Stationary Analysis of an M/M/1 Driven Fluid Queue Subject to Catastrophes and Subsequent Repair

K.V. Vijayashree and A. Anjuka

Abstract—**Fluid models are appropriate in the field of telecommunication for modelling the network traffic where individual units of arrival have less impact on the performance of the network. Such models characterize the traffic as a continuous stream with a parameterized flow rate. For practical design and performance evaluation, it is essential to obtain information about the buffer occupancy distribution. In this paper, we analyze a fluid queue modulated by a single server queueing model subject to catastrophes under steady state conditions. Explicit analytical expression for the joint distribution of the state of the background queueing model and the content of the buffer is presented. A closed form expression for the buffer occupancy distribution is obtained using continued fraction methodology in the transformed domain.**

Keywords: Buffer Content Distribution, Continued Fractions, Laplace Transform, Modified Bessel Function of First Kind

Mathematics Subject Classification: 60K25, 90B22

I. Introduction

The development of telecommunication networks is an exciting and challenging area which requires models and methods for performance studies. One of the major issues of concern is in the traffic regulation mechanism of telecommunication networks. In recent years, fluid queues have been widely accepted as appropriate models for modern telecommunication [1] and manufacturing systems [4]. This modelling approach ignores the discrete nature of the real information flow and treats it as a continuous stream. In particular, in the ATM environment where the fixed cell size is small and the interarrival time between cells at the time of generation is constant for several contiguous cells, this modelling approach has proved to be quite effective. Fluid models play a significant role in ATM networks since the variations on the cell level are almost negligible compared to those on the most important burst level.

The rate at which the information arrives to a switch or multiplexer often fluctuates randomly with a high degree of correlation in time. The information is buffered for service and the service rates may vary randomly. Such behaviour is often described by a single server queueing model in which the rate of information arriving and leaving the switching component is modulated according to a Markov process evolving in the background. Combining this modelling approach with the fluid approximation for information flows lead to *Markov Modulated Fluid Queues*.

Steady state behaviour of Markov driven fluid queues have been extensively studied in the literature. For a fluid queue driven by an $M/M/1$ queueing model, Parthasarathy *etal.* [5] presents an explicit expression for the buffer content distribution in terms of modified Bessel function of first kind using Laplace transforms and continued fractions. In recent years, Silver Soares and Latouche [7] expressed the stationary distribution of a fluid queue with finite buffer as a linear combination of matrix exponential terms using matrix analytic methods. Besides, fluid queues also have successful applications in the field of congestion control [8] and risk processes [6]. Fluid models driven by an $M/M/1/N$ queue with single and multiple exponential vacations were recently studied by Mao *etal.* ([2], [3]) using spectral method.

In this paper, we analyse fluid queues driven by an $M/M/1$ queue subject to catastrophe and subsequent repair. The modulating process is the single server queueing model where customers arrive according to a Poisson process and their service times follow an exponential distribution. With the arrival of a negative customer into the system (*referred to as catastrophe*), the system goes to the state of repair, wherein the repair time also follows exponential distribution. When a negative customer arrives, it induces the positive customers, if any, to immediately leave the system. Further, the arrival of negative customers removes all the unfinished work and leads to server breakdown.

Such queueing models finds the wide range of applications in computer and communication systems. For example, the arrival of virus in the computer networks can be viewed as a negative customer and the performance of all other usual operations through various

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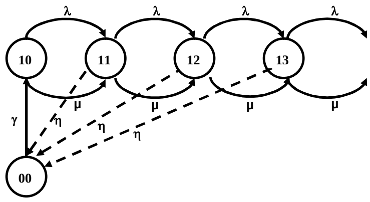

Fig. 1. State Transition Diagram

processors represent positive customers. When the virus affects the system, one or more files may be infected and the system manager may have to go through a number of backups to recover the infected files. The recover time of the infected files can be regarded as the repair time of the server due to an arrival of negative customer. The term queues with negative customers, queues subject to disaster and catastrophe processes are interchangeably used in literature, both in discrete and in continuous time. To the best of our knowledge, the present paper is the first of its kind to analyze fluid models driven by queues subject to disasters, although earlier several authors have studied fluid queues modulated by vacation queueing models.

