$\mathcal{N}$-structures Applied to Finite State Machines

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***Abstract*—In this paper the notion of a $\mathcal{N}$-finite state machine is introduced, as well as the concepts of a $\mathcal{N}$-(immediate) successor, a $\mathcal{N}$-exchange property and a $\mathcal{N}$-subsystem are introduced. Some related properties are discussed. A condition for a $\mathcal{N}$-finite state machine to satisfy the $\mathcal{N}$-exchange property is established. A characterization of a $\mathcal{N}$-subsystem is initiated. We prove that the isomorphism between $\mathcal{N}$-finite state machines is an equivalence relation (resp. partial order relation). We then introduce the concept of $\mathcal{N}$-complete finite state machine and discuss the isomorphism between them.**

***Index Terms*—$\mathcal{N}$-finite state machine, $\mathcal{N}$-immediate, $\mathcal{N}$-sucessor, homomorphism of $\mathcal{N}$-finite state machine, $\mathcal{N}$-complete finite state machine.**

I. Introduction

IN 1965, Zadeh [15] introduced the notion of fuzzy subset of a set. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, artificial intelligence, pattern recognition, robotics, computer networks, decision making and automata theory. The mathematical formulation of a fuzzy automaton was first proposed by Wee [14] in 1967. Santos [13] proposed fuzzy auomata as a model of pattern recognition. Malik et al. [6] introduced the notions of submachine of a fuzzy finite state machine, retrievable, separated and connected fuzzy finite state machines and discussed their basic properties. They also initiated a decomposition theorem for fuzzy finite state machiness in terms of primary submachines. On the other hand, Kumbhojkar and Chaudhari [4] provided several ways of constructing products of fuzzy finite state machines and their mutual relationships, through isomorphism and coverings. Li and Pedrycz [5] indicated that fuzzy finite state automata can be viewed as a mathematical model of computation in fuzzy systems.

Recently, a higher order set with imprecision has been extended to automata. Based on Atanassov's intuitionistic fuzzy sets [1], Jun proposed intuitionistic fuzzy finite state machines in [8] and also intuitionistic fuzzy finite switchboard state machines in [9]. Zhang and Li [18] presented the properties of intuitionistic fuzzy recognizers and intuitionistic fuzzy finite automata. Zhang [16], [17] initiated the concept the concept of bipolar fuzzy sets as a generalization of fuzzy sets which is an extension of fuzzy sets whose membership degree range is $[-1, 1]$. Thus, using the notion of bipolar fuzzy valued sets, the present author [12] introduced the concepts of bipolar fuzzy finite state machines, bipolar successors, bipolar subsystems and studied related properties. In 2012, Kavikumar et al, [2] introduced the notions of bipolar fuzzy finite switchboard state machines and investigated their related properties.

A crisp set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy set relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested for fuzzy automata theory, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [10] introduced and used a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras, and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Jun et al. [11] introduced the notion of a (created) $\mathcal{N}$-ideal of substraction algebras, and investigated several characterizations of $\mathcal{N}$-ideals.

In this paper, the $\mathcal{N}$-structure applied to finite state machines and we introduce the concepts of $\mathcal{N}$-finite state machines as a generalization of fuzzy finite state machines, $\mathcal{N}$-(immediate) successors, $\mathcal{N}$-subsystems, $\mathcal{N}$-homomorphism, $\mathcal{N}$-weak homomorphism and study related properties. We establish a condition for a $\mathcal{N}$-finite state machine to satisfy the $\mathcal{N}$-exchange property. We initiate a characterization of a $\mathcal{N}$-subsystem. We prove that the isomorphism between $\mathcal{N}$-finite state machines is an equivalence relation and also in partial order relation.

II. Preliminaries

Definition 2.1: (Malik et al [6]). A fuzzy finite state machine is a triplet $\mathcal{M} = (Q, X, \mu)$, where Q is a finite nonempty set of states, X is a finite nonempty set of inputs and μ is a fuzzy subset of $Q \times X \times Q$; i.e. $\mu : Q \times X \times Q \to [0, 1]$.

