A Modification of Fan Sub-Equation Method for Nonlinear Partial Differential Equations

Sheng Zhang, Ao-Xue Peng

Abstract—**In this paper, a modification of Fan sub-equation method is proposed to uniformly construct a series of exact solutions of nonlinear partial differential equations. To illustrate the validity of the modification, the (3+1)-dimensional potential YTSF equation is considered. As a result, some new and more general travelling wave solutions are obtained including soliton solutions, rational solutions, triangular periodic solutions, Jacobi and Weierstrass doubly periodic wave solutions. Among them, the Jacobi elliptic periodic wave solutions can degenerate into the soliton solutions at a certain limit condition. It is shown that the modified Fan sub-equation method provides a more effective mathematical tool for solving nonlinear partial differential equations.**

Index Terms—**Nonlinear partial differential equation, Fan sub-equation method, rational solution, triangular periodic solution, Jacobi and Weierstrass doubly periodic wave solution.**

I. Introduction

IT is well known that searching for travelling wave solutions of nonlinear partial differential equations (PDEs) plays an important role in the study of nonlinear physical phenomena in many fields such as fluid dynamics, plasma physics and nonlinear potics. In the past several decades, there has been significant progression in the development of various methods for exactly solving nonlinear PDEs, such as inverse scattering method [1], Bäcklund transformation [2], Darboux transformation [3], [4], Hirota's bilinear method [5], tanh-function method [6], similarity transformation method [7], Painlevé expansion [8], sine-cosine method [9], F-expansion method [10], exp-function method [11], homogeneous balance method [12] and G'/G method [13].

With the development of computer science, recently, solving differential equations analytically or numerically has attracted much attention [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25]. This is due to the availability of symbolic computation systems like *Mathematica* or *Maple* which enable us to perform the complex and tedious computation on computers. Fan sub-equation method [26] is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in φ, and that $\varphi = \varphi(\xi)$ satisfies a first-order nonlinear ordinary differential equation (ODE):

$$(\varphi')^2 = h_0 + h_1\varphi + h_2\varphi^2 + h_3\varphi^3 + h_4\varphi^4, \tag{1}$$

Manuscript received October 10, 2013; revised February 9, 2014. This work was supported by the Natural Science Foundation of Liaoning Province of China (L2012404), the Liaoning BaiQianWan Talents Program (2013921055), the PhD Start-up Fund of Bohai University (bsqd2013025) and the Natural Science Foundation of China (11371071).

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where $\varphi' = \frac{\mathrm{d}\varphi(\xi)}{\mathrm{d}\xi}$, $\xi = x - Vt$, V is a constant, h_0, h_1, h_2, h_3 and h_4 are embedded parameters. The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and the nonlinear terms appearing in the given nonlinear PDEs. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. It was shown that the method present a wider applicability for handling many kinds of nonlinear PDEs like those in [27], [28], [29], [30], [31].

In this paper, we present a modification of Fan sub-equation method to uniformly construct a series of travelling wave solutions including soliton solutions, rational solutions, triangular periodic solutions, Jacobi and Weierstrass doubly periodic wave solutions for general nonlinear PDEs. For illustration, we would like to apply the modified Fan sub-equation method to solve the (3+1)-dimensional potential YTSF equation.

The rest of this paper is organized as follows. In Section 2, we describe a modification of Fan sub-equation method. In Section 3, we use the modified Fan sub-equation method to solve the (3+1)-dimensional potential YTSF equation. In Section 4, some conclusions are given.