II. Model Description

Let us denote the background single server queueing model by $\{X(t), t \geq 0\}$ where $X(t)$ denotes the number of customer in the system at time t. Customers arrive according to a Poisson process at an average rate, λ and the service times are exponentially distributed with parameter μ. Further, the catastrophes are assumed to occur according to a Poisson process with rate η. When the system is met with a catastrophe, all the customers including the one in service is lost and the system transits to a repair state. The repair time of failed server follows exponential distribution with parameter γ. Customers who arrive during the failed state are not allowed to join the queue.

Let $J(t)$ denote the state of the server at time t. Assume that $J(t) = 1$ represents the server is in functional state and $J(t) = 0$ represent that the server is in the state of repair. Then, the state space of the two dimensional process $(J(t), X(t))$ is given by $S = \{(0,0)U(1,k), k = 0,1,2,\cdots\}$. The state transition diagram of the modulating process is shown in Fig 1.

Consider a fluid queue modulated by the above queueing model. Let $\{C(t), t \geq 0\}$ represent the buffer content process, where $C(t)$ denotes the content of the fluid buffer at time t. When the server is in functional state (busy or idle), the fluid accumulates in an infinite capacity buffer at a constant rate $r > 0$. The buffer depletes the fluid during the repair periods of the server at a constant rate $r_0 < 0$ as long as the buffer is nonempty. Hence, the dynamics of the buffer content process is given by

$$\frac{dC(t)}{dt} = \begin{cases} 0, & \text{if } C(t) = 0, J(t) = 0 \\ r_0, & \text{if } C(t) > 0, J(t) = 0 \\ r, & \text{if } C(t) \geq 0, J(t) = 1, X(t) = i, i \geq 0. \end{cases}$$

Clearly, the 3-dimensional process $\{(J(t), X(t), C(t)), t \geq 0\}$ constitutes a Markov process and it possesses a unique stationary distribution under a suitable stability condition. To ensure the stability of the process $\{(J(t), X(t), C(t)), t \geq 0\}$, we assume the mean aggregate input rate to be negative, that is,

$$r_0 \pi_{00} + r \sum_{j=0}^{\infty} \pi_{1j} < 0.$$

The terms π_{00} and π_{1j}, for $j = 0,1,2,\cdots$ denote the stationary probability distribution for the states of the background queueing model given by

$$\pi_{00} = \frac{\eta}{\gamma + \eta} \quad \text{and}$$

$$\pi_{1j} = \left(\frac{\eta}{\gamma + \eta}\right)\left(\frac{\gamma}{\lambda z_2^{j+1}(1 - z_1)}\right), \quad j = 0,1,2,\cdots,$$

where

$$z_1, z_2 = \frac{\lambda + \mu + \eta \mp \sqrt{(\lambda + \mu + \eta)^2 - 4\lambda\mu}}{2\lambda}.$$

Letting

$$Q(t,x) = Pr\{J(t) = 0, X(t) = 0, C(t) \leq x\}, t, x \geq 0,$$

and

$$F_k(t,x) = Pr\{J(t) = 1, X(t) = k, C(t) \leq x\}, t, x \geq 0, \quad k = 0,1,2,\cdots,$$

the Kolmogorov forward equations for the Markov process $\{J(t), X(t), C(t)\}$ are given by

$$\frac{\partial Q(t,x)}{\partial t} + r_0 \frac{\partial Q(t,x)}{\partial x} = -\gamma Q(t,x) + \eta \sum_{k=0}^{\infty} F_k(t,x),$$

$$\frac{\partial F_0(t,x)}{\partial t} + r \frac{\partial F_0(t,x)}{\partial x} = \gamma Q(t,x) - (\lambda + \eta) F_0(t,x) + \mu F_1(t,x),$$

and for $k = 1,2,3,\cdots$

$$\frac{\partial F_k(t,x)}{\partial t} + r \frac{\partial F_k(t,x)}{\partial x} = \lambda F_{k-1}(t,x) - (\lambda + \mu + \eta) F_k(t,x) + \mu F_{k+1}(t,x).$$

