention some ideas that failed to provide optimality. We saw in our examples that all turning points of a best L_1 approximation with $k-2$ sections were turning points of a best L_1 approximation with k sections. Also, the extra 2 turning points of the best L_1 fit with k sections were found in a range of constant components of the best L_1 fit with $k-2$ sections. Hence a best L_1 approximation with $k = 3$ might be obtained by improving the best L_1 monotonic approximation after reducing the search for the turning points to ranges of constant components. Nonetheless, the conjecture has been proved not to be true [6]. Moreover, the extra 2 turning points in our example were located between adjacent turning points of the best approximation with $k-2$ sections. However, the turning points of the best approximation with $k-2$ sections need not be turning points of a best approximation with k sections, as it is shown by [7].

In the absence of any structure, this method requires at least $O(m^3)$ operations when $k \geq 3$, because it is necessary to take account of all possible values of (8), for $\ell = t_{j-1}, \ldots, t_j$, in order to find the integer variables $\{t_j : j = 1, \ldots, k-1\}$. Still, the remarks of the previous paragraph suggest that for at least as many data as in Fig. 3, local improvements of a best fit with a prescribed k may produce an adequate fit without undue computational cost.

Piecewise monotonic approximation reveals the most important turning points (peaks), while interpolating the data at these points. By increasing k, piecewise monotonic approximation has the freedom to make the sum of moduli of residuals smaller, while in practice it maintains the most important turning points. This feature of piecewise monotonic approximation is not shared with wavelet or spline approximation, where it is difficult to represent the data at a peak, because the presence of a peak introduces substantial perturbations into the tail of the approximation. Hence piecewise monotonic approximation provides a considerable advantage over low-pass filtering, which usually results in ringing and blurring artifacts (see, [15], [16], [26]). Weaver [26], with respect to the use of least squares monotonicity algorithms in fMRI, summarizes the primary advantage of the monotonic increasing approximation as follows. It smooths the data as little as possible without blurring the edges; it leaves increases unchanged; both sharp and smooth increases remain unchanged, so no smoothing occurs at all; it avoids Gibb's ringing. Best L_1 approximation shows similar behavior. In addition, since the arithmetic operations involved in the calculation of a L_1 piecewise monotonic approximation are comparisons mainly spent in finding the medians of subranges of data during the monotonic calculations, the L_1 approximation process induces no round-off error in the modified data. The exact arithmetic in the L_1 calculation is an unprecedented advantage to fitting integer data values, as in the case of the data of Fig. 3.

IV. Concluding Remarks

Piecewise monotonic approximation method is relevant to a wide range of applications. In this paper we have presented an application that shows the effectiveness of the best L_1 piecewise monotonic approximation to signal restoration. Despite the large number of local minima that can occur in this optimization calculation, the software package we have developed gives a global solution in cubic complexity with respect to the number of data, but in practice the complexity is about quadratic. This software is suitable for calculations that involve several thousand data points and it would be most useful for real time processing applications.

A similar application of the analogous L_2 problem shows that the L_1 and L_2 algorithms produce closely similar results. Both these algorithms are efficient for the problem they solve, although the L_2 algorithm, in general, is faster by an order of magnitude. However, L_1 norm in data smoothing provides certain advantages and, occasionally, the L_1 piecewise monotonic approximation method can provide a suitable alternative to the use of the L_2 piecewise monotonic approximation method.

In order to apply the piecewise monotonic approximation method effectively, it would be very helpful to try to solve particular signal processing problems, so as to receive guidance from numerical results and from processing practices. Moreover, it is an important practical question to decide how large to make k. Signals in practice are piecewise monotonic, but usually the number of monotonic sections they contain is not known in advance. Prior knowledge about the geometry of the signal may provide good estimates of k. In certain applications in MR spectroscopy [16] we often have good estimates of k that can be utilized by our piecewise monotonic approximation methods. Furthermore, there are some features of our examples that can guide the development of new piecewise monotonic approximation algorithms for signal restoration calculations.

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