II. A Modification of Fan sub-equation method

For a given nonlinear PDE, say, in four variables x, y, z and t:

$$P(x, y, z, t, u, u_x, u_y, u_z, u_t, \cdots) = 0, \tag{2}$$

we use the following transformation

$$u = u(\xi), \quad \xi = ax + by + cz - \omega t, \tag{3}$$

where a, b, c and ω are undetermined constants, then (2) is reduced into an ODE [32]:

$$Q(x, y, z, t, u^{(r)}, u^{(r+1)}, \cdots) = 0, \tag{4}$$

where $u^{(r)} = \frac{\mathrm{d}^r u}{\mathrm{d}\xi^r}$, $u^{(r+1)} = \frac{\mathrm{d}^{r+1} u}{\mathrm{d}\xi^{r+1}}$, $r \geq 1$, and r is the least order of derivatives in the equation. To keep the solution process as simple as possible, the function Q should not be a total ξ-derivative of another function. Otherwise, taking integration with respect to ξ further reduces the transformed equation.

We further introduce

$$u^{(r)}(\xi) = v(\xi) = \sum_{i=1}^{n} \alpha_i \varphi^i + \alpha_0, \tag{5}$$

where $\varphi = \varphi(\xi)$ satisfies (1), while α_0, $\alpha_i (i = 1, 2, \cdots, n)$ are constants to be determined later.

To determine u explicitly, we take the following four steps:

Step 1. Determine the integer n by substituting (5) along with (1) into (4) and then balancing the highest-order nonlinear term(s) and the highest-order partial derivative.

Step 2. Substitute (5) given the value of n determined in Step 1 along with (1) into (4) to derive a polynomial in φ, and then set all the coefficients of the polynomial to zero to derive a set of algebraic equations for a, b, c, ω, α_0 and $\alpha_i (i = 1, 2, \cdots, n)$.

Step 3. Solve the system of algebraic equations derived in Step 2 for a, b, c, ω, α_0 and $\alpha_i (i = 1, 2, \cdots, n)$ by use of *Mathematica*.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions $v(\xi)$ of (4) depending on φ, since the solutions of (1) have been well known [26], then we can obtain exact solutions of (2) by integrating each of the obtained fundamental solutions $v(\xi)$ with respect to ξ, r times:

$$u = \int^{\xi}\int^{\xi_r}\cdots\int^{\xi_2} v(\xi_1)\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{r-1}\mathrm{d}\xi_r + \sum_{j=1}^{r} d_j\xi^{r-j}, \quad (6)$$

where $d_j (j = 1, 2, \cdots, r)$ are arbitrary constants.

III. Application to potential YTSF equation

Let us consider in this section the (3+1)-dimensional potential YTSF equation [33]

$$-4u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx}u_z + 3u_{yy} = 0, \quad (7)$$

which can be derived from the (3+1)-dimensional YTSF equations:

$$\begin{aligned} &[-4v_t + \Phi(v)v_z]_x + 3u_{yy} = 0, \\ &\Phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1}, \end{aligned} \quad (8)$$

by using the potential $v = u_x$. It was Yu et al. [32] who extended the (2+1)-dimensional Bogoyavlenskii–Schiff equations:

$$\begin{aligned} &v_t + \Phi(v)v_z = 0, \\ &\Phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1}, \end{aligned} \quad (9)$$

to the (3+1)-dimensional nonlinear PDE in the form of (7).

Using the transformation (3), we reduce (7) into an ODE equation in the form:

$$a^3cu^{(4)} + 6a^2cu'u'' + (4a\omega + 3b^2)u'' = 0. \quad (10)$$

Integrating (10) once with respect to ξ and setting the integration constant to zero yields

$$a^3cu^{(3)} + 3a^2c(u')^2 + (4a\omega + 3b^2)u' = 0. \quad (11)$$

Further setting $r = 1$ and $u' = v$, we have

$$a^3cv'' + 3a^2cv^2 + (4a\omega + 3b^2)v = 0. \quad (12)$$

According to Step 1, we get $n + 2 = 2n$, hence $n = 2$. We then suppose that (12) has the formal solution:

$$v = \alpha_2\varphi^2 + \alpha_1\varphi + \alpha_0. \quad (13)$$

Substituting (13) along with (1) into (12) and collecting all terms with the same order of φ together, the left-hand side of (12) is converted into a polynomial in φ. Setting each coefficient of the polynomial to zero, we derive a set of algebraic equations for a, b, c, ω, α_0, α_1 and α_2 as follows:

$$\varphi^0 : a^3c\alpha_1h_1 + 4a^3c\alpha_2h_0 + 6a^2c\alpha_0^2 + 8a\omega\alpha_0 + 6b^2\alpha_0 = 0,$$

$$\begin{aligned} \varphi^1 : &a^3c\alpha_1h_2 + 3a^3c\alpha_2h_1 + 6a^2c\alpha_0\alpha_1 \\ &+4a\omega\alpha_1 + 6b^2\alpha_1 = 0, \end{aligned}$$

$$\begin{aligned} \varphi^2 : &3a^3c\alpha_1h_3 + 8a^3c\alpha_2h_2 + 6a^2c\alpha_1^2 + 12a^2c\alpha_0\alpha_2 \\ &+8a\omega\alpha_2 + 6b^2\alpha_2 = 0, \end{aligned}$$

$$\varphi^3 : 2a^3c\alpha_1h_4 + 5a^3c\alpha_2h_3 + 6a^2c\alpha_1\alpha_2 = 0,$$

$$\varphi^4 : 2a^3c\alpha_2h_4 + a^2c\alpha_2^2 = 0.$$

Solving the set of algebraic equations by use of *Mathematica*, we obtain five cases as follows.

Case 3.1: When $h_3 = h_1 = h_0 = 0$, we have

$$\begin{aligned} &\alpha_2 = -2ah_4, \quad \alpha_1 = 0, \quad \alpha_0 = 0, \\ &\omega = \frac{-3b^2 - 4a^3ch_2}{4a}, \end{aligned} \quad (14)$$

and

$$\begin{aligned} &\alpha_2 = -2ah_4, \quad \alpha_1 = 0, \quad \alpha_0 = -\tfrac{4}{3}ah_2, \\ &\omega = \frac{-3b^2 + 4a^3ch_2}{4a}. \end{aligned} \quad (15)$$

We, therefore, have

$$v = -2ah_4\varphi^2, \quad \omega = \frac{-3b^2 - 4a^3ch_2}{4a}, \quad (16)$$

and

$$v = -2ah_4\varphi^2 - \frac{4}{3}ah_2, \quad \omega = \frac{-3b^2 + 4a^3ch_2}{4a}. \quad (17)$$

Substituting the general solutions [26] of (1) into (16) and (17), respectively, and using (6), we obtain three types of travelling wave solutions of (7).

(i) If $h_2 > 0, h_4 < 0$, we obtain two kink shaped soliton solutions:

$$u = 2a\sqrt{h_2}\tanh(\sqrt{h_2}\xi) + d_1, \quad (18)$$

where $\xi = ax + by + cz + \frac{3b^2+4a^3ch_2}{4a}t$, d_1 is an arbitrary constant;

$$u = 2a\sqrt{h_2}\tanh(\sqrt{h_2}\xi) - \frac{4}{3}ah_2\xi + d_1, \quad (19)$$

where $\xi = ax + by + cz + \frac{3b^2-4a^3ch_2}{4a}t$, d_1 is an arbitrary constant.

(ii) If $h_2 < 0, h_4 > 0$, we obtain two triangular solutions:

$$u = -2a\sqrt{-h_2}\tan(\sqrt{-h_2}\xi) + d_1, \quad (20)$$

where $\xi = ax + by + cz + \frac{3b^2+4a^3ch_2}{4a}t$, d_1 is an arbitrary constant;

$$u = 2a\sqrt{-h_2}\tan(\sqrt{-h_2}\xi) - \frac{4}{3}ah_2\xi + d_1, \quad (21)$$

where $\xi = ax + by + cz + \frac{3b^2-4a^3ch_2}{4a}t$, d_1 is an arbitrary constant.