When the process is in equilibrium, the above system then reduces to

$$r_0 \frac{dQ(x)}{dx} = -\gamma Q(x) + \eta \sum_{k=0}^{\infty} F_k(x), \tag{II.1}$$

$$r \frac{dF_0(x)}{dx} = \gamma Q(x) - (\lambda + \eta) F_0(x) + \mu F_1(x), \tag{II.2}$$

and

$$r\frac{dF_k(x)}{dx} = \lambda F_{k-1}(x) - (\lambda+\mu+\eta)F_k(x) + \mu F_{k+1}(x) \quad k = 1,2,3\cdots, \tag{II.3}$$

subject to the boundary conditions,

$$\begin{aligned} F_k(0) &= 0, k = 0,1,2,\cdots \quad \text{and} \\ Q(0) &= a, \text{for some constant } \ 0 < a < 1. \end{aligned}$$

Ťhe condition $Q(0) = a$ suggest that with some positive probability, say a, the buffer content remains empty when the server in the background queuing model is under repair.

Now, taking Laplace transform of equations (II.1) to (II.3) leads to,

$$r_0[s\hat{Q}(s) - Q(0)] = -\gamma\hat{Q}(s) + \eta\sum_{k=0}^{\infty}\hat{F}_k(s),$$

$$(rs+\lambda+\eta)\hat{F}_0(s) - \mu\hat{F}_1(s) = \gamma\hat{Q}(s) \qquad \text{and}$$

$$(rs+\lambda+\mu+\eta)\hat{F}_k(s) - \mu\hat{F}_{k+1}(s) = \lambda\hat{F}_{k-1}(s), \quad k = 1,2,3,\cdots,$$

which upon simplification yields

$$\hat{Q}(s) = \frac{ar_0}{r_0 s+\gamma} + \frac{\eta\sum_{k=0}^{\infty}\hat{F}_k(s)}{r_0 s+\gamma}, \tag{II.4}$$

$$\hat{F}_0(s) = \frac{\gamma\hat{Q}(s)}{rs+\lambda+\eta-\frac{\mu\hat{F}_1(s)}{\hat{F}_0(s)}} \quad \text{and} \tag{II.5}$$

$$\frac{\hat{F}_k(s)}{\hat{F}_{k-1}(s)} = \frac{\lambda}{rs+\lambda+\mu+\eta-\frac{\mu\hat{F}_{k+1}(s)}{\hat{F}_k(s)}}. \tag{II.6}$$

The last equation yields a continued fraction representation given by

$$\frac{\hat{F}_k(s)}{\hat{F}_{k-1}(s)} = \frac{\lambda}{rs+\lambda+\mu+\eta-} \quad \frac{\lambda\mu}{rs+\lambda+\mu+\eta-} \quad \frac{\lambda\mu}{rs+\lambda+\mu+\eta-}\cdots \tag{II.7}$$

Assume

$$\begin{aligned} f(s) &= \frac{\lambda\mu}{rs+\lambda+\mu+\eta-} \quad \frac{\lambda\mu}{rs+\lambda+\mu+\eta-}\cdots, \\ &= \frac{\lambda\mu}{rs+\lambda+\mu+\eta-f(s)} \\ &= \frac{\frac{\lambda\mu}{r}}{s+\frac{\lambda+\mu+\eta}{r}-\frac{f(s)}{r}}, \end{aligned}$$

which leads to the quadratic equations

$$\frac{f(s)^2}{r} - \left(s+\frac{\lambda+\mu+\eta}{r}\right)f(s) + \frac{\lambda\mu}{r} = 0.$$

Upon solving the above equation, we get

$$f(s) = \frac{p-\sqrt{p^2-\alpha^2}}{\frac{2}{r}} \tag{II.8}$$

where $p = s + \frac{\lambda+\mu+\eta}{r}$ and $\alpha = \frac{2\sqrt{\lambda\mu}}{r}$. Using the continued fraction representation of equation (II.7) in equation (II.5), leads to

$$\hat{F}_0(s) = \frac{\gamma\hat{Q}(s)}{rs+\lambda+\eta-f(s)}. \tag{II.9}$$

Similarly,

$$\frac{\hat{F}_k(s)}{\hat{F}_{k-1}(s)} = \frac{1}{\mu}f(s)$$

and hence

$$\hat{F}_k(s) = \frac{f(s)}{\mu}\hat{F}_{k-1}(s) = \left(\frac{f(s)}{\mu}\right)^k\hat{F}_0(s). \tag{II.10}$$

