Preconditioned Parallel Multisplitting USAOR Methods for $H-$matrices Linear Systems

Xuezhong Wang, Zuowei Wang

***Abstract*—In this paper, three preconditioned parallel multisplitting methods are established for solving the large sparse linear systems, including local preconditioned parallel multisplitting relaxation method, global preconditioned parallel multisplitting relaxation method and global preconditioned nonstationary parallel multisplitting relaxation method. The convergence and comparison results of the methods associated with USAOR multisplitting are given when the coefficient matrices of the linear systems are H-matrices. We prove three preconditioned parallel multisplitting USAOR method are better than parallel multisplitting USAOR methods for M-matrices linear systems. Finally, numerical examples are given to illustrate the methods are valid.**

***Index Terms*—Preconditioned, Relaxation parallel multisplitting method, Convergence, USAOR, H-matrix.**

I. Introduction

FOR the linear system

$$Ax = b, \tag{1}$$

where A is an $n\times\ n$ square matrix, and x and b are n-dimensional vectors. The basic iterative method for solving equation (1) is

$$Mx^{k+1} = Nx^k + b,\ \ k = 0, 1, \cdots, \tag{2}$$

where $A = M - N$ and M is nonsingular. Thus (2) can be written as

$$x^{k+1} = Tx^k + c,\ \ k = 0, 1, \cdots, \tag{3}$$

where $T = M^{-1}N,\ c = M^{-1}b$.

The original systems (1) can be transformed into the preconditioned form

$$PAx = Pb. \tag{4}$$

Then, we can define the basic iterative scheme:

$$M_p x^{k+1} = N_p x^k + Pb,\ \ k = 0, 1, \cdots, \tag{5}$$

where $PA = M_p - N_p$ and M_p is nonsingular. Thus (5) can also be written as

$$x^{k+1} = Tx^k + c,\ \ k = 0, 1, \cdots,$$

where $T = M_p^{-1}N_p,\ c = M_p^{-1}Pb$.

Without loss of generality, we assume that A has unit diagonal elements. In the literature, various authors have suggested different models of $(I+S)$-type preconditioner [1-7, 24-28] for linear systems (1). These preconditioners have effectiveness and low construction cost. For example, in this paper, we consider the preconditioner of $(I+S)$-type with the following form

$$P = I + S_\alpha - S_\beta, \tag{6}$$

where

$$I + S_\alpha = \begin{pmatrix} 1 & & & \alpha_{1r} & & \\ & 1 & & & \alpha_{2s} & \\ & & \ddots & & & \\ & \alpha_{iu} & & \ddots & & \\ & & & & \ddots & \\ & & \alpha_{nt} & & & 1 \end{pmatrix},$$

and

$$S_\beta = \begin{pmatrix} 0 & & & \beta_1 a_{1r} & & \\ & 0 & & & \beta_2 a_{2s} & \\ & & \ddots & & & \\ & \beta_i a_{iu} & & \ddots & & \\ & & & & \ddots & \\ & & \beta_n a_{nt} & & & 0 \end{pmatrix}.$$

O'Leary and White [8] invented the matrix multisplitting method in 1985 for parallely solving the large sparse linear systems on the multiprocessor systems and it was further studied by many authors [9-19]. For example, Neumann and Plemmons [12] developed some more refined convergence results for one of the cases considered in [8]. Elsner [13] established the comparison theorems about the asymptotic convergence rate of this case. Frommer and Mayer [14] discussed the successive overrelaxation (SOR) method in the sense of multisplitting. White [15] studied the convergence properties of the above matrix multisplitting methods for the symmetric positive definite matrix class. Zhang, Huang, et al. [9] presented local relaxed parallel multisplitting method and global relaxed parallel multisplitting method for H-matrices, and so on.

A collection of triples (M_k, N_k, E_k), $k = 1, 2, \ldots, \alpha$, is called a multisplitting of A. If $A = M_k - N_k$ is a splitting of A for $k = 1, 2, \ldots, \alpha$, and E_k's, called weighting matrices, are nonnegative diagonal matrices such that $\sum_{k=1}^{\alpha} e_{ii}^{(k)} = 1$. The multisplitting method associated with this multisplitting for solving the linear system (1) is as follows.

Suppose that we have a multiprocessor with α processors connected to a host processor, that is, the same number of processors as splittings, and all the processors have the last update vector x^k, then the kth processor only computes those entries of the vector

$$M_k^{-1}N_k x^k + M_k^{-1}b,$$

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X. Z. Wang is with School of Mathematics and Statistics, Hexi University, Zhangye, Gansu, 734000, P. R. C., e-mail: mail2011wang@gmail.com.

Z. W. Wang is with College of Foreign Language and Literature, Hexi University, Zhangye, Gansu, 734000, P. R. C., e-mail: wzwrichard@yahoo.com.

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which correspond to the diagonal entries $E_{ii}^{(k)}$ of the matrix E_k. The processor then scales these entries so as to be able to deliver the vector

$$E_K(M_k^{-1}N_kx^k+M_k^{-1}b),$$

to the host processor, performing the parallel multisplitting scheme

$$\begin{aligned}x^{m+1}&=\sum_{k=1}^{\alpha}E_kM_k^{-1}N_kx^m+\sum_{k=1}^{\alpha}E_kM_k^{-1}b\\&=Hx^m+Gb,\ m=0,1,2,\ldots.\end{aligned}$$

Under certain conditions, we establish local preconditioned parallel multisplitting relaxation method, global preconditioned parallel multisplitting relaxation method and global preconditioned nonstationary parallel multisplitting relaxation method for solving the large sparse linear systems and study the convergence of our methods associated with USAOR multisplitting when the coefficient matrices of the linear systems are H-matrices.

II. Establishments of the methods

Let $A=I-L_k-U_k,\ k=1,2,\ldots,\alpha$, where I is a identity matrix, $L_k,\ k=1,2,\ldots,\alpha$, are strictly lower triangular matrices, and $U_k,\ k=1,2,\ldots,\alpha$, are general matrices, respectively, then parallel multisplitting relaxation USAOR methods are defined [9] as follows.

Algorithm 2.1. (local parallel multisplitting relaxation method)

Given the initial vector. For $m=0,1,2,\ldots$ repeat (I) and (II), until convergence.

(I) For $k=1,2,\ldots,\alpha$, (parallel) solving y_k:

$$M_ky_k=N_kx^m+b.$$

(II) Computing

$$x^{m+1}=\sum_{k=1}^{\alpha}E_ky_k.$$

Algorithm 2.1 associated with LUSAOR method can be written as

$$x^{m+1}=H_{LUSAOR}x^m+G_{LUSAOR}b,\ \ m=0,1,\cdots,$$

where

$$\begin{aligned}H_{LUSAOR}&=\sum_{k=1}^{\alpha}E_kU_{\omega_2r_2}(k)L_{\omega_1r_1}(k),\\U_{\omega_2r_2}(k)&=(I-r_2U_k)^{-1}[(1-\omega_2)I\\&\quad+(\omega_2-r_2)U_k\\&\quad+\omega_2L_k],\\L_{\omega_1r_1}(k)&=(I-r_1L_k)^{-1}[(1-\omega_1)I+(\omega_1-r_1)L_k\\&\quad+\omega_1U_k],\\G_{LUSAOR}&=\sum_{k=1}^{\alpha}E_k(I-r_2U_k)^{-1}[(\omega_1+\omega_2\\&\quad-\omega_1\omega_2)I+\omega_2(\omega_1-r_1)L_k\\&\quad+\omega_1(\omega_2-r_2)U_k](I-r_1L_k)^{-1}.\end{aligned}$$

By using a suitable positive relaxation parameter β, global parallel multisplitting relaxation USAOR method will be established in the following, which is based on Algorithm 2.1.