(iii) If $h_2 = 0, h_4 > 0$, we obtain two rational solutions:

$$u = 2a\xi^{-1} + d_1, \quad (22)$$

where $\xi = ax + by + cz + \frac{3b^2}{4a}t$, d_1 is an arbitrary constant;

$$u = 2a\xi^{-1} - \frac{4}{3}ah_2\xi + d_1, \quad (23)$$

where $\xi = ax + by + cz + \frac{3b^2}{4a}t$, d_1 is an arbitrary constant.

Case 3.2: When $h_3 = h_1 = 0, h_0 = \frac{h_2^2}{4h_4}$, we have

$$\alpha_2 = -2ah_4, \quad \alpha_1 = 0, \quad \alpha_0 = -ah_2, \quad \omega = \frac{-3b^2 + 2a^3ch_2}{4a}, \tag{24}$$

and

$$\alpha_2 = -2ah_4, \quad \alpha_1 = 0, \quad \alpha_0 = -\frac{1}{3}ah_2, \quad \omega = \frac{-3b^2 - 2a^3ch_2}{4a}. \tag{25}$$

We, therefore, have

$$v = -2ah_4\varphi^2 - ah_2, \quad \omega = \frac{-3b^2 + 2a^3ch_2}{4a}, \tag{26}$$

and

$$v = -2ah_4\varphi^2 - \frac{1}{3}ah_2, \quad \omega = \frac{-3b^2 - 2a^3ch_2}{4a}. \tag{27}$$

Substituting the general solutions [26] of (1) into (26) and (27), respectively, and using (6), we obtain two types of travelling wave solutions of (7).

(i) If $h_2 < 0, h_4 > 0$, we obtain two kink shaped soliton solutions:

$$u = a\sqrt{-2h_2}\tanh(\sqrt{-\frac{h_2}{2}}\xi) + d_1, \tag{28}$$

where $\xi = ax + by + cz + \frac{3b^2 - 2a^3ch_2}{4a}t$, d_1 is an arbitrary constant;

$$u = a\sqrt{-2h_2}\tanh(\sqrt{-\frac{h_2}{2}}\xi) + \frac{2}{3}ah_2\xi + d_1, \tag{29}$$

where $\xi = ax + by + cz + \frac{3b^2 + 2a^3ch_2}{4a}t$, d_1 is an arbitrary constant.

(ii) If $h_2 > 0, h_4 > 0$, we obtain two triangular solutions:

$$u = -2a\sqrt{2h_2}\tan(\sqrt{\frac{h_2}{2}}\xi) + ah_2\xi + d_1, \tag{30}$$

where $\xi = ax + by + cz + \frac{3b^2 - 2a^3ch_2}{4a}t$, d_1 is an arbitrary constant;

$$u = -2a\sqrt{2h_2}\tan(\sqrt{\frac{h_2}{2}}\xi) + \frac{5}{3}ah_2\xi + d_1, \tag{31}$$

where $\xi = ax + by + cz + \frac{3b^2 + 2a^3ch_2}{4a}t$, d_1 is an arbitrary constant.

Case 3.3: When $h_3 = h_1 = 0$, we have

$$\alpha_2 = -2ah_4, \quad \alpha_1 = 0, \quad \alpha_0 = \frac{2}{3}(-ah_2 \mp \sqrt{a^2h_2^2 - 3a^2h_0h_4}), \quad \omega = \frac{-3b^2 \pm 4a^2c\sqrt{a^2(h_2^2 - 3h_0h_4)}}{4a}. \tag{32}$$

We, therefore, have

$$v = -2ah_4\varphi^2 - \frac{2}{3}(ah_2 \pm \sqrt{a^2h_2^2 - 3a^2h_0h_4}), \quad \omega = \frac{-3b^2 \pm 4a^2c\sqrt{a^2(h_2^2 - 3h_0h_4)}}{4a}. \tag{33}$$

Substituting the general solutions [26] of (1) into (33), respectively, and using (6), we obtain three types of travelling wave solutions of (7).

