On Solutions of a System of Wiener-Hopf Integral Equations

Amir T. Payandeh Najafabadi and Dan Z. Kucerovsky

***Abstract*—Consider the problem of solving a system of Wiener-Hopf integral equations**

$$\lambda_i g_i(x) - \sum_{j=1}^{n} \int_0^\infty g_j(\theta) k_{ij}(x-\theta) d\theta = f_i(x), \textbf{and } x \geq 0,$$

where for $i = 1, 2, \cdots, n$, $\lambda_i \in \mathbb{R}$, and $k_{ij}(\cdot)$ and $f_i(\cdot)$ are given functions and $g_j(\cdot)$ are to be determined. This article provides solutions for such system of Wiener-Hopf integral equations.

***Index Terms*—Wiener-Hopf integral equations; Hölder condition; Fourier transform; Convolution theorem; Shannon sampling theorem.**

I. INTRODUCTION

CONSIDER the problem of solving a system of Wiener-Hopf integral equations

$$\lambda_i g_i(x) - \sum_{j=1}^{n} \int_0^\infty g_j(\theta) k_{ij}(x-\theta) d\theta \quad = \quad f_i(x), \qquad (1)$$

where $x \geq 0$, $i = 1, 2, \cdots, n$, $\lambda_i \in \mathbb{R}$, $g_j(\cdot)$ are to be determined, and $k_{ij}(\cdot)$ and $f_i(\cdot)$ are given functions that: **(i)** go to zero faster than some power; **(ii)** satisfy $k_{ii}(-x) = k_{ii}(x)$; and **(iii)** the Fourier transform of k_{ij}, say $\hat{k}_{ij}$, satisfy $\hat{k}_{ij} \equiv \bar{\hat{k}}_{ji}$, for all $i, j = 1, 2, \cdots, n$; and where $\bar{k}_{ji}$ stands for the conjugate of k_{ji}.

Solving a system of integral equations is a practical mathematical problem which studied by several authors. For instance [1] employed an integral form of the method of moving planes to study positive solutions of the following system of integral equations in $\mathbb{R}^n$

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} |x-y|^{\alpha-n} v^q(y) dy \\ v(x) &= \int_{\mathbb{R}^n} |x-y|^{\alpha-n} u^q(y) dy, \end{aligned}$$

where $(q+1)^{-1}+(q+1)^{-1} = 1-\alpha/n$, $u \in L_{p+1}(\mathbb{R}^n)$ and $v \in L_{q+1}(\mathbb{R}^n)$. [2] used a Taylor-series expansion method to solve a second kind Fredholm integral equations system with smooth or weakly singular kernels. [3] implemented Adomian-Pade (Modified Adomian-Pade) technique along with the Pade approximation to solve linear and nonlinear systems of Volterra functional equations. [4] based upon the calculus of variations solved a class of linear and nonlinear system of Volterra integral equations of the first and the second kinds. [5] derived solutions (and some asymptotic properties of solutions) of a singularly perturbed nonlinear system fractional integral equations. [6] employed the Taylor collocation method to approximate solutions of a system of Volterra-Fredholm integral equations in terms of Taylor polynomials. [7] studied application of Laplace and inverse Laplace transforms to approximate solutions of a system of Volterra integral equations of the first kind with highly oscillatory Bessel kernels. [8] compared two well known variational iteration and modified variational iteration methods for approximating solution of a system of the first kind Volterra integral equations. [9] used a hybrid functional approximation method to solve a system of nonlinear mixed Volterra-Fredholm integral equations. [10] employed a generalized Single-Term Walsh Series method to solve systems of linear Volterra integral equations of the second kind. [11] introduced two direct quadrature methods based on linear rational interpolation for solving general system of the second kind Volterra integral equations.

This article employs the matrix Riemann-Hilbert problem along with the well-known Shannon sampling theorem to provide an exact solutions for a class of system of Wiener-Hopf integral equations. Section 2 collects some useful elements for other sections. Exact solutions for such system of Wiener-Hopf integral equations accompanied with an error estimate for the case of approximate solutions and some real examples are given in Section 3. Section 4 reviews the findings and discusses a situation where the given functions in the corresponding matrix Riemann-Hilbert problem are non-exponential-type functions.

II. PRELIMINARIES

Now, we collect some useful elements for the rest of this article.

Definition 1. A function f in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is said to be an exponential type T function on the domain $D \subset \mathbb{C}$ if there are positive constants M and T such that $|f(\omega)| \leq M \exp\{T|\omega|\}$, for $\omega \in D$. An $n \times m$ matrix function $\mathbf{f}$ is said to be of exponential type T in a domain D if its components are of exponential type T or better.