Algorithm 2.2. (global parallel multisplitting relaxation method)

Given the initial vector. For $m=0,1,2,\ldots$ repeat (I) and (II), until convergence.

(I) For $k=1,2,\ldots,\alpha$, (parallel) solving y_k:

$$M_ky_k=N_kx^m+b.$$

(II) Computing

$$x^{m+1}=\beta\sum_{k=1}^{\alpha}E_ky_k+(1-\beta)x^m.$$

Algorithm 2.2 associated with GUSAOR method can be written as

$$x^{m+1}=H_{GUSAOR}x^m+\beta G_{LUSAOR}b,\ m=0,1,\cdots,$$

where $H_{GBUSAOR}=\beta H_{LBUSAOR}+(1-\beta)I$.

In the standard multisplitting method, each local approximation is updated exactly once by using the same previous iterate x^m. On the other hand, it is possible to update the local approximations more than once, by using different iterates computed earlier. In this case, we call this method a nonstationary multisplitting method [17,18,19]. The main idea of the nonstationary method is that at the mth iteration each processor k solves the system $q(m,k)$ times, using each time the new calculated vector to update the right-hand side, i.e., we have the following algorithm:

Algorithm 2.3. (global nonstationary parallel multisplitting relaxation method)

Given the initial vector. For $m=0,1,2,\ldots$ repeat (I) and (II), until convergence.

(I) For $i=1,2,\ldots,q(m,k)$, (parallel) solving $y_k^{(i)}$:

$$M_ky_k^{(i)}=N_ky_k^{(i-1)}+b.$$

(II) Computing

$$x^{m+1}=\beta\sum_{k=1}^{\alpha}E_ky_k^{q(m,k)}+(1-\beta)x^m.$$

Algorithm 2.3 associated with GNUSAOR method can be written as

$$x^{m+1}=H_{GNUSAOR}x^m+\beta G_{GNUSAOR}b,\ \ m=0,1,\cdots,$$

where

$$\begin{aligned}H_{GNUSAOR}&=\beta\sum_{k=1}^{\alpha}E_k(P_{\omega r}Q_{\xi\eta})^{q(m,k)}+(1-\beta)I\\P_{\omega r}&=(I-r_kU_k)^{-1}[(1-\omega_k)I+(\omega_k-r_k)U_k\\&\quad+\omega_kL_k]=W_r^{-1}R_{\omega r},\\Q_{\xi\eta}&=(I-\eta_kL_k)^{-1}[(1-\xi_k)I+(\xi_k-\eta_k)L_k\\&\quad+\xi_kU_k]=V_\eta^{-1}F_{\xi\eta},\\G_{GNUSAOR}&=\beta\sum_{k=1}^{\alpha}E_k[\sum_{i=1}^{q(m,k)-1}(W_rV_\eta)^{-1}\\&\quad(F_{\xi\eta}R_{\omega r})^i](W_rV_\eta)^{-1}\omega_k\xi_k.\end{aligned}$$

It follows that when $q(m,k)=1,\ \omega_k=\omega_2,\ r_k=r_2,\ \xi_k=\omega_1$ and $\eta_k=r_1$ for $1<k<\alpha,\ m=0,1,2....$, Algorithm 2.3 reduces to Algorithm 2.2.

Let $\tilde{A}=PA=\tilde{D}-\tilde{L}_k-\tilde{U}_k,\ k=1,2,\ldots,\alpha$, where $\tilde{D}$ is a diagonal matrix, $\tilde{L}_k,\ k=1,2,\ldots,\alpha$, are strictly lower triangular matrices, and $\tilde{U}_k,\ k=1,2,\ldots,\alpha$, are general matrices, respectively. Then Algorithm 2.1 associated with local preconditioned parallel multisplitting relaxation method (LPUSAOR) can be written as

$$x^{m+1} = \tilde{H}_{LPUSAOR} x^m + \tilde{G}_{LPUSAOR} b, \ m = 0,1,\cdots,$$

where

$$\begin{aligned}
\tilde{H}_{LPUSAOR} &= \sum_{k=1}^{\alpha} E_k \tilde{U}_{\omega_2 r_2}(k) \tilde{L}_{\omega_1 r_1}(k),\\
\tilde{U}_{\omega_2 r_2}(k) &= (\tilde{D} - r_2 \tilde{U}_k)^{-1}[(1-\omega_2)\tilde{D} \\
&\quad +(\omega_2 - r_2)\tilde{U}_k + \omega_2 \tilde{L}_k],\\
\tilde{L}_{\omega_1 r_1}(k) &= (\tilde{D} - r_1 \tilde{L}_k)^{-1}[(1-\omega_1)\tilde{D} \\
&\quad +(\omega_1 - r_1)\tilde{L}_k + \omega_1 \tilde{U}_k],\\
\tilde{G}_{LPUSAOR} &= \sum_{k=1}^{\alpha} E_k (\tilde{D} - r_2 \tilde{U}_k)^{-1}[(\omega_1 + \omega_2 \\
&\quad -\omega_1\omega_2)\tilde{D} + \omega_2(\omega_1 - r_1)\tilde{L}_k \\
&\quad +\omega_1(\omega_2 - r_2)\tilde{U}_k](\tilde{D} - r_1 \tilde{L}_k)^{-1}.
\end{aligned}$$

Algorithm 2.2 associated with global preconditioned parallel multisplitting relaxation method (GPUSAOR) method can be written as

$$x^{m+1} = \tilde{H}_{GPUSAOR} x^m + \beta \tilde{G}_{LPUSAOR} b, \ m = 0,1,\cdots, \quad (7)$$

where $\tilde{H}_{GPUSAOR} = \beta \tilde{H}_{LPUSAOR} + (1-\beta) I$.

Algorithm 2.3 associated with global preconditioned nonstationary parallel multisplitting relaxation method (GPNUSAOR) can be written as

$$x^{m+1} = \tilde{H}_{GPNUSAOR} x^m + \beta \tilde{G}_{GPNUSAOR} b, \ m = 0,1,\cdots,$$

where

$$\begin{aligned}
\tilde{H}_{GPNUSAOR} &= \beta \sum_{k=1}^{\alpha} E_k (\tilde{P}_{\omega r} \tilde{Q}_{\xi\eta})^{q(m,k)} \\
&\quad +(1-\beta) I\\
\tilde{P}_{\omega r} &= (\tilde{D} - r_k \tilde{U}_k)^{-1}[(1-\omega_k)\tilde{D} \\
&\quad +(\omega_k - r_k)\tilde{U}_k + \omega_k \tilde{L}_k] = \tilde{W}_r^{-1} \tilde{R}_{\omega r},\\
\tilde{Q}_{\xi\eta} &= (\tilde{D} - \eta_k \tilde{L}_k)^{-1}[(1-\xi_k)\tilde{D} + \\
&\quad (\xi_k - \eta_k)\tilde{L}_k + \xi_k \tilde{U}_k] = \tilde{V}_\eta^{-1} \tilde{F}_{\xi\eta},\\
\tilde{G}_{GPNUSAOR} &= \beta \sum_{k=1}^{\alpha} E_k [\sum_{i=1}^{q(m,k)-1} (\tilde{W}_\omega \tilde{V}_\eta)^{-1} \\
&\quad (\tilde{F}_{\xi\eta} \tilde{R}_{\omega r})^i](\tilde{W}_\omega \tilde{V}_\eta)^{-1} \omega_k \xi_k.
\end{aligned}$$

Obviously, when $q(m,k) = 1$, $\omega_k = \omega_2$, $r_k = r_2$, $\xi_k = \omega_1$ and $\eta_k = r_1$ for $1 < k < \alpha$, $m = 0,1,2....$, GPNUSAOR method reduces to GPUSAOR method.