Definition 2.2: (Mordeson et al [7]). Let $\mathcal{M} = (Q, X, \mu)$ be an fuzzy finite state machine. Define $\mu^* : Q \times X \times Q \to [0, 1]$ by

$$\mu^*(q, \lambda, p) = \begin{cases} 1 & \text{if } q = p, \\ 0 & \text{if } q \neq p. \end{cases}$$

and

$$\mu^*(q, xa, p) = \vee\{\mu^*(q, x, r) \wedge \mu(r, a, p) \mid r \in Q\}$$

for all $q, p \in Q$, $x \in X^*$, $a \in X$.

Definition 2.3: (Kim et al. [3]). Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ be two fuzzy finite state machines.

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Let $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$ be two mappings. Then the pair (α,β) is called a fuzzy finite state machine homomorphism, symbolically $(\alpha,\beta) : \mathcal{M}_1 \to \mathcal{M}_2$, if $\mu_1(p, a, q) \leq \mu_2(\alpha(p), \beta(a), \alpha(q))$, $\forall p, q \in Q$ and $a \in X_1$.

The homomorphism $(\alpha,\beta) : \mathcal{M}_1 \to \mathcal{M}_2$ is called *monomorphism (epimorphism, isomorphism)*, if both the mappings α and β are injective (surjective, bijective, respectively).

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that, an element of $\mathcal{F}(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure we mean an ordered pair (X,ξ), where ξ is an $\mathcal{N}$-function on X.

III. $\mathcal{N}$-FINITE STATE MACHINES

Definition 3.1: An $\mathcal{N}$-finite state machine is a triple $\mathcal{M} = (Q, X,\xi)$, where Q and X are finite nonemptysets, called the set of states and the set of input symbols, respectively, and ξ is an $\mathcal{N}$-structure on $Q \times X \times Q$.

Let X^* denote the set of all words of elements of X of finite length. Let λ denote the empty word in X^* and $| x |$ denote the length of x for every $x \in X^*$.

Definition 3.2: Let $\mathcal{M} = (Q, X,\xi)$ be an $\mathcal{N}$-finite state machine. Define $\xi^* : Q \times X^* \times Q \to [-1, 0]$ by

$$\xi^*(q, \lambda, p) = \begin{cases} -1 & \text{if } q = p, \\ 0 & \text{if } q \neq p. \end{cases}$$

and

$$\xi^*(q, xa, p) = \inf_{r \in Q} [\xi^*(q, x, r) \vee \xi(r, a, p)]$$

for all $q, p \in Q$, $x \in X^*$, $a \in X$.

Lemma 3.3: Let $\mathcal{M} = (Q, X,\xi)$ be an $\mathcal{N}$-finite state machine. Then

$$\xi^*(q, xy, p) = \inf_{r \in Q} [\xi^*(q, x, r) \vee \xi^*(r, y, p)]$$

for all $p, q \in Q$ and $x, y \in X^*$.

Proof: Let $p, q \in Q$ and $x, y \in X^*$. We prove the result by induction on $| y |= n$. If $n = 0$, then $y = \lambda$ and so $xy = x\lambda = x$. Hence

$$\begin{aligned} &\inf_{r\in Q}[\xi^*(q, x, r) \vee \xi^*(r, y, p)] \\ &= \inf_{r\in Q}[\xi^*(q, x, r) \vee \xi^*(r, \lambda, p)] \\ &= \xi^*(q, x, p) = \xi^*(q, xy, p) \end{aligned}$$

Thus the result hols for $n = 0$. Suppose that the result is true for all $u \in X^*$ such that $| u |= n - 1$, $n > 0$. Let $y = ua$ where $u \in X^*$ and $a \in X$, and $| u |= n - 1$. Then

$$\begin{aligned} &\xi^*(q, xy, p) = \xi^*(q, xua, p) \\ &= \inf_{r\in Q}[\xi^*(q, xu, r \vee \xi^*(r, a, p)] \\ &= \inf_{r\in Q}[\inf_{s\in Q}[\xi^*(q, x, s) \vee \xi^*(s, u, r)] \vee \xi(r, a, p)] \\ &= \inf_{r,s\in Q}[\xi^*(q, x, s) \vee \xi^*(s, u, r) \vee \xi(r, a, p)] \\ &= \inf_{s\in Q}[\xi^*(q, x, s) \vee (\inf_{r\in Q}[\xi^*(s, u, r) \vee \xi(r, a, p)])] \\ &= \inf_{s\in Q}[\xi^*(q, x, s) \vee \xi^*(s, ua, p)] \\ &= \inf_{s\in Q}[\xi^*(q, x, s) \vee \xi^*(s, y, p)] \end{aligned}$$