The following lemma collects some useful properties of a matrix function.

Lemma 1. Suppose $g(x)$ stands for the pointwise operator norm of a matrix function $\mathbf{F} := [f_{ij}]$, i.e., $g(x) := \|[f_{ij}(x)]\|$. Then,

(a) $|f_{ij}| \leq g(x)$ for all coefficient functions f_{ij};
(b) $g(x)$ is an exponential type function if and only if $\mathbf{F}$ is an exponential type matrix-valued function.

Proof. For part (a), suppose E_{ij} denote the elementary matrix units, which satisfy $E_{ij}\mathbf{F}E_{jk} = E_{ik}f_{ik}$. Now observe

Manuscript received August, 29, 2013; revised April, 17, 2014. This work was supported in part by Shahid Beheshti University and Natural Sciences and Engineering Research Council (NSERC) of Canada.

A. T. Payandeh is with Department of Mathematical Sciences, Shahid Beheshti University, G.C. Evin, 1983963113, Tehran, Iran (Email address: amirtpayandeh@sbu.ac.ir).

D. Z. Kucerovsky is with Department of Mathematics and Statistics, University of New Brunswick, Fredericton, N.B. CANADA E3B 5A3 (Email address: dkucerov@unb.ca).

that

$$\begin{aligned}|f_{ik}| &= ||E_{ik}f_{ik}|| \\ &= ||E_{ij}\mathbf{F}E_{jk}|| \\ &\leq g(x),\end{aligned}$$

where the last inequality comes from the submultiplicative property of the operator norm. Proof of part (b) comes by an application of part (a). □

The well known Paley-Wiener theorem states that the Fourier transform of an $L_2(\mathbb{R})$ function vanishes outside of an interval $[-T, T]$, if and only if the function is an exponential type T, see [12] for more detail. The Fourier transforms of exponential type functions are continuous functions which are infinitely differentiable everywhere and are given by a Taylor series expansion over every compact interval, see [13] and [14]. These functions are also called band-limited functions, see [15] for more details about band-limited functions (which are equivalent to exponential-type functions under the Fourier transform by the above stated Paley-Wiener theorem).

Lemma 2. Suppose h is a bounded function which goes to zero faster than some power (i.e., $h(\omega) = o(|\omega|^{-\alpha})$, for some positive α, as $|\omega| \to \infty$). Then, the Fourier transform of h satisfies the Hölder condition on $\mathbb{C}$ for some positive exponent λ.

Proof. Suppose $\hat{h}$ stands for the Fourier transform of h. Now, observe that

$$\begin{aligned}|\hat{h}(\omega_2) - \hat{h}(\omega_1)| &\leq \int_{-\infty}^{\infty} |e^{it\omega_2} - e^{it\omega_1}|\,|h(x)|dx \\ &\leq a|\omega_2 - \omega_1|^{\lambda} \int_{-\infty}^{\infty} |x^{\lambda}h(x)|dx \\ &\leq aM|\omega_2 - \omega_1|^{\lambda},\end{aligned}$$

where the second inequality comes from the fact that $e^{it\omega}$ satisfies the Hölder condition and the third inequality comes from the fact that h is bounded function that goes to zero faster than some power. □

Lemma 3. Suppose h satisfies the Hölder condition on a disk around 0 in the complex plane. Then, h and $\ln(h)$ are exponential-type functions.

Proof. Since h satisfies the Hölder condition, one may conclude that

$$\begin{aligned}|h(\omega)| &\leq |h(\omega) - h(0)| + |h(0)| \\ &\leq a|\omega|^{\lambda} + M \\ &= ae^{\lambda \ln(|\omega|)} + M \\ &\leq (a + M)e^{\lambda|\omega|},\end{aligned}$$

where the last inequality comes from the fact that $\ln(|\omega|) \leq |\omega| - 1 < |\omega|$ and that since λ is positive $e^{\lambda|\omega|} \geq 1$. However, $|\ln(h(\omega))| \leq |h(\omega)|$, for all $\omega \in \mathbb{C}$. □

Definition 2. A complex-valued square matrix function $M(t)$ is Hermitian, whenever its conjugate transpose, say $M^*(t)$, is equal to $M(t)$.

Hermitian matrices can be understood as an analogue of real symmetric matrices (see [16]). Moreover, a Hermitian matrix $M(t)$ is non-degenerate on a smooth oriented curve Γ if and only if $det(M(t)) \neq 0$, for all $t \in \Gamma$ (see [17]). [18] and [19], as well as others, showed that all the partial indices (see below for definition) of a Hermitian matrix function are zero.