III. Preliminaries

We shall use the following notations and lemmas. A matrix A is called nonnegative (positive) if each entry of A is nonnegative (positive), respectively. We denote them by $A \geq 0$ ($A > 0$). Similarly, for n-dimensional vector x, by identifying them with $n \times 1$ matrix, we can also define $x \geq 0$ ($x > 0$). A matrix $A = (a_{ij})$ is called a Z-matrix if for any $i \neq j$, $a_{ij} \leq 0$. A Z-matrix is a nonsingular M-matrix if A is nonsingular and if $A^{-1} \geq 0$. We call $\langle A \rangle = (\bar{a}_{ij})$ its comparison matrix, if $(\bar{a}_{ij}) = |a_{ij}|$ for $i = j$, if $(\bar{a}_{ij}) = -|a_{ij}|$ for $i \neq j$. If $\langle A \rangle$ is a nonsingular M-matrix, then A is called an H-matrix. $A = M - N$ is said to be a splitting of A if M is nonsingular, $A = M - N$ is said to be regular if $M^{-1} \geq 0$ and $N \geq 0$, and weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$. Additionally, we denote the spectral radius of A by $\rho(A)$. It is well-known that if $A \geq 0$ and there exists a vector $x > 0$, such that $Ax < \alpha x$, then $\rho(A) < \alpha$.

Some basic properties are given below, which will be used in the paper.

Lemma 3.1 [21]. Let A be a Z-matrix. Then the following statements are equivalent:
(a) A is an M-matrix.
(b) There is a positive vector x such that $Ax > 0$.
(c) $A^{-1} \geq 0$.
(d) All principal submatrices of A are M-matrices.
(e) All principal minors are positive.

Lemma 3.2 [20]. Let $A = M - N$ be an M-splitting of A, then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.

Lemma 3.3 [20]. Let A and B be two $n \times n$ matrices with $0 \leq B \leq A$, then $\rho(B) \leq \rho(A)$.

Lemma 3.4 [22]. If A is an H−matrix, then $|A^{-1}| \leq \langle A \rangle^{-1}$.

Lemma 3.5 [12]. Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splitting of monotone matrices A_1 and A_2 respectively, such that $M_2^{-1} \geq M_1^{-1}$. If there exists a positive vector x such that $0 \leq A_1 x \leq A_2 x$, then for the monotone norm associated with x,

$$\| M_2^{-1} N_2 \|_x \leq \| M_1^{-1} N_1 \|_x . \quad (8)$$

In particular, if $M_1^{-1} N_1$ has a positive perron vector, then

$$\rho(M_2^{-1} N_2) \leq \rho(M_1^{-1} N_1). \quad (9)$$

Moreover if x is a Perron vector of $M_1^{-1} N_1$ and strictly inequality holds in (8), then strictly inequality holds in (9).

Lemma 3.6 [9]. Let A be an H-matrix, and for $k = 1,2,\ldots,\alpha$, L_k be strictly lower triangular matrices. Define the matrices U_k, $k = 1,2,\ldots,\alpha$, such that $A = D - L_k - U_k$. Assume that $\langle A \rangle = |D| - |L_k| - |U_k| = |D| - |B|$. If

$$0 < \omega_1, \omega_2 < \frac{2}{1+\rho}, \ 0 \leq r_1 \leq \omega_1, \ 0 \leq r_2 \leq \omega_2,$$

then LUSAOR method converges for any initial vector x^0, where $\rho = \rho(J) = \rho(|D|^{-1}|B|)$.

Lemma 3.7 [9]. Let A be an H-matrix, and for $k = 1,2,\ldots,\alpha$, L_k be strictly lower triangular matrices. Define the matrices U_k, $k = 1,2,\ldots,\alpha$, such that $A = D - L_k - U_k$. Assume that $\langle A \rangle = |D| - |L_k| - |U_k| = |D| - |B|$. If

$$0 < \omega_1, \omega_2 < \frac{2}{1+\rho}, \ 0 \leq r_1 \leq \omega_1, \ 0 \leq r_2 \leq \omega_2, \ 0 < \beta < \frac{2}{1+\theta},$$

then GUSAOR method converges for any initial vector x^0, where $\rho = \rho(J) = \rho(|D|^{-1}|B|)$ and

$$\theta = max\{|1-\omega_1| + \omega_1\rho, \ |1-\omega_2| + \omega_2\rho\}.$$

Lemma 3.8 [9]. Let A be an H-matrix, and for $k = 1,2,\ldots,\alpha$, L_k be strictly lower triangular matrices. Define the matrices U_k, $k = 1,2,\ldots,\alpha$, such that $A = D - L_k - U_k$. Assume that $\langle A \rangle = |D| - |L_k| - |U_k| = |D| - |B|$. If

$$0 < \omega_k, \xi_k < \frac{2}{1+\rho}, \ 0 \leq r_k \leq \omega_k, \ 0 \leq \eta_k \leq \xi_k, \ 0 < \beta < \frac{2}{1+\theta'}$$

then GUSAOR method converges for any initial vector x^0, where $\rho = \rho(J) = \rho(|D|^{-1}|B|)$ and

$$\theta' = max\{|1-\omega_k| + \omega_k\rho, \ |1-\xi_k| + \xi_k\rho\}.$$

Let $A = M - N = F - Q$ are two splittings of A. If we set

$$T = F^{-1} Q M^{-1} N.$$

Benzi and Szyld [23] have established the following result.

Lemma 3.9 [23]. Let $A^{-1} \geq 0$. If the splitting $A = M - N = F - Q$ are weak regular, then $\rho(T) < 1$ and the unique splitting $A = B - C$ induced by T is weak regular, where $B = F(M + P - A)^{-1}M$ and $C = B - A$.

IV. Convergence

For Algorithms 1, 2 and 3, we give convergence theorems for $H-$matrices. For convenience, let $t_i = \frac{(\langle A\rangle x)_i}{2x_m - (\langle A\rangle x)_m}$, where $x = (x_1, x_2, \cdots, x_n)^T$ denotes a vector.

Theorem 4.1. Let A be an $H-$matrix with unit diagonal elements. Assume that there exists a positive vector $x = (x_1, x_2, \cdots, x_n)^T$, such that $\langle A\rangle x > 0$. If $|\alpha_{im} - \beta_i a_{im}| \leq t_i$, $i = 1, 2, \cdots, n$, then PA is an $H-$matrix.

Proof. Let $(PA)_{ij} = a_{ij} + (\alpha_{im} - \beta_i a_{im}) a_{mj}$, $i, j = 1, 2, \cdots, n, m = r, s, \cdots, t$, and $x = \left(x_{1,} x_{2,} \cdots, x_n\right)^T$, then

$$\begin{aligned}
(\langle PA\rangle x)_i &= \left|1 + \left(\alpha_{im} - \beta_i a_{im}\right) a_{mi}\right| x_i \\
&\quad - \left|a_{im} + \left(\alpha_{im} - \beta_i a_{im}\right)\right| x_m \\
&\quad - \sum_{j \neq i,m} \left|a_{ij} + \left(\alpha_{im} - \beta_i a_{im}\right) a_{mj}\right| x_j \\
&\geq x_i - \left|\alpha_{im} - \beta_i a_{im}\right| |a_{mi}| x_i - |a_{im}| x_m \\
&\quad - \left|\alpha_{im} - \beta_i a_{im}\right| x_m - \sum_{j \neq i,m} \left|a_{ij}\right| x_j \\
&\quad - \sum_{j \neq i,m} \left|\alpha_{im} - \beta_i a_{im}\right| \left|a_{mj}\right| x_j \\
&\triangleq d.
\end{aligned}$$