Hence the result is valid for $| y |= n$. This completes the proof. ■

Definition 3.4: Let $\mathcal{M} = (Q, X,\xi)$ be an $\mathcal{N}$-finite state machine and let $p, q \in Q$. The p is called a $\mathcal{N}$-immediate successor of q if the following condition holds:

$$(\exists a \in X)(\xi(q, a, p) < 0).$$

We say that p is a $\mathcal{N}$-successor of q if the following condition holds:

$$(\exists a \in X^*)(\xi^*(q, x, p) < 0).$$

We denote by $\mathcal{S}(q)$ the set of all $\mathcal{N}$-successors of q. For any subset T of Q, the set of all $\mathcal{N}$-successors of T, denoted by $\mathcal{S}(T)$, is denoted to be the set

$$\mathcal{S}(T) = \cup\{\mathcal{S}(q) \mid q \in T\}.$$

Proposition 3.5: For any $\mathcal{N}$-finite state machine $\mathcal{M} = (Q, X,\xi)$, we have the following properties:

(1) $(\forall q \in Q)(q \in \mathcal{S}(q))$.
(2) $(\forall p, q, r \in Q)(p \in \mathcal{S}(q), r \in \mathcal{S}(p) \Rightarrow r \in \mathcal{S}(q))$.

Proof:

(1) Since $\xi^*(q, \lambda, q) = -1 < 0$, we have $q \in \mathcal{S}(q)$.
(2) Let $p \in \mathcal{S}(q)$ and $r \in \mathcal{S}(p)$. Then there exist $x, y \in X^*$ such that $\xi^($q, x, p$) < 0$, and $\xi^*(p, y, r) < 0$. Using Lemma 3.3, we have

$$\begin{aligned} \xi^*(q, xy, r) &= \inf_{s\in Q}[\xi^*(q, x, s) \vee \xi^*(s, y, r)] \\ &\leq \xi^*(q, x, p) \vee \xi^*(p, y, r) < 0 \end{aligned}$$

Hence $r \in \mathcal{S}(q)$. ■

Proposition 3.6: Let $\mathcal{M} = (Q, X,\xi)$ be a $\mathcal{N}$-finite state machine. For any subsets A and B of Q, the following assertions hold.

(1) $A \subseteq B \Rightarrow \mathcal{S}(A) \subseteq \mathcal{S}(B)$.
(2) $A \subseteq \mathcal{S}(A)$.
(3) $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$.
(4) $\mathcal{S}(A \cup B) = \mathcal{S}(A) \cup \mathcal{S}(B)$.
(5) $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A) \cap \mathcal{S}(B)$.

Proof: The proofs of (1), (2), (4) and (5) are straightforward. For (3), obviously $\mathcal{S}(A) \subseteq \mathcal{S}(\mathcal{S}(A))$. If $q \in \mathcal{S}(\mathcal{S}(A))$, then $q \in \mathcal{S}(p)$ for some $p \in \mathcal{S}(A)$. From $p \in \mathcal{S}(A)$, there exists $r \in A$ such that $p \in \mathcal{S}(r)$. It follows from Proposition 3.5 (2) that $q \in \mathcal{S}(r) \subseteq \mathcal{S}(A)$ so that $\mathcal{S}(\mathcal{S}(A)) \subseteq \mathcal{S}(A)$. Thus (3) is valid. ■

Definition 3.7: $\mathcal{M} = (Q, X,\xi)$ be a $\mathcal{N}$-finite state machine. We say that $\mathcal{M}$ satisfies the $\mathcal{N}$-exchange property if the following condition holds:

$$(\forall p, q \in Q)(\forall T \subseteq Q)(p \in \mathcal{S}(T\cup\{q\})\backslash\mathcal{S}(T) \Rightarrow q \in \mathcal{S}(T\cup\{p\})).$$

Theorem 3.8: Let $\mathcal{M} = (Q, X,\xi)$ be a $\mathcal{N}$-finite state machine. Then the following assertions are equivalent.