Lemma 4. (Payandeh & Kucerovsky, 2014) Suppose $\mathbf{g}\colon \mathbb{R} \longrightarrow M_n(\mathbb{C})$ be a Hermitian matrix function with either $\mathbf{g}(-x) = \mathbf{g}(x)$ or $\mathbf{g}(-x) = -\mathbf{g}(x)$. Then, the (inverse) Fourier transform of $\mathbf{g}$ is a scalar multiple of a Hermitian matrix function.

Henceforth, logarithms and exponentials of matrix-valued functions are defined by the resolvent functional calculus from operator theory. In general, such logarithms and exponentials do not coincide with the componentwise logarithms and exponentials, except of course in the case of 1-by-1 matrices.

Lemma 5. Suppose matrix function $\mathbf{G}$ is a Hermitian matrix at every point. Moreover, suppose that the spectrum of $\mathbf{G}(x)$ is bounded below by some fixed positive real number, say a. Then,

(a) the operator logarithm of $\mathbf{G}$ exists;
(b) the logarithm of $\mathbf{G}(x)$ is an exponential type function, whenever $\mathbf{G}(x)$ is an exponential type function.

Proof. For part (a), we take a contour that encloses the spectrum of $\mathbf{G}$ but does not enclose zero. For part (b), suppose, $\lambda(x)$ and $\mu(x)$, respectively, represent the largest and the smallest eigenvalues of $\mathbf{G}(x)$. The largest and smallest eigenvalues of the Hermitian matrix function $\ln(\mathbf{G}(x))$ are then $\ln(\lambda(x))$ and $\ln(\mu(x))$. For Hermitian matrix functions, the pointwise operator norm can be given in terms of the spectrum at each point. In particular, then, $\|\ln(\mathbf{G}(x))\|$ is bounded by $\max\{|\ln(\lambda(x))|, |\ln(\mu(x))|\}$. But, since $\lambda(x)$ is equal to the pointwise operator norm of $\mathbf{G}$, which is an exponential-type, and $\mu(x) \geq a > 0$ (From part a). One may conclude that $\max\{|\ln(\lambda(x))|, |\ln(\mu(x))|\}$ is certainly an exponential-type. Then the pointwise operator norm of $\ln(\mathbf{G}(x))$ is of exponential-type and hence by part (b) of Lemma 1 the matrix function $\ln(\mathbf{G}(x))$ is of exponential-type. □

A matrix Riemann-Hilbert problem is, in general, far more complicated than a scalar Riemann-Hilbert problems. It is the function-theoretical problem of finding a vector of functions $\mathbf{\Phi}$ which are sectionally analytic, bounded, and having a prescribed jump discontinuity on $\mathbb{R}$, i.e.,

$$\mathbf{\Phi}_+(\omega) = \mathbf{G}(\omega)\mathbf{\Phi}_-(\omega) + \mathbf{F}(\omega), \quad \text{for } \omega \in \mathbb{R}, \tag{2}$$

where $\mathbf{\Phi}_+$ and $\mathbf{\Phi}_-$ are one-sided limits of $\mathbf{\Phi}$ at the discontinuities, the kernel $\mathbf{G}$ and the nonhomogeneous vector $\mathbf{F}$ are given complex-valued and continuous matrix functions whose elements satisfy a Hölder condition on $\mathbb{R}$ and $\det \mathbf{G}(t)$ does not vanish on $\mathbb{R}$.

The continuity and non-vanishing properties are quite restrictive conditions. In some cases, the Riemann-Hilbert problem can be extended to handle cases with vanishing $\mathbf{G}$ or jump discontinuities of $\mathbf{F}$, see [20] for more detail.