Case 1. $\beta_i a_{im} \leq \alpha_{im} \leq \beta_i a_{im} + t_i$

$$\begin{aligned}
d &\geq x_i - (\alpha_{im} - \beta_i a_{im}) |a_{mi}| x_i - |a_{im}| x_m \\
&\quad -(\alpha_{im} - \beta_i a_{im}) x_m - \sum_{j \neq i,m} \left|a_{ij}\right| x_j \\
&\quad - \sum_{j \neq i,m} (\alpha_{im} - \beta_i a_{im}) \left|a_{mj}\right| x_j \\
&= x_i - |a_{im}| x_m - \sum_{j \neq i,m} \left|a_{ij}\right| x_j - (\alpha_{im} - \beta_i a_{im}) |a_{mi}| x_i \\
&\quad -(\alpha_{im} - \beta_i a_{im}) x_m - \sum_{j \neq i,m} (\alpha_{im} - \beta_i a_{im}) \left|a_{mj}\right| x_j \\
&= (\langle A\rangle x)_i + \left(\alpha_{im} - \beta_i a_{im}\right)(-x_m - \sum_{j \neq m} \left|a_{mj}\right| x_j) \\
&= (\langle A\rangle x)_i + \left(\alpha_{im} - \beta_i a_{im}\right)(-\sum_{j \neq m} \left|a_{mj}\right| x_j + x_m - 2x_m) \\
&= (\langle A\rangle x)_i + \left(\alpha_{im} - \beta_i a_{im}\right)[(\langle A\rangle x)_m - 2x_m] \\
&> 0.
\end{aligned}$$

Case 2. $\beta_i a_{im} - t_i \leq \alpha_{im} \leq \beta_i a_{im}$

$$\begin{aligned}
d &\geq x_i + \left(\alpha_{im} - \beta_i a_{im}\right) |a_{mi}| x_i - |a_{im}| x_m \\
&\quad + \left(\alpha_{im} - \beta_i a_{im}\right) x_m - \sum_{j \neq i,m} \left|a_{ij}\right| x_j \\
&\quad + \sum_{j \neq i,m} \left(\alpha_{im} - \beta_i a_{im}\right) \left|a_{mj}\right| x_j \\
&= (\langle A\rangle x)_i + \left(\alpha_{im} - \beta_i a_{im}\right)(\sum_{j \neq m} \left|a_{mj}\right| x_j - x_m + 2x_m) \\
&= (\langle A\rangle x)_i + \left(\alpha_{im} - \beta_i a_{im}\right)[2x_m - (\langle A\rangle x)_m] \\
&> 0.
\end{aligned}$$

Therefore, $\langle PA\rangle$ is an M-matrix, and PA is an H-matrix.

Together with Lemma 3.7, Lemma 3.8, Lemma 3.9 and Theorem 4.1, we can obtain the following results.

Theorem 4.2. Let A be an H-matrix with unit diagonal elements, and for $k = 1, 2, \ldots, \alpha$, $\tilde{L}_k$ be strictly lower triangular matrices. Define the matrices $\tilde{U}_k$, $k = 1, 2, \ldots, \alpha$, such that $\tilde{A} = PA = \tilde{D} - \tilde{L}_k - \tilde{U}_k$. Assume that $\langle\tilde{A}\rangle = |\tilde{D}| - |\tilde{L}_k| - |\tilde{U}_k| = |\tilde{D}| - |\tilde{B}|$. If $|\alpha_{im} - \beta_i a_{im}| \leq t_i$, $i = 1, 2, \cdots, n$, and

$$0 < \omega_1, \omega_2 < \frac{2}{1 + \tilde{\rho}},\ 0 \leq r_1 \leq \omega_1,\ 0 \leq r_2 \leq \omega_2,$$

then LPUSAOR method converges for any initial vector x^0, where $\tilde{\rho} = \rho(\tilde{J}) = \rho(|\tilde{D}|^{-1}|\tilde{B}|)$.

Theorem 4.3. Let A be an H-matrix with unit diagonal elements, and for $k = 1, 2, \ldots, \alpha$, $\tilde{L}_k$ be strictly lower triangular matrices. Define the matrices $\tilde{U}_k$, $k = 1, 2, \ldots, \alpha$, such that $\tilde{A} = PA = \tilde{D} - \tilde{L}_k - \tilde{U}_k$. Assume that $\langle\tilde{A}\rangle = |\tilde{D}| - |\tilde{L}_k| - |\tilde{U}_k| = |\tilde{D}| - |\tilde{B}|$. If $|\alpha_{im} - \beta_i a_{im}| \leq t_i$, $i = 1, 2, \cdots, n$, and

$$0 < \omega_1, \omega_2 < \frac{2}{1 + \tilde{\rho}},\ 0 \leq r_1 \leq \omega_1,\ 0 \leq r_2 \leq \omega_2,\ 0 < \beta < \frac{2}{1 + \tilde{\theta}},$$

then GPUSAOR method converges for any initial vector x^0, where $\tilde{\rho} = \rho(\tilde{J}) = \rho(|\tilde{D}|^{-1}|\tilde{B}|)$ and

$$\tilde{\theta} = max\{|1 - \omega_1| + \omega_1\tilde{\rho},\ |1 - \omega_2| + \omega_2\tilde{\rho}\}.$$

Theorem 4.4. Let A be an H-matrix with unit diagonal elements, and for $k = 1, 2, \ldots, \alpha$, $\tilde{L}_k$ be strictly lower triangular matrices. Define the matrices $\tilde{U}_k$, $k = 1, 2, \ldots, \alpha$, such that $\tilde{A} = PA = \tilde{D} - \tilde{L}_k - \tilde{U}_k$. Assume that $\langle\tilde{A}\rangle = |\tilde{D}| - |\tilde{L}_k| - |\tilde{U}_k| = |\tilde{D}| - |\tilde{B}|$. If $|\alpha_{im} - \beta_i a_{im}| \leq t_i$, $i = 1, 2, \cdots, n$, and

$$0 < \omega_k, \xi_k < \frac{2}{1 + \tilde{\rho}},\ 0 \leq r_k \leq \omega_k,\ 0 \leq \eta_k \leq \xi_2,\ 0 < \beta < \frac{2}{1 + \tilde{\theta}'},$$

then GPNUSAOR method converges for any initial vector x^0, where $\tilde{\rho} = \rho(\tilde{J}) = \rho(|\tilde{D}|^{-1}|\tilde{B}|)$ and

$$\tilde{\theta}' = max\{|1 - \omega_k| + \omega_k\tilde{\rho},\ |1 - \xi_k| + \xi_k\tilde{\rho}\}.$$

The proofs of Theorem 4.2, Theorem 4.3 and Theorem 4.4 are similar to Lemma 3.7, Lemma 3.8 and Lemma 3.9, respectively. So omitted.