(1) $\mathcal{M}$ satisfies the $\mathcal{N}$-exchange property.
(2) $(\forall p, q \in Q)$ $(p \in \mathcal{S}(q) \Longleftrightarrow q \in \mathcal{S}(p))$.

Proof: Assume that $\mathcal{M}$ satisfies the $\mathcal{N}$-exchange property. Let $p, q \in Q$ be such that $p \in \mathcal{S}(q) = \mathcal{S}(\emptyset \cup \{q\})$. Note that $p \notin \mathcal{S}(\emptyset)$ and so $q \in \mathcal{S}(\emptyset \cup \{p\}) = \mathcal{S}(p)$. Similarly if $q \in \mathcal{S}(p)$ then $p \in \mathcal{S}(q)$. Conversely suppose that (2) is valid. Let $p, q \in Q$ and $T \subseteq Q$. If $p \in \mathcal{S}(T \cup \{q\}) \setminus \mathcal{S}(T)$, then $p \in \mathcal{S}(q)$. It follows from (2) that $q \in \mathcal{S}(p) \subseteq \mathcal{S}(T \cup \{p\})$. Hence $\mathcal{M}$ satisfies the $\mathcal{N}$-exchange property. ■

Definition 3.9: Let $\mathcal{M} = (Q, X,\xi)$ be a $\mathcal{N}$-finite state machine. Let ξ_Q be a $\mathcal{N}$-fuzzy set in Q. Then (Q,ξ_Q, X, ξ) is called a $\mathcal{N}$-subsystem of $\mathcal{M}$ if for all $p, q \in Q$ and $a \in X$,

$$\xi_Q(q) \leq \xi_Q(p) \vee \xi(p, a, q).$$

Example 3.10: Let $Q = \{p, q\}$, $X = \{a\}$, $\xi(r, a, t) = -\frac{1}{2}$ for $r, t \in Q$. Let ξ_Q be given by $\xi_Q(q) = -\frac{1}{3}$ and $\xi_Q(p) = -\frac{6}{7}$. Then

$$\xi_Q(q) \vee \xi(q,a,p) = (-\frac{1}{3}) \vee (-\frac{1}{2}) = -\frac{1}{3} > -\frac{6}{7} = \xi_Q(p).$$

Hence (Q, ξ_Q, X, ξ) is a $\mathcal{N}$-subsystem.

Theorem 3.11: Let $\mathcal{M} = (Q, X, \xi)$ be a $\mathcal{N}$-finite state machine and let ξ_Q be a $\mathcal{N}$-fuzzy set in Q. Then (Q, ξ_Q, X, ξ) is a $\mathcal{N}$-subsystem of $\mathcal{M}$ if and only if

$$\xi_Q(q) \leq \xi_Q(p) \vee \xi^*(p,x,q)$$

for all $p, q \in Q$ and $x \in X^*$.

Proof: Suppose that (Q, ξ_Q, X, ξ) is a $\mathcal{N}$-subsystem of $\mathcal{M}$. Let $p, q \in Q$ and $x \in X^*$. The proof is induction on $\mid x \mid = n$. If $n = 0$, then $x = \lambda$. Now if $p = q$, then

$$\xi_Q(q) \vee \xi^*(q, \lambda, q) = \xi_Q(q).$$

If $p \neq p$, then

$$\xi_Q(q) \vee \xi^*(p, \lambda, q) = 0 \geq \xi_Q(q).$$

Thus the result is true for $n = 0$. Suppose the result is valid for all $y \in X^*$ with $\mid y \mid = n-1$, $n > 0$. For the y above, let $x = ya$ where $a \in X$. Then

$$\begin{aligned} &\xi_Q(p) \vee \xi^*(p,x,q) \\ &= \xi_Q(p) \vee \xi^*(p,ya,q) \\ &= \xi_Q(p) \vee (\inf_{r \in Q}[\xi^*(p,y,r) \vee \xi(r,a,q)]) \\ &= \inf_{r \in Q}[\xi_Q(p) \vee \xi^*(p,y,r) \vee \xi(r,a,q)] \\ &\geq \inf_{r \in Q}[\xi_Q(r) \vee \xi(r,a,q)] \geq \xi_Q(q) \end{aligned}$$