(i) If $h_4 < 0, h_2 > 0, h_0 = \frac{1-m^2}{(2m^2-1)^2}$, we obtain a Jacobi elliptic function solution:

$$u = \frac{2ah_2m^2}{2m^2-1}\int^{\xi}\mathrm{cn}^2(\sqrt{\frac{h_2}{2m^2-1}}\xi_1, m)\mathrm{d}\xi_1 - \frac{2}{3}(ah_2 \pm \sqrt{a^2h_2^2 - 3a^2h_0h_4})\xi + d_1, \tag{34}$$

where $\xi = ax + by + cz + \frac{3b^2 \mp 4a^2c\sqrt{a^2(h_2^2 - 3h_0h_4)}}{4a}t$, d_1 is an arbitrary constant.

(ii) If $h_4 > 0, h_2 < 0, h_0 = \frac{h_2^2m^2}{2h_4(m^2+1)}$, we obtain a Jacobi elliptic function solution:

$$u = \frac{2ah_2m^2}{m^2+1}\int^{\xi}\mathrm{sn}^2(\sqrt{-\frac{h_2}{m^2+1}}\xi_1, m)\mathrm{d}\xi_1 - \frac{2}{3}(ah_2 \pm \sqrt{a^2h_2^2 - 3a^2h_0h_4})\xi + d_1, \tag{35}$$

where $\xi = ax + by + cz + \frac{3b^2 \mp 4a^2c\sqrt{a^2(h_2^2 - 3h_0h_4)}}{4a}t$, d_1 is an arbitrary constant.

As $m \to 1$, the Jacobi doubly periodic solutions (34) and (35) can degenerate into soliton solutions but we omit them here for the sake of simplicity.

Case 3.4: When $h_4 = h_1 = h_0 = 0$, we have

$$\alpha_2 = 0, \quad \alpha_1 = 0, \quad \alpha_0 = -\frac{1}{3}ah_0, \quad \omega = \frac{-3b^2 + a^3ch_2}{4a}, \tag{36}$$

and

$$\alpha_2 = 0, \quad \alpha_1 = -\frac{1}{2}ah_3, \quad \alpha_0 = 0, \quad \omega = \frac{-3b^2 - a^3ch_2}{4a}, \tag{37}$$

and

$$\alpha_2 = 0, \quad \alpha_1 = -\frac{1}{2}ah_3, \quad \alpha_0 = -\frac{1}{3}ah_2, \quad \omega = \frac{-3b^2 + a^3ch_2}{4a}. \tag{38}$$

We, therefore, have

$$v = -\frac{1}{3}ah_0, \quad \omega = \frac{-3b^2 + a^3ch_2}{4a}, \tag{39}$$

and

$$v = -\frac{1}{2}ah_3\varphi, \quad \omega = \frac{-3b^2 - a^3ch_2}{4a}, \tag{40}$$

and

$$v = -\frac{1}{2}ah_3\varphi - \frac{1}{3}ah_2, \quad \omega = \frac{-3b^2 + a^3ch_2}{4a}. \tag{41}$$

Substituting the general solutions [26] of (1) into (39), (40) and (41) respectively, and using (6), we obtain three types of travelling wave solutions of (7).

$$u = -\frac{1}{3}ah_0\xi + d_1, \tag{42}$$

where $\xi = ax + by + cz + \frac{3b^2 - a^3ch_2}{4a}t$, d_1 is an arbitrary constant.