V. Comparison results of spectral radius

In what follows we will give some comparison results on the spectral radius of preconditioned parallel multisplitting relaxation USAOR iteration matrices with preconditioner P. Let

$$\begin{aligned}
\langle A\rangle &= \hat{M}_k - \hat{N}_k = \tfrac{1}{\omega_1}(|D| - r_1|L_k|) \\
&\quad - \tfrac{1}{\omega_1}[(1 - \omega_1)I + (\omega_1 - r_1)|L_k| + \omega_1|U_k|] \\
&= \hat{\hat{M}}_k - \hat{\hat{N}}_k = \tfrac{1}{\omega_2}(|D| - r_2|U_k|) \\
&\quad - \tfrac{1}{\omega_2}[(1 - \omega_2)I + (\omega_2 - r_2)|U_k| + \omega_2|L_k|],
\end{aligned}$$

where

$$\hat{M}_k = \frac{1}{\omega_1}(|D| - r_1|L_k|),$$

$$\hat{N}_k = \frac{1}{\omega_1}[(1 - \omega_1)|D| + (\omega_1 - r_1)|L_k| + \omega_1|U_k|],$$

and

$$\hat{\hat{M}}_k = \frac{1}{\omega_2}(|D| - r_2|U_k|),$$

$$\hat{\hat{N}}_k = \frac{1}{\omega_2}[(1 - \omega_2)|D| + (\omega_2 - r_2)|U_k| + \omega_2|L_k|],$$

and then the iteration matrix of local parallel multisplitting relaxation USAOR method for $\langle A\rangle$ is as follows

$$\hat{H}_{LUSAOR} = \sum_{k=1}^{\alpha} E_k \hat{\hat{M}}_k^{-1} \hat{\hat{N}}_k \hat{M}_k^{-1} \hat{N}_k.$$

Let

$$\begin{aligned}\langle PA\rangle &= \tilde{M}_k-\tilde{N}_k=\tfrac{1}{\omega_1}(|\tilde{D}|-r_1|\tilde{L}_k|)-\tfrac{1}{\omega_1}[(1-\omega_1)|\tilde{D}| \\ &\quad +(\omega_1-r_1)|\tilde{L}_k|+\omega_1|\tilde{U}_k|] \\ &= \tilde{\tilde{M}}_k-\tilde{\tilde{N}}_k=\tfrac{1}{\omega_2}(|\tilde{D}|-r_2|\tilde{U}_k|)-\tfrac{1}{\omega_2}[(1-\omega_2)|\tilde{D}| \\ &\quad +(\omega_2-r_2)|\tilde{U}_k|+\omega_2|\tilde{L}_k|],\end{aligned}$$

where

$$\tilde{M}_k=\frac{1}{\omega_1}(|\tilde{D}|-r_1|\tilde{L}_k|),$$

$$\tilde{N}_k=\frac{1}{\omega_1}[(1-\omega_1)|\tilde{D}|+(\omega_1-r_1)|\tilde{L}_k|+\omega_1|\tilde{U}_k|],$$

and

$$\tilde{\tilde{M}}_k=\frac{1}{\omega_2}(|\tilde{D}|-r_2|\tilde{U}_k|),$$

$$\tilde{\tilde{N}}_k=\frac{1}{\omega_2}[(1-\omega_2)|\tilde{D}|+(\omega_2-r_2)|\tilde{U}_k|+\omega_2|\tilde{L}_k|],$$

and then the iteration matrix of local preconditioned parallel multisplitting relaxation USAOR method for $\langle PA\rangle$ is as follows

$$\ddot{H}_{LPUSAOR}=\sum_{k=1}^{\alpha}E_k\tilde{\tilde{M}}_k^{-1}\tilde{\tilde{N}}_k\tilde{M}_k^{-1}\tilde{N}_k.$$

Theorem 5.1. Let A be an H-matrix with unit diagonal elements, and for $k=1,2,\ldots,\alpha$, $\tilde{L}_k$ and L_k be strictly lower triangular matrices. Define the matrices $\tilde{U}_k$ and U_k, $k=1,2,\ldots,\alpha$, such that $\tilde{A}=PA=\tilde{D}-\tilde{L}_k-\tilde{U}_k$ and $A=I-L_k-U_k$. Assume that $\langle\tilde{A}\rangle=\langle PA\rangle=|\tilde{D}|-|\tilde{L}_k|-|\tilde{U}_k|$ and $\langle A\rangle=I-|L_k|-|U_k|$. If $|\alpha_{im}-\beta_i a_{im}|\le t_i$, $i=1,2,\cdots,n$, and

$$0<\omega_1,\omega_2<1,\ 0\le r_1\le\omega_1,\ 0\le r_2\le\omega_2,$$

then $\rho(\tilde{H}_{LPUSAOR})\le\rho(\ddot{H}_{LPUSAOR})\le\rho(\hat{H}_{LUSAOR})$.

Proof. Since $\langle A\rangle$ is a nonsingular M-matrix, it is easy to show $\langle A\rangle=\hat{M}_k-\hat{N}_k=\hat{\hat{M}}_k-\hat{\hat{N}}_k$ are two weak regular splittings. From Lemma 3.10, the unique splitting $\langle A\rangle=\hat{B}_k-\hat{C}_k$ induced by $\hat{\hat{M}}_k^{-1}\hat{\hat{N}}_k\hat{M}_k^{-1}\hat{N}_k$ is weak regular splitting, where $\hat{B}_k=\hat{\hat{M}}(\hat{M}+\hat{\hat{M}}-\langle A\rangle)^{-1}\hat{M}$ and $\hat{C}_k=\hat{B}_k-\langle A\rangle$, and then the iteration matrix of LUSAOR method for $\langle A\rangle$ can be rewritten as $\hat{H}_{LUSAOR}=\sum_{k=1}^{\alpha}E_k\hat{B}_k^{-1}\hat{C}_k$.

By Theorem 4.1, $\langle PA\rangle$ is a nonsingular M-matrix, and then $\langle PA\rangle=\tilde{M}_k-\tilde{N}_k=\tilde{\tilde{M}}_k-\tilde{\tilde{N}}_k$ are two weak regular splittings. Similar to the above analysis, we have the unique splitting $\langle PA\rangle=\tilde{B}_k-\tilde{C}_k$ induced by $\tilde{\tilde{M}}_k^{-1}\tilde{\tilde{N}}_k\tilde{M}_k^{-1}\tilde{N}_k$, which is a weak regular splitting, where $\tilde{B}_k=\tilde{\tilde{M}}_k(\tilde{M}_k+\tilde{\tilde{M}}_k-\langle PA\rangle)^{-1}\tilde{M}_k$ and $\tilde{C}_k=\tilde{B}_k-\langle PA\rangle$. From

$$\tilde{L}_{\omega_1 r_1}(k)=(\tilde{D}-r_1\tilde{L}_k)^{-1}[(1-\omega_1)\tilde{D}+(\omega_1-r_1)\tilde{L}_k+\omega_1\tilde{U}_k],$$

we have

$$\begin{aligned}|\tilde{L}_{\omega_1 r_1}(k)| &= |(\tilde{D}-r_1\tilde{L}_k)^{-1}[(1-\omega_1)\tilde{D}+(\omega_1-r_1)\tilde{L}_k \\ &\quad +\omega_1\tilde{U}_k]| \\ &\le |(\tilde{D}-r_1\tilde{L}_k)^{-1}||(1-\omega_1)\tilde{D}+(\omega_1-r_1)\tilde{L}_k \\ &\quad +\omega_1\tilde{U}_k| \\ &\le |(\tilde{D}-r_1\tilde{L}_k)^{-1}||(1-\omega_1)\tilde{D}+(\omega_1-r_1)\tilde{L}_k \\ &\quad +\omega_1\tilde{U}_k| \\ &\le (|\tilde{D}|-r_1|\tilde{L}_k|)^{-1}|[(1-\omega_1)|\tilde{D}|+(\omega_1-r_1)|\tilde{L}_k| \\ &\quad +\omega_1|\tilde{U}_k|] \\ &= \tilde{M}_k^{-1}\tilde{N}_k.\end{aligned}$$