The converse is trivial, completing the proof. ■

Definition 3.12: Let $\mathcal{M} = (Q, X, \xi)$ be an $\mathcal{N}$-finite state machine. An $\mathcal{N}$-finite state machine $\mathcal{M}^s = (Q^s, X^s, \xi^s)$ is called sub $\mathcal{N}$-finite state machine of $\mathcal{M}$, if

(a) $Q^s \subseteq Q$, $X^s \subseteq X$ and
(b) $\xi \mid_{Q^s \times X^s \times Q^s = \xi^s}$.

Theorem 3.13: An $\mathcal{N}$-finite state machine relation " $\subseteq$ " is reflexive, anti-symmetric and transitive.

Proof: The proof is straight forward. ■

IV. Homomorphism of $\mathcal{N}$-finite state machines

Definition 4.1: Let $\mathcal{M}_1 = (Q_1, X_1, \xi_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \xi_2)$ be an $\mathcal{N}$-finite state machines. A pair (α, β) of mappings, $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$ is called a homomorphism, written $(\alpha, \beta) : \mathcal{M}_1 \to \mathcal{M}_2$, if

$$\xi_1(q,x,p) \geq \xi_2(\alpha(q), \beta(x), \alpha(p))$$

$\forall q, p \in Q_1$ and $\forall x \in X_1$. The pair (α, β) is called a weak homomorphism if

$$\xi_2(\alpha(q), \beta(x), \alpha(p)) = \inf_{t \in Q_1} \{\xi_1(q,x,t) \mid \alpha(t) = \alpha(p)\}$$

$\forall q, p \in Q_1$ and $\forall x \in X_1$.

A bijective homomorphism (weak homomorphism) with the property

$$\xi_2(\alpha(q), \beta(x), \alpha(p)) = \xi_1(q,x,p)$$

is called an isomorphism (weak isomorphism).

Example 4.2: Let $\mathcal{M}_1 = (Q_1, X_1, \xi_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \xi_2)$ be an $\mathcal{N}$-finite state machines, where $Q_1 = \{q_1, q_2, q_3\}$, $X_1 = \{a, b\}$, and $Q_2 = \{p_1, p_2\}$, $X_2 = \{a, b\}$ and ξ_1, ξ_2 are defined as follows:

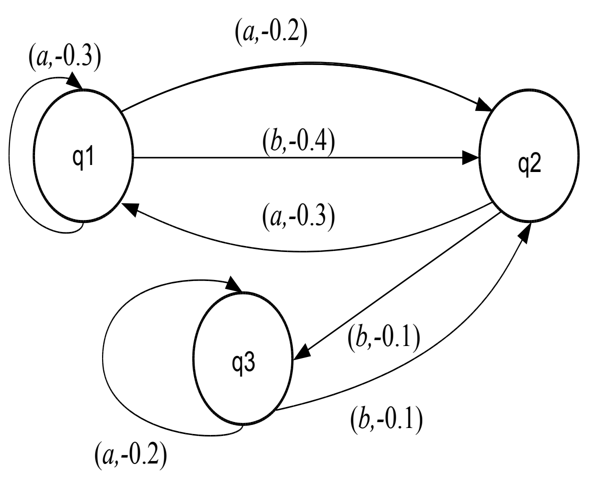

Fig. 1. $\mathcal{N}$-finite state machine $\mathcal{M}_1$

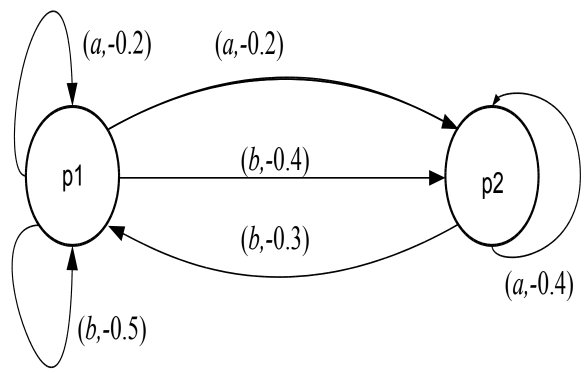

Fig. 2. $\mathcal{N}$-finite state machine $\mathcal{M}_2$ (Homomorphic image of $\mathcal{M}_1$)

Define $\alpha = Q_1 \to Q_2$ and $\beta : X_1 \to X_2$ as follows $\alpha(q_1) = \alpha(q_2) = p_1$, $\alpha(q_3) = p_2$, $\beta(a) = a$ and $\beta(b) = b$.

Proposition 4.3: An isomorphism between $\mathcal{N}$-finite state machines is an equivalence relation.

Proof: The relfexivity and symmetry are obvious. To prove the transitivity, we let $(\alpha_1, \beta_1) : \mathcal{M}_1 \to \mathcal{M}_2$ and $(\alpha_2, \beta_2) : \mathcal{M}_2 \to \mathcal{M}_3$, where $\alpha_1 : Q_1 \to Q_2$, $\alpha_2 : Q_2 \to Q_3$, $\beta_1 : X_1 \to X_2$ and $\beta_2 : X_2 \to X_3$ be the isomorphisms of $\mathcal{M}_1$ onto $\mathcal{M}_2$ and $\mathcal{M}_2$ onto $\mathcal{M}_3$, respectively. Then $(\alpha_2, \beta_2) \circ (\alpha_1, \beta_1) : \mathcal{M}_1 \to \mathcal{M}_3$ is a bijective map from $\mathcal{M}_1$ to $\mathcal{M}_3$, where $\Big((\alpha_2, \beta_2) \circ (\alpha_1, \beta_1)\Big)(q_1, x_1, p_1) = (\alpha_2, \beta_2)\Big((\alpha_1, \beta_1)(q_1, x_1, p_1)\Big)$ $\forall p_1, q_1 \in Q_1$ and $\forall x_1 \in X_1$. Since a map $(\alpha_1, \beta_1) : \mathcal{M}_1 \to \mathcal{M}_2$ defined by $\alpha_1(q_1) = q_2$, $\beta_1(x_1) = x_2$, and $\alpha_1(p_1) = p_2$ for $p_1, q_1 \in Q_1$ and $\forall x_1 \in X_1$ is an isomorphism, so we have

$$\begin{aligned} \xi_1(q_1, x_1, p_1) &= \xi_2(\alpha_1(q_1), \beta_1(x_1), \alpha_1(p_1)) \\ &= \xi_2(q_2, x_2, p_2) \end{aligned} \quad (1)$$

$\forall p_1, q_1 \in Q_1$ and $\forall x_1 \in X_1$.

Since a map $(\alpha_2, \beta_2) : \mathcal{M}_2 \to \mathcal{M}_3$ defined by $\alpha_2(q_2) = q_3$, $\beta_2(x_2) = x_3$, and $\alpha_2(p_2) = p_3$ for $p_2, q_2 \in Q_2$ and $\forall x_2 \in X_2$ is an isomorphism, so we have

$$\begin{aligned} \xi_2(q_2, x_2, p_2) &= \xi_3(\alpha_2(q_2), \beta_2(x_2), \alpha_2(p_2)) \\ &= \xi_3(q_3, x_3, p_3) \end{aligned} \quad (2)$$

$\forall p_2, q_2 \in Q_2$ and $\forall x_2 \in X_2$.