Computing the partial indices of a matrix Riemann-Hilbert problem is usually a key step in determining the existence and number of solutions of a matrix Riemann-Hilbert problem. The index of a complex-valued smooth scalar kernel G on a smooth oriented curve Γ is defined to be the winding number of $G(\Gamma)$ about the origin. In contrast to the case of a scalar kernel, the indices of a matrix kernel apparently cannot be completely determined by any *a priori* investigation. To determine such indices, one must solve the corresponding homogeneous Riemann-Hilbert problem. Thus, to evaluate the partial indices of an n-by-n matrix kernel $\mathbf{G}$ on a smooth oriented curve Γ, one has to find a fundamental solution matrix $\mathbf{X}$ that is componentwise sectionally analytical in the upper and lower complex half-planes, and such that the lower and upper radial limits, say $\mathbf{X}_{\pm}$, respectively, satisfy $\mathbf{G}(t) = \mathbf{X}_{+}(t)\mathbf{X}_{-}^{-1}(t)$, for all $t \in \Gamma$. Then, the partial indices $\kappa_1 \cdots, \kappa_n$ are defined by investigating the behavior of $X_{11}(t), \cdots, X_{nn}(t)$ at infinity. Some limited *a priori* information on partial indices can be found using the fact that $\sum_{i=1}^{n} \kappa_i = \text{ind}_{\Gamma} \det(\mathbf{G}(t))$, so that we may at least easily find the sum of the partial indices: more detail can be found in [21]. We are primarily interested in the matrix Riemann-Hilbert problems with a Hermitian kernel whose partial indices are all zero (see [18] and [19]).

Solutions of a matrix Riemann-Hilbert problem whose kernel has a so-called commutative factorization (a commutative kernel) can be restated in term of the Sokhotskyi-Plemelj integrals, see [22], [23], and [24] for more details. Unfortunately, the majority of matrix Riemann-Hilbert problems which appear in practical problems simply do not have commutative factorizations. Using the Shanonn sampling theorem along with *Carlemann's method* [25] provided exact solution for a wide class of matrix Riemann-Hilbert problems. Since, commutative factorization is not required for Payandeh & Kucerovsky's method, one may employ their method for a wide class of practical problems. The following two theorems represent their findings.

Theorem 1. (Payandeh & Kucerovsky, 2014) Suppose all the partial indices of matrix Riemann-Hilbert problem 2 are zero. Moreover, suppose $\ln(\mathbf{G})$ and $\mathbf{F}$, in the matrix Riemann-Hilbert problem 2 are two exponential-type T and T^* matrix functions. Then, unique solution of the matrix Riemann-Hilbert problem 2 can be explicitly determined by

$$\mathbf{\Phi}_{\pm}(\omega) = \pm\exp\left\{\pm\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(\frac{2n}{T}))\frac{e^{\pm i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\} \times\left[\sum_{m=-\infty}^{\infty}\mathbf{H}(\frac{2m}{T^*})\frac{e^{\pm i\pi(T^*\omega-2m)}-1}{2i\pi(T^*\omega-2m)}\}\right],$$

where

$$\mathbf{H}(\omega) := \exp\{-\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(\frac{2n}{T}))\frac{e^{i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\}\times\mathbf{F}(\omega).$$

The following corollary represents [25]'s results under some weak conditions.

Corollary 1. Suppose $\mathbf{G}$ and $\mathbf{F}$ in matrix Riemann-Hilbert problem 2 are given, Hermitian, and exponential type T function and T^* matrix functions. Moreover, suppose also that the spectrum of $\mathbf{G}(x)$ is uniformly bounded away from zero. Then, unique solution of the matrix Riemann-Hilbert problem 2 can be explicitly determined by

$$\mathbf{\Phi}_{\pm}(\omega) = \pm\exp\left\{\pm\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(\frac{2n}{T}))\frac{e^{\pm i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\} \times\left[\sum_{m=-\infty}^{\infty}\mathbf{H}(\frac{2m}{T^*})\frac{e^{\pm i\pi(T^*\omega-2m)}-1}{2i\pi(T^*\omega-2m)}\}\right],$$

where

$$\mathbf{H}(\omega) := \exp\{-\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(\frac{2n}{T}))\frac{e^{i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\}\times\mathbf{F}(\omega).$$

Proof. Since $\mathbf{G}$ and $\mathbf{F}$ are two Hermitian matrices, from [18] and [19]'s result, one may conclude that: all the partial indices of matrix Riemann-Hilbert problem 2 are zero. From By part (b) of Lemma 1 and Lemma 5 observe that $\ln(\mathbf{G})$ is an exponential-type function. Now, an application of Theorem 1 leads to the desired conclusion. □

The above corollary extends [25]'s result to any matrix Riemann-Hilbert problem that: (**1**) The operator norm of its corresponding kernel and inhomogeneous matrix functions, say $\mathbf{G}$ and $\mathbf{F}$, respectively, are exponential-type matrices; (**2**) $\mathbf{G}$ and $\mathbf{F}$ are Hermitian matrices; and (**3**) The spectrum of $\mathbf{G}$, at each point, is uniformly bounded below by some positive real number a.

The following theorem gives the error bound for our approximate solutions to matrix Riemann-Hilbert problem 2.