Similar to the above proving process, we have

$$\begin{aligned}|\tilde{U}_{\omega_2 r_2}(k)| &= |(\tilde{D}-r_2\tilde{U}_k)^{-1}[(1-\omega_2)\tilde{D}+(\omega_2-r_2)\tilde{U}_k \\ &\quad +\omega_2\tilde{L}_k]| \\ &\le \tilde{\tilde{M}}_k^{-1}\tilde{\tilde{N}}_k,\end{aligned}$$

and then

$$\begin{aligned}|\tilde{H}_{LPUSAOR}| &= |\sum_{k=1}^{\alpha}E_k\tilde{U}_{\omega_2 r_2}(k)\tilde{L}_{\omega_1 r_1}(k)| \\ &\le \sum_{k=1}^{\alpha}E_k|\tilde{U}_{\omega_2 r_2}(k)||\tilde{L}_{\omega_1 r_1}(k)| \\ &\le \sum_{k=1}^{\alpha}E_k\tilde{\tilde{M}}_k^{-1}\tilde{\tilde{N}}_k\tilde{M}_k^{-1}\tilde{N}_k \\ &= \sum_{k=1}^{\alpha}E_k\tilde{B}_k^{-1}\tilde{C}_k \\ &= \ddot{H}_{LPUSAOR}.\end{aligned} \tag{10}$$

Note that $\tilde{B}_k^{-1}\langle PA\rangle=I-\tilde{B}_k^{-1}\tilde{C}_k$ and $\sum_{k=1}^{\alpha}E_k\tilde{B}_k^{-1}\langle PA\rangle=I-\sum_{k=1}^{\alpha}E_k\tilde{B}_k^{-1}\tilde{C}_k=I-\ddot{H}_{LPUSAOR}$. Similarly, we have $\hat{B}_k^{-1}\langle A\rangle=I-\hat{B}_k^{-1}\hat{C}_k$ and $\sum_{k=1}^{\alpha}E_k\hat{B}_k^{-1}\langle A\rangle=I-\sum_{k=1}^{\alpha}E_k\hat{B}_k^{-1}\hat{C}_k=I-\hat{H}_{LUSAOR}$. From $\tilde{B}_k=\tilde{\tilde{M}}_k(\tilde{M}_k+\tilde{\tilde{M}}_k-\langle PA\rangle)^{-1}\tilde{M}_k$ and $\hat{B}_k=\hat{\hat{M}}_k(\hat{M}_k+\hat{\hat{M}}_k-\langle A\rangle)^{-1}\hat{M}$, we have

$$\tilde{B}_k^{-1}=\tilde{M}_k^{-1}(\tilde{M}_k+\tilde{\tilde{M}}_k-\langle PA\rangle)\tilde{\tilde{M}}_k^{-1},$$

and

$$\hat{B}_k^{-1}=\hat{M}_k^{-1}(\hat{M}_k+\hat{\hat{M}}_k-\langle A\rangle)\hat{\hat{M}}_k^{-1}.$$

Since

$$\langle PA\rangle=(I+|S|)\langle A\rangle,$$

by simple calculation, we have

$$\tilde{B}_k^{-1}\ge\hat{B}_k^{-1}\ge 0.$$

Let $x=\langle A\rangle^{-1}e>0$, then

$$(\langle PA\rangle-\langle A\rangle)x=(I+|S|)e>0,$$

and then

$$\begin{aligned}(\sum_{k=1}^{\alpha}E_k\tilde{B}_k^{-1}\langle PA\rangle)x &= (I-\ddot{H}_{LPUSAOR})x \\ &\ge (\sum_{k=1}^{\alpha}E_k\hat{B}_k^{-1}\langle A\rangle)x \\ &= (I-\hat{H}_{LUSAOR})x.\end{aligned}$$

Thus, it follows that

$$\|\ddot{H}_{LPUSAOR}\|_x\le\|\hat{H}_{LUSAOR}\|_x.$$

As $\hat{H}_{LUSAOR}$ is a nonnegative matrix, there exists a positive perron vector y. By Lemma 3.10, the following inequality holds:

$$\rho(\ddot{H}_{LPUSAOR})\le\rho(\hat{H}_{LUSAOR}).$$

From (10) and Lemma 3.3, we have

$$\rho(\tilde{H}_{LPUSAOR})\le\rho(|\tilde{H}_{LPUSAOR}|)\le\rho(\ddot{H}_{LPUSAOR}),$$

and then

$$\rho(\tilde{H}_{LPUSAOR})\le\rho(\ddot{H}_{LPUSAOR})\le\rho(\hat{H}_{LUSAOR}).$$

Using GPUSAOR method, we can also get the following results.

Theorem 5.2. Let A be an H-matrix with unit diagonal elements, and, for $k=1,2,\ldots,\alpha$, $\tilde{L}_k$ and L_k be strictly lower triangular matrices. Define the matrices $\tilde{U}_k$ and U_k, $k=1,2,\ldots,\alpha$, such that $\tilde{A}=PA=\tilde{D}-\tilde{L}_k-\tilde{U}_k$ and $A=I-L_k-U_k$. Assume that $\langle\tilde{A}\rangle=\langle PA\rangle=|\tilde{D}|-|\tilde{L}_k|-|\tilde{U}_k|$ and $\langle A\rangle=I-|L_k|-|U_k|$. If $|\alpha_{im}-\beta_i a_{im}|\le t_i$, $i=1,2,\cdots,n$, and

$$0<\omega_1,\omega_2<1,\ 0\le r_1\le\omega_1,\ 0\le r_2\le\omega_2,\ 0<\beta\le 1,$$

then $\rho(\tilde{H}_{GPUSAOR})\le\rho(\ddot{H}_{GPUSAOR})\le\rho(\hat{H}_{GUSAOR})$.

Proof. Since $\rho(H_{GBUSAOR})\le\rho(|H_{GBUSAOR}|)$, we only need to show

$$\rho(|\tilde{H}_{GPUSAOR}|)\le\rho(\ddot{H}_{GPUSAOR})\le\rho(\hat{H}_{GUSAOR}).$$

From (7), we have

$$\begin{array}{rcl}|\tilde{H}_{GPUSAOR}| &=& |\beta\tilde{H}_{LPUSAOR}+(1-\beta)I| \\ &\le& |\beta\tilde{H}_{LPUSAOR}|+|1-\beta|I.\end{array}$$

By Theorem 5.1, we know

$$|\tilde{H}_{LPUSAOR}|\le\ddot{H}_{LPUSAOR}\le\rho\hat{H}_{LUSAOR},$$

and together with $0<\beta\le 1$, we can obtain

$$\begin{array}{rcl}|\tilde{H}_{GPUSAOR}| &\le& |\beta\tilde{H}_{LPUSAOR}|+|1-\beta|I \\ &\le& \beta\ddot{H}_{LPUSAOR}+(1-\beta)I \\ &=& \ddot{H}_{GPUSAOR} \\ &\le& \beta\hat{H}_{LUSAOR}+(1-\beta)I \\ &=& \hat{H}_{GUSAOR},\end{array}$$

By Lemma 3.3, we have

$$\rho(|\tilde{H}_{GPUSAOR}|)\le\rho(\ddot{H}_{GPUSAOR})\le\rho(\hat{H}_{GUSAOR}),$$

and then

$$\rho(\tilde{H}_{GPUSAOR})\le\rho(\ddot{H}_{GPUSAOR})\le\rho(\hat{H}_{GUSAOR}).$$

Theorem 5.3. Let A be an H-matrix with unit diagonal elements, and, for $k=1,2,\ldots,\alpha$, $\tilde{L}_k$ and L_k be strictly lower triangular matrices. Define the matrices $\tilde{U}_k$ and U_k, $k=1,2,\ldots,\alpha$, such that $\tilde{A}=PA=\tilde{D}-\tilde{L}_k-\tilde{U}_k$ and $A=I-L_k-U_k$. Assume that $\langle\tilde{A}\rangle=\langle PA\rangle=|\tilde{D}|-|\tilde{L}_k|-|\tilde{U}_k|$ and $\langle A\rangle=I-|L_k|-|U_k|$. If $|\alpha_{im}-\beta_i a_{im}|\le t_i$, $i=1,2,\cdots,n$, and

$$0<\omega_k,\xi_k\le 1,\ 0\le r_k\le\omega_k,\ 0\le\eta_k\le\xi_2,\ 0<\beta\le 1,$$

then $\rho(\tilde{H}_{GPNUSAOR})\le\rho(\ddot{H}_{GPNUSAOR})\le\rho(\hat{H}_{GNUSAOR})$.