Now, substituting for $\hat{F}_k(s)$ in $\hat{Q}(s)$ given by equation (II.4) yields

$$\hat{Q}(s) = \frac{ar_0}{r_0 s+\gamma} + \frac{\mu\eta\hat{F}_0(s)}{(r_0 s+\gamma)(\mu-f(s))}.$$

Substituting for $\hat{F}_0(s)$ from equation (II.9) and upon simplification leads to,

$$\hat{Q}(s) = \frac{a}{s} - \frac{a\gamma/r_0}{s(s+\frac{\eta}{r}+\frac{\gamma}{r_0})}, \tag{II.11}$$

which on inversion yields,

$$Q(x) = a - \frac{a\gamma}{r_0}\int_0^x e^{-\left(\frac{\eta}{r}+\frac{\gamma}{r_0}\right)y}dy. \tag{II.12}$$

Substituting for $\hat{Q}(s)$ from equation (II.11) in equation (II.9), after simplification yields

$$\begin{aligned} \hat{F}_0(s) &= \frac{a\gamma}{r\mu}\frac{(\mu-f(s))}{s(s+\frac{\eta}{r}+\frac{\gamma}{r_0})}, \\ &= \frac{a\gamma r_0}{\mu}\frac{(\mu-f(s))}{(\eta r_0+\gamma r)}\left[\frac{1}{s} - \frac{1}{s+\frac{\eta}{r}+\frac{\gamma}{r_0}}\right], \end{aligned}$$

which on inversion yields,

$$\begin{aligned} F_0(x) = \frac{a\gamma r_0}{\mu(\gamma r+\eta r_0)}\Bigg[\mu - \mu e^{-\left(\frac{\eta}{r}+\frac{\gamma}{r_0}\right)x} &- \int_0^x e^{-\left(\frac{\lambda+\mu+\eta}{r}\right)x}\frac{I_1(\alpha x)\alpha}{x}dx \\ &+ \int_0^x e^{-\left(\frac{\eta}{r}+\frac{\gamma}{r_0}\right)u}e^{-\left(\frac{\lambda+\mu+\eta}{r}\right)(x-u)} \frac{I_1(\alpha(x-u))\alpha}{x-u}du\Bigg]. \end{aligned} \tag{II.13}$$

The other steady state probabilities are computed from equation (II.10) as follows

$$\begin{aligned} \hat{F}_k(s) &= \left(\frac{f(s)}{\mu}\right)^k\hat{F}_0(s) \\ &= \left(\frac{p-\sqrt{p^2-\alpha^2}}{\frac{2\mu}{r}}\right)^k\hat{F}_0(s) \\ &= \frac{r^k}{(2\mu)^k}\left[p-\sqrt{p^2-\alpha^2}\right]^k\hat{F}_0(s) \end{aligned}$$

which on the inversion yields,

$$F_k(x) = \left(\frac{r}{2\mu}\right)^k e^{-(\frac{\lambda+\mu+\eta}{r})x} \frac{kI_k(\alpha x)\alpha^k}{x} * F_0(x) \quad \text{(II.14)}$$

where $F_0(x)$ is given by equation (II.13). The constant a which represents $Q(0)$ is given by

$$Q(0) = a = \frac{r}{r_0} + (1 - \frac{r}{r_0})\frac{\eta}{\gamma+\eta} \quad \text{(II.15)}$$

Remark :

The stationary buffer occupancy distribution is given by

$$F(x) = \lim_{t\to\infty} Pr\{C(t) < x\} = Q(x) + \sum_{k=0}^{\infty} \hat{F}_k(x)$$

$$= a\left[1 + \frac{\gamma}{\eta} - \frac{\gamma r}{(\eta r_0 + \gamma r)} + \frac{\gamma^2 r}{\eta(\eta r_0 + \gamma r)}\right] - \frac{2a\gamma}{\eta} e^{\frac{-\eta}{r}x} + \frac{a\gamma(r_0 + r)}{(\eta r_0 + \gamma r)} e^{-\left(\frac{\eta}{r} + \frac{\gamma}{r_0}\right)x}.$$

III. Conclusion

We provide explicit analytical expressions for the joint system size probabilities for the state of the background queueing model and the content of the buffer, under steady state, for the fluid queue modulated by an $M/M/1$ queueing model subject to catastrophes and subsequent repair. Such closed form expressions will greatly aid in an indepth analysis of the physical model for the practitioners. Further extension to the present work may include the time dependent analysis of the model under consideration.

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