From (1), (2) and $\alpha_1(q_1) = q_2$, $\beta_1(x_1) = x_2$, and $\alpha_1(p_1) = p_2$ for $p_1, q_1 \in Q_1$ and $\forall x_1 \in X_1$, we have

$$\begin{aligned} \xi_1(q_1, x_1, p_1) &= \xi_2\big(\alpha_1(q_1), \beta_1(x_1), \alpha_1(p_1)\big) \\ &= \xi_2(q_2, x_2, p_2) \\ &= \xi_3\Big((\alpha_2, \beta_2)(q_2, x_2, p_2)\Big) \\ &= \xi_3\Big((\alpha_2, \beta_2)\Big((\alpha_1, \beta_1)(q_1, x_1, p_1)\Big)\Big) \end{aligned}$$

$\forall p_1, q_1 \in Q_1$ and $\forall x_1 \in X_1$.

Therefore, $(\alpha_2, \beta_2) \circ (\alpha_1, \beta_1)$ is an isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_3$. This completes the proof. ∎

Proposition 4.4: A weak isomorphism between $\mathcal{N}$-finite state machines is a partial ordering relation.

Proof: The reflexivity and transitivity are obvious. To prove the anti symmetry, we let $(\alpha_1, \beta_1) : \mathcal{M}_1 \to \mathcal{M}_2$ be a weak isomorphism of $\mathcal{M}_1$ onto $\mathcal{M}_2$. Then (α_1, β_1) is a bijective map defined by $\alpha_1(q_1) = q_2$, $\beta_1(x_1) = x_2$, and $\alpha_1(p_1) = p_2$ for $p_1, q_1 \in Q_1$ and $\forall x_1 \in X_1$ satisfying

$$\begin{aligned} \xi_1(q_1, x_1, p_1) &= \xi_2(\alpha_1(q_1), \beta_1(x_1), \alpha_1(p_1)) \\ &\geq \xi_2(\alpha_1(q_1), \beta_1(x_1), \alpha_1(p_1)) \end{aligned} \quad (3)$$

$\forall p_1, q_1 \in Q_1$ and $\forall x_1 \in X_1$.

Let $(\alpha_2, \beta_2) : \mathcal{M}_2 \to \mathcal{M}_3$ be a weak isomorphism of $\mathcal{M}_2$ onto $\mathcal{M}_3$. Then (α_2, β_2) is a bijective map defined by $\alpha_2(q_2) = q_3$, $\beta_2(x_2) = x_3$, and $\alpha_2(p_2) = p_3$ for $p_2, q_2 \in Q_2$ and $\forall x_2 \in X_2$ satisfying

$$\begin{aligned} \xi_2(q_2, x_2, p_2) &= \xi_3(\alpha_2(q_2), \beta_2(x_2), \alpha_2(p_2)) \\ &\geq \xi_3(\alpha_2(q_2), \beta_2(x_2), \alpha_2(p_2)) \end{aligned} \quad (4)$$

$\forall p_2, q_2 \in Q_2$ and $\forall x_2 \in X_2$.

The inequalities (3) and (4) hold on the finite sets $\mathcal{M}_1$ and $\mathcal{M}_2$ only when Q_1 and Q_2 have same number of set of states and X_1 and X_2 have same number of set of input symbols. Hence $\mathcal{M}_1$ and $\mathcal{M}_2$ are identical. Therefore $(\alpha_2, \beta_2) \circ (\alpha_1, \beta_1)$ is a weak isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_3$. This completes the proof. ∎

Lemma 4.5: Let $\mathcal{M}_1 = (Q_1, X_1, \xi_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \xi_2)$ be an $\mathcal{N}$-finite state machines. Let $(\alpha,\beta) : \mathcal{M}_1 \to \mathcal{M}_2$ be a weak homomorphism. Then $\forall q, r \in Q_1$ and $\forall x \in X_1$, if $\xi_2\big(\alpha(q), \beta(x), \alpha(r)\big) < 0$, then $\exists t \in Q_1$ such that $\xi_1(q, x, t) < 0$ and $\alpha(t) = \alpha(r)$. Furthermore, $\forall p \in Q$ if $\alpha(p) = \alpha(q)$, then $\xi_1(q, x, t) \leq \xi_(p, x, r)$.

Proof: Let $p, q, r \in Q_1$ and $x \in X_1$.