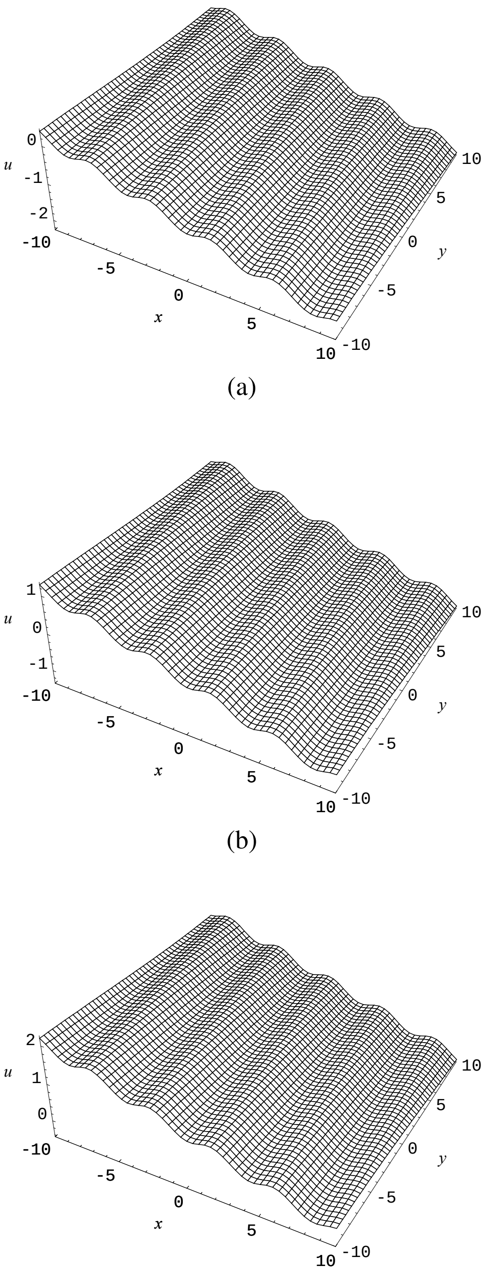

Fig. 1. Asymptotical property of Jacobi doubly periodic solution (35) with (+) branch for parameters $a = 1$, $b = 0.1$, $c = 2$, $d_1 = 0$, $h_2 = -1$, $h_4 = 2$, $m = 0.5$, $z = 0$ at different times: (a) $t = -5$, (b) $t = 0$, (c) $t = 5$.

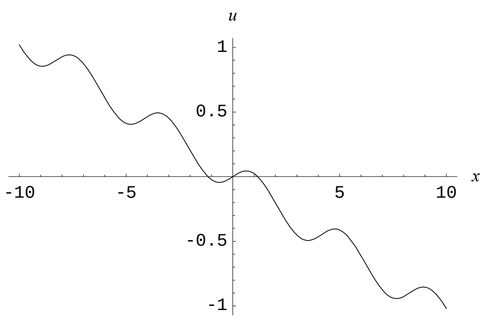

Fig. 2. Asymptotical property of Jacobi doubly periodic solution (35) with (+) branch for parameters $a = 1$, $b = 0.1$, $c = 2$, $d_1 = 0$, $h_2 = -1$, $h_4 = 2$, $m = 0.5$, $y = 0$, $z = 0$ at time $t = 0$.

(i) If $h_2 > 0$, we obtain two kink shaped soliton solutions:

$$u = \frac{a}{2}\sqrt{2h_2}\tanh(\sqrt{\frac{h_2}{2}}\xi) + d_1, \tag{43}$$

where $\xi = ax + by + cz + \frac{3b^2+a^3ch_2}{4a}t$, d_1 is an arbitrary constant;

$$u = \frac{a}{2}\sqrt{2h_2}\tanh(\sqrt{\frac{h_2}{2}}\xi) - \frac{1}{3}ah_2\xi + d_1, \tag{44}$$

where $\xi = ax + by + cz + \frac{3b^2-a^3ch_2}{4a}t$, d_1 is an arbitrary constant.