Theorem 2. (Payandeh & Kucerovsky, 2014) Suppose $\mathbf{G}$ and $\mathbf{F}$ in matrix Riemann-Hilbert problem 2 are given, Hermitian, exponential-type (T and T^*, respectively) matrix functions, and the spectrum of $\mathbf{G}(x)$ is uniformly bounded away from zero. Moreover, suppose $\mathbf{G}^{(m)}$ and $\mathbf{F}^{(m)}$ are two sequence of Hermitian, exponential-type (T and T^*, respectively) matrix functions, and the spectrum of $\mathbf{G}^{(m)}(x)$ is uniformly bounded away from zero. Then, the approximate solutions of matrix Riemann-Hilbert problem (2) can be explicitly given as:

$$\mathbf{\Phi}_{\pm}^{(m)}(\omega) = \pm\exp\left\{\pm\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}^{(m)}(\frac{2n}{T}))\frac{e^{\pm i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\} \times\left[\sum_{k=-\infty}^{\infty}\mathbf{H}^{(m)}(\frac{2k}{T^*})\frac{e^{\pm i\pi(T^*\omega-2k)}-1}{2i\pi(T^*\omega-2k)}\}\right],$$

where

$$\mathbf{H}^{(m)}(\omega) := \exp\{-\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}^{(m)}(\frac{2n}{T}))\frac{e^{i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\} \times\mathbf{F}^{(m)}(\omega);$$

and satisfy the error bound

$$|\mathbf{\Phi}_{\pm}^{(m)}-\mathbf{\Phi}_{\pm}| \leq \|\ln(\mathbf{G}^{(m)})-\ln(\mathbf{G})\|\|\mathbf{H}^{(m)}-\mathbf{H}\|,$$

where the norm is defined by $\|M\| := \sup_{ij}\left\{\int_{-\infty}^{\infty}|M_{ij}(x)|^2\,dx\right\}^{1/2}$.

III. MAIN RESULTS

Now consider solving the system of Wiener-Hopf integral equation given by (1) in functional vector $\mathbf{g} = (g_1, g_2 \cdots, g_n)'$.

Lemma 6. The system of integral Equation (1) can be converted to the following matrix Riemann-Hilbert problem.

$$\mathbf{G}(\omega)\mathbf{\Phi}_{-}(\omega) = \mathbf{\Phi}_{+}(\omega)+\mathbf{F}(\omega), \omega\in\mathbb{R}, \quad (3)$$

where $\mathbf{F}$ is the Fourier transform of vector function $\mathbf{f}$ and elements of the kernel matrix $\mathbf{G} = [g_{ij}]_{n\times n}$ are $g_{ij} = -\hat{k}_{ij}$ and $g_{ii} = \lambda_i - \hat{k}_{ii}$, for $i \neq j = 1, 2, \cdots, n$.

Proof. For $x \leq 0$, an unknown vector function $\mathbf{h} = (h_1, h_2, \cdots, h_n)'$ can be defined as $h_i(x) := \lambda_i g_i(x) - \sum_{j=1}^{n}\int_0^{\infty} g_j(x)k_{ij}(x-\theta)d\theta$, for $i = 1, 2, \cdots, n$. By adding in such an unknown vector function $\mathbf{h}$ to integral equation 1, each elements of integral equation 1 can be extended to the whole of the real line $\mathbb{R}$ as

$$\lambda_i g_i^*(x) - \sum_{j=1}^{n}\int_0^{\infty} g_j^*(x)k_{ij}(x-\theta)d\theta = f^*(x)+h^*(x),$$

where $(g_i^*(x), f_i^*(x), h_i^*(x)) = (0,0,h(x))I_{(-\infty,0)}(x) + (g_i(x), f_i(x), 0)I_{[0,\infty)}(x)$, for $x \in \mathbb{R}$, and $i = 1, 2, \cdots, n$. The Fourier transform from both sides of the above extended integral equations along with the convolution theorem and the Paley-Wiener theorem leads to desired results. □

Theorem 3. Solution of the integral equation 1 is given by the inverse Fourier transform of

$$\mathbf{\Phi}_-(\omega) = -\exp\left\{-\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(\frac{2n}{T}))\frac{e^{-i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\} \times\left[\sum_{m=-\infty}^{\infty}\mathbf{H}(\frac{2m}{T^*})\frac{e^{-i\pi(T\omega-2m)}-1}{2i\pi(T\omega-2m)}\right],$$

where

$$\mathbf{H}(\omega) := \exp\{-\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(\frac{2n}{T}))\frac{e^{i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\}\times\mathbf{F}(\omega).$$