Proof. Similar to the proofs of Theorem 5.1 and Theorem 5.2, we can prove Theorem 5.3.

When A is an M-matrix, we can obtain the following Corollaries.

Corollary 5.1. Let A be an M-matrix with unit diagonal elements, and, for $k=1,2,\ldots,\alpha$, $\tilde{L}_k$ and L_k be strictly lower triangular matrices. Define the matrices $\tilde{U}_k$ and U_k, $k=1,2,\ldots,\alpha$, such that $\tilde{A}=PA=\tilde{D}-\tilde{L}_k-\tilde{U}_k$ and $A=I-L_k-U_k$. Assume that $\langle\tilde{A}\rangle=\langle PA\rangle=|\tilde{D}|-|\tilde{L}_k|-|\tilde{U}_k|$ and $A=I-|L_k|-|U_k|$. If $|\alpha_{im}-\beta_i a_{im}|\le t_i$, $i=1,2,\cdots,n$, and

$$0<\omega_1,\omega_2<1,\ 0\le r_1\le\omega_1,\ 0\le r_2\le\omega_2,$$

then $\rho(\tilde{H}_{LPUSAOR})\le\rho(\ddot{H}_{LPUSAOR})\le\rho(H_{LUSAOR})$.

Remark 1. Corollary 5.1 shows that local preconditioned parallel multisplitting relaxation USAOR method is better than local parallel multisplitting relaxation USAOR method for M-matrices linear systems.

Corollary 5.2. Let A be an M-matrix with unit diagonal elements, and, for $k=1,2,\ldots,\alpha$, $\tilde{L}_k$ and L_k be strictly lower triangular matrices. Define the matrices $\tilde{U}_k$ and U_k, $k=1,2,\ldots,\alpha$, such that $\tilde{A}=PA=\tilde{D}-\tilde{L}_k-\tilde{U}_k$ and $A=I-L_k-U_k$. Assume that $\langle\tilde{A}\rangle=\langle PA\rangle=|\tilde{D}|-|\tilde{L}_k|-|\tilde{U}_k|$ and $A=I-|L_k|-|U_k|$. If $|\alpha_{im}-\beta_i a_{im}|\le t_i$, $i=1,2,\cdots,n$, and

$$0<\omega_1,\omega_2<1,\ 0\le r_1\le\omega_1,\ 0\le r_2\le\omega_2,\ 0<\beta\le 1,$$

then $\rho(\tilde{H}_{GPUSAOR})\le\rho(\ddot{H}_{GPUSAOR})\le\rho(H_{GUSAOR})$.

Remark 2. Corollary 5.2 shows that global preconditioned parallel multisplitting relaxation USAOR method is better than global parallel multisplitting relaxation USAOR method for M-matrices linear systems.

Corollary 5.3. Let A be an M-matrix with unit diagonal elements, and, for $k=1,2,\ldots,\alpha$, $\tilde{L}_k$ and L_k be strictly lower triangular matrices. Define the matrices $\tilde{U}_k$ and U_k, $k=1,2,\ldots,\alpha$, such that $\tilde{A}=PA=\tilde{D}-\tilde{L}_k-\tilde{U}_k$ and $A=I-L_k-U_k$. Assume that $\langle\tilde{A}\rangle=\langle PA\rangle=|\tilde{D}|-|\tilde{L}_k|-|\tilde{U}_k|$ and $A=I-|L_k|-|U_k|$. If $|\alpha_{im}-\beta_i a_{im}|\le t_i$, $i=1,2,\cdots,n$, and

$$0<\omega_k,\xi_k\le 1,\ 0\le r_k\le\omega_k,\ 0\le\eta_k\le\xi_2,\ 0<\beta\le 1,$$

then $\rho(\tilde{H}_{GPNUSAOR})\le\rho(\ddot{H}_{GPNUSAOR})\le\rho(H_{GNUSAOR})$.

Remark 3. Corollary 5.3 shows that global preconditioned nonstationary parallel multisplitting relaxation USAOR method is better than global nonstationary parallel multisplitting relaxation USAOR method for M-matrices linear systems.

VI. Numerical example

In this section, we present some numerical examples which compare the performance of our method (GP-NUSAOR) with global non-stationary parallel multisplitting relaxation USAOR method by considering the linear system [1,4]

$$Ax=b, \tag{11}$$

where

$$A=\begin{pmatrix}1 & -\frac{1}{4} & & & \\ -\frac{1}{4} & 1 & -\frac{1}{4} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{4} & 1 & -\frac{1}{4} \\ & & & -\frac{1}{4} & 1\end{pmatrix},$$

and the right hand side vector b is chosen as

$$b^T=(1,\frac{1}{4},\ldots,\frac{1}{n^2}).$$

We take

$$P=\begin{pmatrix}1 & \alpha_{12}+\frac{1}{4}\beta_1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \alpha_{n-1,n}+\frac{1}{4}\beta_{n-1} \\ 0 & \cdots & \alpha_{n,n-1}+\frac{1}{4}\beta_n & 1\end{pmatrix},$$

and $r=\langle A\rangle^{-1}e$, where $e=(1,1,\ldots,1)^T$, then $t_i=\frac{1}{2r_m-1}$, α_i and β_i meet the inequality $|\alpha_i+\frac{1}{4}\beta_i|\le t_i$, $(i=1,2,\ldots,n)$.

Take $\alpha=2$, $q(m,1)=2$, $q(m,2)=1$ and $\alpha=2$, $q(m,1)=3$, $q(m,2)=2$, respectively.

$$J_1=\{1,2,\ldots,m_1\},\quad J_2=\{m_2,m_2+1,\ldots,n\},$$

with two positive integers m_1 and m_2 satisfying $1<m_2<m_1<n$. We determine $(I-L_k,U_k,E_k)$ and $(I-U_k,L_k,E_k)$, $k=1,2$, of the matrix A in accordance with the following way:

$$L_1=(\mathscr{L}_{ij}^{(1)}),\ \mathscr{L}_{ij}^{(1)}=\begin{cases}1 & \text{for } j=i-1 \text{ and } 2\le i\le m_1,\\ 0 & \text{otherwise,}\end{cases}$$

$$U_1=(\mathscr{U}_{ij}^{(1)}),\ \mathscr{U}_{ij}^{(1)}=\begin{cases}1 & \text{for } j=i-1 \text{ and } m_1+1\le i\le n,\\ 1 & \text{for } j=i+1 \text{ and } 1\le i\le n-1,\\ 0 & \text{otherwise,}\end{cases}$$

$$L_2=(\mathscr{L}_{ij}^{(2)}),\ \mathscr{L}_{ij}^{(2)}=\begin{cases}1 & \text{for } j=i-1 \text{ and } m_2\le i\le n,\\ 0 & \text{otherwise,}\end{cases}$$

$$U_2=(\mathscr{U}_{ij}^{(2)}),\ \mathscr{U}_{ij}^{(2)}=\begin{cases}1 & \text{for } j=i-1 \text{ and } 2\le i\le m_2-1,\\ 1 & \text{for } j=i+1 \text{ and } 1\le i\le n-1,\\ 0 & \text{otherwise,}\end{cases}$$

$$E_k=\ \text{diag}\ (E_{11}^{(k)},E_{22}^{(k)}\ldots,E_{nn}^{(k)}),\ k=1,2,$$

$$E_{ii}^{(1)}=\begin{cases}1 & \text{for } 1\le i\le m_2,\\ \frac{1}{2} & \text{for } m_2\le i\le m_1,\\ 0 & \text{for } m_1<i\le n.\end{cases}\qquad(12)$$

$$E_{ii}^{(2)}=\begin{cases}0 & \text{for } \le i\le m_2,\\ \frac{1}{2} & \text{for } m_2\le i\le m_1,\\ 1 & \text{for } m_1<i\le n.\end{cases}\qquad(13)$$