(ii) If $h_2 < 0$, we obtain two triangular solutions:

$$u = -\frac{a}{2}\sqrt{-2h_2}\tan(\sqrt{-\frac{h_2}{2}}\xi) + d_1, \tag{45}$$

where $\xi = ax + by + cz + \frac{3b^2+a^3ch_2}{4a}t$, d_1 is an arbitrary constant;

$$u = -\frac{a}{2}\sqrt{-2h_2}\tan(\sqrt{-\frac{h_2}{2}}\xi) - \frac{1}{3}ah_2\xi + d_1, \tag{46}$$

where $\xi = ax + by + cz + \frac{3b^2-a^3ch_2}{4a}t$, d_1 is an arbitrary constant.

(iii) If $h_2 = 0$, we obtain two rational solutions:

$$u = -\frac{a}{2}\xi^{-1} + d_1, \tag{47}$$

where $\xi = ax + by + cz + \frac{3b^2}{4a}t$, d_1 is an arbitrary constant;

$$u = -\frac{a}{2}\xi^{-1} - \frac{1}{3}ah_2\xi + d_1, \tag{48}$$

where $\xi = ax + by + cz + \frac{3b^2}{4a}t$, d_1 is an arbitrary constant.

Case 3.5: When $h_4 = h_2 = 0, h_3 > 0$, we have

$$\begin{aligned}&\alpha_2 = 0, \quad \alpha_1 = -\frac{ah_3}{2}, \quad \alpha_0 = \pm\frac{\sqrt{3}a\sqrt{h_1h_3}}{6}i,\\ &\omega = \frac{\pm 3b^2 + \sqrt{3}a^3c\sqrt{h_1h_3}i}{4a}.\end{aligned} \tag{49}$$

We, therefore, have

$$\begin{aligned}v &= -\frac{ah_3}{2}\varphi \pm \frac{\sqrt{3}a\sqrt{h_1h_3}}{6}i,\\ \omega &= \frac{\pm 3b^2 + \sqrt{3}a^3c\sqrt{h_1h_3}i}{4a},\end{aligned} \tag{50}$$

Substituting the general solutions [26] of (1) into (50), respectively, and using (6), we obtain a Weierstrass elliptic function solution of (7):

$$\begin{aligned}u = &-\frac{ah_3}{2}\int^{\xi}\wp(\frac{\sqrt{h_3}}{2}\xi_1, g_2, g_3)\mathrm{d}\xi_1\\ &\pm\frac{\sqrt{3}a\sqrt{h_1h_3}i}{6}\xi + d_1,\end{aligned} \tag{51}$$

where $\xi = ax + by + cz \mp \frac{3b^2 \pm \sqrt{3}a^3c\sqrt{h_1h_3}i}{4a}t$, $g_2 = -\frac{4h_1}{h_3}$, $g_3 = -\frac{4h_0}{h_3}$ and d_1 is an arbitrary constant.

IV. Conclusion

In summary, we have proposed and used a modification of Fan sub-equation method with symbolic computation to construct a series of travelling wave solutions for the (3+1)-dimensional potential YTSF equation (1) including soliton solutions, triangular periodic solutions, rational solutions, Weierstrass and Jacobi doubly periodic wave solutions. Some of the obtained solutions contain an explicit linear function ξ of the variables x, y, z and t. It may be important to explain some physical phenomena though the physical relevance of soliton solutions and periodic solutions is clear to us. Solutions (19), (21), (29)–(31), (34), (35), (44), (46) and (51) can not be obtained by Fan sub-equation method [26] and its existing improvements like [27] if we do not transform (11) into (12) but directly solving (11). To the best of our knowledge, they have not been reported in literatures. The paper shows the effectiveness and advantages of the modified Fan sub-equation method in handling the solution process of nonlinear PDEs. Employing it to study other nonlinear PDEs is our task in the future.

Acknowledgment

The authors would like to express their sincere thanks to anonymous reviewers for the valuable suggestions and comments.

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