Proof. Using Lemma 6, one may restate the system of Wiener-Hopf integral equation (1) as the matrix Riemann-Hilbert problem (3). Our conditions on the k_{ij} functions imply that the kernel $\mathbf{G}(\omega)$ is a Hermitian matrix function. Therefore, all of the partial indices of matrix Riemann-Hilbert problem (3) are zero. The desired proof completed by an application of Corollary 1. □

Theorem 4. Suppose given functions k_{ij} and f_i in integral Equations (1) replaced with by a sequence functions $k_{ij}^{(m)}$ and $f_i^{(m)}$, where, for each specified m, $k_{ij}^{(m)}$ and $f_i^{(m)}$: **(i)** go to zero faster than some power; **(ii)** their Fourier transform satisfy the Hölder condition on $\mathbb{R}$; **(iii)** $k_{ij}^{(m)}\equiv\bar{k}_{ji}^{(m)}$, for all $i,j=1,2,\cdots,n$; and **(iv)** all functions $k_{ij}(\cdot)$ satisfy either $k_{ij}^{(m)}(-x)=k_{ij}^{(m)}(x)$ or $k_{ij}^{(m)}(-x)=-k_{ij}^{(m)}(x)$, where $\bar{k}_{ji}^{(m)}$ stands for conjugate of $k_{ji}^{(m)}$. Then, solution of integral equations

$$\lambda_i g_i^{(m)}(x) = \sum_{j=1}^{n}\int_0^\infty g_j^{(m)}(\theta)k_{ij}^{(m)}(x-\theta)d\theta + f_i^{(m)}(x),\ x\geq 0 \quad (4)$$

given by the inverse Fourier transform of

$$\mathbf{\Phi}_-^{(m)}(\omega) = -\exp\left\{-\sum_{n=-\infty}^{\infty}\ln(A^{(m)}(\frac{2n}{T}))\frac{e^{-i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\} \times\left[\sum_{m=-\infty}^{\infty}\mathbf{H}^{(m)}(\frac{2m}{T^*})\frac{e^{-i\pi(T\omega-2m)}-1}{2i\pi(T\omega-2m)}\right],$$

where

$$\mathbf{H}^{(m)}(\omega) := exp\left\{-\sum_{n=-\infty}^{\infty}\ln(A^{(m)}(\frac{2n}{T}))\frac{e^{i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\} \times\mathbf{F}^{(m)}(\omega);$$

and satisfy the error bound

$$||\mathbf{g}^{(m)}-\mathbf{g}||_{L_2} \leq ||\ln(\mathbf{G}^{(m)})-\ln(\mathbf{G})||_{L_2}||\mathbf{H}^{(m)}-\mathbf{H}||_{L_2},$$

where the norm is defined by $\|M\| := \sup_{ij}\left\{\int_{-\infty}^{\infty}|M_{ij}(x)|^2\,dx\right\}^{1/2}$.

The following represents a practical application of our findings.

[26] considered the following system of Wiener-Hopf equations

$$\begin{aligned} f_1(x) &= g_1(x)+\frac{\alpha}{\pi}\int_0^\infty\frac{\sin(x-t)}{x-t}g_1(t)dt -\frac{\alpha}{i\pi}\int_0^\infty\frac{\cos(x-t)}{x-t}g_2(t)dt\\ f_2(x) &= g_2(x)+\frac{1}{i\pi\alpha}\int_0^\infty\frac{\cos(x-t)}{x-t}g_1(t)dt -\frac{1}{\pi\alpha}\int_0^\infty\frac{\sin(x-t)}{x-t}g_2(t)dt, \end{aligned} \quad (5)$$

where f_1 and f_2 are two real-valued and given function which go to zero faster than some power, $\alpha\neq 0$, and g_1 and g_2 are two functions which should be determined.

[26] stated that "System of Wiener-Hopf equations, unlike single equations, cannot be solved in closed form." Then, he found an asymptotic solution of the Wiener-Hopf Equation (5). The following lemma establishes an exact solution for the Wiener-Hopf Equation (5).