Let $\tilde{A}=PA=(\tilde{a}_{ij})$, we determine $(\tilde{D}-\tilde{L}_k,\tilde{U}_k,E_k)$ and $(\tilde{D}-\tilde{U}_k,\tilde{L}_k,E_k)$, $k=1,2$, of the matrix PA in accordance with the following way:

$$\tilde{D}\ =\ \text{diag}(\tilde{a}_{11},\tilde{a}_{22},\ldots,\tilde{a}_{nn}),$$

$$\tilde{L}_1=(\tilde{\mathscr{L}}_{ij}^{(1)}),\ \tilde{\mathscr{L}}_{ij}^{(1)}=\begin{cases}\tilde{a}_{ij} & \text{for } j=i-1 \text{ and } 2\le i\le m_1,\\ 0 & \text{otherwise,}\end{cases}$$

$$\tilde{U}_1=(\tilde{\mathscr{U}}_{ij}^{(1)}),\ \tilde{\mathscr{U}}_{ij}^{(1)}=\begin{cases}\tilde{a}_{ij} & \text{for } j=i-1 \text{ and } m_1+1\le i\le n,\\ \tilde{a}_{ij} & \text{for } j=i+1 \text{ and } 1\le i\le n-1,\\ 0 & \text{otherwise,}\end{cases}$$

$$\tilde{L}_2=(\tilde{\mathscr{L}}_{ij}^{(2)}),\ \tilde{\mathscr{L}}_{ij}^{(2)}=\begin{cases}\tilde{a}_{ij} & \text{for } j=i-1 \text{ and } m_2\le i\le n,\\ 0 & \text{otherwise,}\end{cases}$$

$$\tilde{U}_2=(\tilde{\mathscr{U}}_{ij}^{(2)}),\ \tilde{\mathscr{U}}_{ij}^{(2)}=\begin{cases}\tilde{a}_{ij} & \text{for } j=i-1 \text{ and } 2\le i\le m_2-1,\\ \tilde{a}_{ij} & \text{for } j=i+1 \text{ and } 1\le i\le n-1,\\ 0 & \text{otherwise,}\end{cases}$$

$$E_k=\ \text{diag}\ (E_{11}^{(k)},E_{22}^{(k)}\ldots,E_{nn}^{(k)}),\ k=1,2,$$

are same as in (12) and (13). In particular, we select the positive integer pair (m_1,m_2) to be $m_1=\text{Int}(\frac{4n}{5})$, $m_2=\text{Int}(\frac{n}{5})$, and then we can get two kinds of concrete cases of the weighting matrices E_1 and E_2, here, $\text{Int}(\cdot)$ denotes the integer part of the corresponding real number.

For convenience, we assume that $\omega_k=\omega_2$, $r_k=r_2$, $\xi_k=\omega_1$ and $\eta_k=r_1$, and let $\rho(\cdot)$ denote the spectral radius of the corresponding iteration matrices. we take $\alpha_{12}=\alpha_{23}=\ldots=\alpha_{n-1,n}=\alpha_{n,n-1}=0.25$ and $\beta_i=0.3333$, $i=1,2,\ldots,n$, namely

$$P=\begin{pmatrix}1 & \frac{1}{3} & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & 1 & \frac{1}{3}\\ 0 & \cdots & \frac{1}{3} & 1\end{pmatrix}.$$

In the following, we make three groups of experiments. In Figure 1, we test the relation between ρ and r, when $n=100$, $\omega_1=\omega_2=1$, $r_1=r_2$, $\alpha=2$, $q(m,1)=3$, $q(m,2)=2$ and $\beta=1$, where "$\circ$", "$+$" and "$*$" denotes the spectral radius of PA, $\langle PA\rangle$ and A, respectively. In Table I and Table II, we report the spectral radius of iteration matrices with GPNUSAOR and GNUSAOR. The meaning of notations $\rho(\hat{H})$, $\rho(\ddot{H})$ and $\rho(H)$ denotes the spectral radius of PA, $\langle PA\rangle$ and A, respectively.

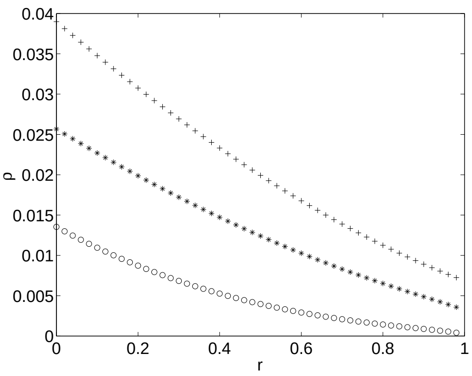

Fig. 1. The Relation Between ρ and r, When $n=100$, $\omega_1=\omega_2=1$.

From Figure 1, Table I and Table II, we easily see that the preconditioned multisplitting relaxation USAOR methods discussed in this paper substantially have better numerical behaviours than the multisplitting relaxation USAOR methods studied in [9], which shows that our new methods are applicable and efficient.

TABLE I
COMPARISON OF SPECTRAL RADIUS WHEN $\alpha=2$, $q(m,1)=2$, $q(m,2)=1$

n	ω_1,r_1	ω_2,r_2	β	$\rho(\hat{H})$	$\rho(\ddot{H})$	$\rho(H)$
50	0.8,0.4	0.7,0.6	0.5	0.0098	0.0211	0.0542
50	1.0,0.6	0.9,0.6	0.8	0.0074	0.0172	0.0437
50	1.0,0.8	1.0,0.7	1	0.0059	0.0154	0.0286
100	0.8,0.4	0.7,0.6	0.5	0.0102	0.0255	0.0563
100	1.0,0.6	0.9,0.6	0.8	0.0085	0.0179	0.0475
100	1.0,0.8	1.0,0.7	1	0.0067	0.0172	0.0304
1000	0.8,0.4	0.7,0.6	0.5	0.0157	0.0312	0.0687
1000	1.0,0.6	0.9,0.6	0.8	0.0105	0.0234	0.0551
1000	1.0,0.8	1.0,0.7	1	0.0089	0.0202	0.0456

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TABLE II
COMPARISON OF SPECTRAL RADIUS WHEN $\alpha = 2$, $q(m,1) = 3$, $q(m,2) = 2$

n	ω_1, r_1	ω_2, r_2	β	$\rho(\tilde{H})$	$\rho(\ddot{H})$	$\rho(H)$
50	0.8,0.4	0.7,0.6	0.5	0.0025	0.0123	0.0354
50	1.0,0.6	0.9,0.6	0.8	0.0021	0.0111	0.0298
50	1.0,0.8	1.0,0.7	1	0.0018	0.0095	0.0213
100	0.8,0.4	0.7,0.6	0.5	0.0028	0.0145	0.0388
100	1.0,0.6	0.9,0.6	0.8	0.0026	0.0140	0.0364
100	1.0,0.8	1.0,0.7	1	0.0025	0.0136	0.0325
1000	0.8,0.4	0.7,0.6	0.5	0.0030	0.0151	0.0407
1000	1.0,0.6	0.9,0.6	0.8	0.0029	0.0146	0.0390
1000	1.0,0.8	1.0,0.7	1	0.0027	0.0142	0.0380

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