Let $\xi_2\big(\alpha(q), \beta(x), \alpha(r)\big) < 0$. Then

$$\inf_{s \in Q_1} \big\{\xi_1(q, x, s) \mid \alpha(s) = \alpha(r)\big\} < 0.$$

Since Q_1 is finite, $\exists t \in Q_1$ such that $\alpha(t) = \alpha(r)$ and

$$\xi_1(q, x, t) = \inf_{s \in Q_1} \big\{\xi_1(q, x, s) \mid \alpha(s) = \alpha(r)\big\} < 0.$$

Suppose $\alpha(p) = \alpha(q)$. Then

$$\begin{aligned} \xi_1(q, x, t) &= \xi_2\big(\alpha(q), \beta(x), \alpha(r)\big) \\ &= \xi_2\big(\alpha(p), \beta(x), \alpha(r)\big) \\ &\leq \xi_1(p, x, r). \end{aligned}$$

∎

Definition 4.6: Let $\mathcal{M} = (Q, X,\xi)$ be an $\mathcal{N}$-finite state machine. $\overline{\mathcal{M}}$ is called $\mathcal{N}$-complete if for all $q \in Q$ and $a \in X$, there exists $p \in Q$ such that $\xi(q, a, p) < 0$.

Definition 4.7: Let $\mathcal{M} = (Q, X,\xi)$ be an $\mathcal{N}$-finite state machine. An $\mathcal{N}$- structure finite state machine $\mathcal{M}^c = (Q^c, X^c, \xi^c)$ is called an $\mathcal{N}$-completion of $\mathcal{M}$, if the following conditions hold:

(a) $\mathcal{M}^c$ is a $\mathcal{N}$-complete finite state machine, and
(b) $\mathcal{M} \subseteq \mathcal{M}^c$.

Let $\mathcal{M} = (Q, X,\xi)$ be an $\mathcal{N}$-finite state machine that is $\mathcal{N}$-incomplete. Consider $\mathcal{M}' = (Q', X', \xi')$, where $Q' = Q \cup \{z\}$, $z \notin Q$ and

$$\xi'(q, u, p) = \begin{cases} \xi(q, u, p) & \text{if } q, p \in Q \text{ and } \xi(q, u, p) \neq 0, \\ -1 & \text{if either } \xi(q, u, r) = 0 \ \forall r \in Q \\ & \text{and } p = z \text{ or } p = q = z, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{M}' = (Q', X', \xi')$ is called the smallest $\mathcal{N}$-completion of $\mathcal{M}$

Proposition 4.8: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two $\mathcal{N}$-finite state machines. Then $\mathcal{M}_1 \cong \mathcal{M}_2$ if and only if $\mathcal{M}_1^c \cong \mathcal{M}_2^c$.

Proof: Assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic, there exists a bijective map $(\alpha,\beta) : \mathcal{M}_1 \to \mathcal{M}_2$ satisfying

$$\xi_1(q, x, p) \geq \xi_2\big(\alpha(q), \beta(x), \alpha(p)\big)$$

$\forall q, p \in Q$ and $x \in X_1$. By definition of $\mathcal{N}$-completion, we have $\mathcal{M}_1 \subseteq \mathcal{M}_1^c$ and $\mathcal{M}_2 \subseteq \mathcal{M}_2^c$. Hence $\mathcal{M}_1^c \cong \mathcal{M}_2^c$. The proof of converse part is straightforward. This completes the proof. ∎

The following Propositions are obvious.

Proposition 4.9: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two $\mathcal{N}$-finite state machines. If there is a weak isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$, then there is a weak isomorphism between $\mathcal{M}_1^c$ and $\mathcal{M}_2^c$.

Proposition 4.10: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two $\mathcal{N}$-finite state machines. If there is a weak isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$, then there is a homomorphism between $\mathcal{M}_1^c$ and $\mathcal{M}_2^c$.

V. Conclusion

Fuzzy automata theory have supported a wealth of important applications in many fields. In this paper, the concepts of $\mathcal{N}$-finite state machine have been generalized by substituting the interval [-1,0] and studied some of its related properties. Based on the results, more studies are connected with rough finite state machines and bipolar fuzzy finite state machines.

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