Lemma 7. The Wiener-Hopf integral Equation (5) have an unique solution, which is given by the inverse Fourier transform of

$$\mathbf{\Phi}_-(\omega) = -\exp\left\{-\sum_{n=-\infty}^{\infty}\ln(A(2n))\frac{e^{-i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\} \times\left[\sum_{m=-\infty}^{\infty}\mathbf{H}(2m)\frac{e^{-i\pi(T\omega-2m)}-1}{2i\pi(T\omega-2m)}\right],$$

where $\mathbf{H}(\omega):=\mathbf{F}(\omega)\exp\{-\sum_{n=-\infty}^{\infty}\ln(A(2n))\frac{\exp\{i\pi(T\omega-2n)\}-1}{2i\pi(T\omega-2n)}\}$,

$$\mathbf{G}(\omega) := (1+\alpha I_{[-1,1]}(\omega))\begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix} +sgn(1-\omega)[1-I_{[-1,1]}(\omega)]\alpha\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\mathbf{F}(\omega) := \begin{pmatrix}\hat{f}_1(\omega)\\ \alpha^2\hat{f}_2(\omega)\end{pmatrix}.$$

Proof. Using the fact that $\alpha\neq 0$, one may restate Equations (5) as $f_1(x)=g_1(x)+\frac{\alpha}{\pi}\int_0^\infty k_1(x-t)g_1(t)dt+\frac{\alpha i}{\pi}\int_0^\infty k_2(x-t)g_2(t)dt$ and $\alpha^2 f_2(x)=\alpha^2 g_2(x)-\frac{\alpha i}{\pi}\int_0^\infty k_2(x-t)g_1(t)dt-\frac{\alpha}{\pi}\int_0^\infty k_1(x-t)g_2(t)dt$, where $k_1(x)=\sin(x)/x$ and $k_2(x)=\cos(x)/x$. The corresponding matrix Riemann-Hilbert problem for 5 is given by

$$\mathbf{G}(\omega)\begin{pmatrix}\Phi_1^-(\omega)\\ \Phi_2^-(\omega)\end{pmatrix} = \begin{pmatrix}\Phi_1^+(\omega)\\ \Phi_2^+(\omega)\end{pmatrix}+\mathbf{F}(\omega),\quad \omega\in\mathbb{R}.$$

The determination of kernel of the above 2×2 matrix Riemann-Hilbert problem does not vanishes nowhere on $\mathbb{R}$, i.e., $Det(\mathbf{K})(\omega)=2\alpha^2+(\alpha^3-2\alpha^2-\alpha)I_{[-1,1]}(\omega)\neq 0$, for all $\omega\in\mathbb{R}$. On the other hand, all elements of the kernel $\mathbf{K}$ are exponential-type-1 functions that satisfy the Hölder condition. Application of Theorem 3 along with the fact that $\hat{\mathbf{f}}=(\hat{f}_1,\hat{f}_2)'$ is an exponential-type-T (from Lemma 2 and 3) vector function complete the desired proof. □

[27] considered a class of single integral equation

$$\int_0^\infty k(x-t)g(t)dt=f(x),\ x\geq 0,$$

where $k(x)=\mathrm{sgn}(x)f(x)$ and $f(x)$ is an even, real-valued, bounded and given function which goes to zero faster than some power and g is to be determined.

The following develops [27]'s findings to a 2×2 system of integral equations.

Example 1. Consider the following system of integral equations.

$$\begin{aligned} f_1(x) &= \int_0^\infty k_1(x-t)g_1(t)dt \\ f_2(x) &= \int_0^\infty k_2(x-t)g_2(t)dt \end{aligned} \quad (6)$$

where $x \geq 0$, $k_i(x) = \text{sgn}(x)f_i(x)$, $f_1(x) = \exp\{-x^2/2\}/\sqrt(2\pi)$, and $f_2(x) = \exp\{-x\}/(1+\exp\{-x\})^2$. The corresponding 2×2 matrix Riemann-Hilbert problem is given by $\begin{pmatrix} \frac{-2i}{\sqrt{\pi}}Daw(\frac{\omega}{\sqrt{2}}) & 0 \\ 0 & 1-2\frac{\omega(\omega-i)LPhi(-1;1;-i\omega)}{i\omega-1}+\frac{\pi\omega}{\sinh(\pi\omega)} \end{pmatrix} \times \begin{pmatrix} \Phi_1^-(\omega) \\ \Phi_2^-(\omega) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-\frac{\omega^2}{2}} - \frac{2i}{\sqrt{\pi}}Daw(\frac{\omega}{\sqrt{2}}) \\ \frac{1}{2} - i\omega LPhi(-1;1;-i\omega) \end{pmatrix} + \begin{pmatrix} \Phi_1^+(\omega) \\ \Phi_2^+(\omega) \end{pmatrix}$,where $Daw(\cdot)$ is the Dawson function given by $Daw(y) = e^{-y^2}\int_0^y e^{t^2}dt$ and $LPhi(\cdot,;\cdot;\cdot)$ is the general Lerch Phi function given by $LPhi(\zeta;\alpha;\beta)=\sum_{n=0}^\infty \zeta^n/(n+\beta)^\alpha$. The determination of kernel $\mathbf{K}$vanishes at $\omega=0$. To remove this barrier, one may reduce the above 2×2 matrix Riemann-Hilbert problem to the following 2×2 matrix Riemann-Hilbert problem which all require conditions are held.

$$\mathbf{G}(\omega)\begin{pmatrix} \Phi_1^-(\omega) \\ \Phi_2^-(\omega) \end{pmatrix} = \begin{pmatrix} \frac{1}{i\omega}\Phi_1^+(\omega) \\ \frac{1}{i\omega}\Phi_2^+(\omega) \end{pmatrix} + \mathbf{F}(\omega),$$

where

$$\begin{aligned} \mathbf{G}(\omega) &:= \begin{pmatrix} \frac{-2i}{\sqrt{\pi}\omega}Daw(\frac{\omega}{\sqrt{2}}) & 0 \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\omega} - 2\frac{(\omega-i)LPhi(-1;1;-i\omega)}{i\omega-1} + \frac{\pi}{\sinh(\pi\omega)} \end{pmatrix} \\ \mathbf{F}(\omega) &:= \begin{pmatrix} \frac{1}{2i\omega}\exp\{-\frac{\omega^2}{2}\} - \frac{2}{\sqrt{\pi}\omega}Daw(\frac{\omega}{\sqrt{2}}) \\ \frac{1}{2\omega} - iLPhi(-1;1;-i\omega) \end{pmatrix}. \end{aligned}$$

Using the fact that both kernel and nonhomogeneous vector functions are exponential type 1 functions, one may conclude that an approximate solution for the system of integral Equations (6) is given by the inverse Fourier transform of

$$\begin{aligned} \mathbf{\Phi}_-(\omega) &= -\exp\left\{-\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(2n))\frac{e^{-i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\} \\ &\times \left[\sum_{m=-\infty}^{\infty}\mathbf{H}(2m)\frac{e^{-i\pi(T\omega-2m)}-1}{2i\pi(T\omega-2m)}\right], \end{aligned}$$

where $\mathbf{H}(\omega) := \mathbf{F}(\omega)\exp\{-\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(2n))\frac{\exp\{i\pi(T\omega-2n)\}-1}{2i\pi(T\omega-2n)}\}$.
□

IV. Conclusion and suggestions

This article considers a class of system of Wiener-Hopf integral equations

$$\lambda_i g_i(x) - \sum_{j=1}^n \int_0^\infty g_j(\theta)k_{ij}(x-\theta)d\theta = f_i(x),$$

where $x\geq 0$, $i=1,2,\cdots,n$, g_i-s are to be determined, and k_{ij}-s and f_i-s are given functions with some mild conditions. Exact and approximated solution for such class of system of Wiener-Hopf integral equations are given.

In the case of $\lambda_i=0$, for $i=1,2,\cdots,n$, one may replace two above conditions (**ii**) and (**iii**) by matrix function $\mathbf{k}=[k_{ij}]$ be a Hermitian matrix which either $k_{ij}(-x)=k_{ij}(x)$ or $k_{ij}(-x)=-k_{ij}(x)$. Application of Lemma 4 leads to desired conditions on the corresponding matrix Riemann-Hilbert problem. Moreover, result of Example 1 may be extended to

$$\int_0^\infty g_i(\theta)k_i(x-\theta)d\theta = f_i(x),\ x\geq 0,$$

where either $k_i(-x)=k_i(x)$ or $k_i(-x)=-k_i(x)$, for all $i=1,2,\cdots,n$.

Acknowledgment

The support of Shahid Beheshti University and Natural Sciences and Engineering Research Council (NSERC) of Canada are gratefully acknowledged by authors. Thanks to anonymous reviewers for their constructive comments.

References

[1] W. E. Chen, C. O. Li, and B. Ou, "Classification of solutions for a system of integral equations," *Communications in Partial Difference Equations*, vol. 30, no. 1, pp. 59–65, 2005.

[2] K. Maleknejad, N. Aghazadeh, and M. Rabbani, "Numerical solution of second kind fredholm integral equations system by using a taylor-series expansion method," *Applied Mathematics and Computation*, vol. 175, no. 2, pp. 1229–1234, 2006.

[3] M. Dehghan, M. Shakourifar, and A. Hamidi, "The solution of linear and nonlinear systems of volterra functional equations using adomian-pade technique," *Chaos, Solitons and Fractals*, vol. 39, no. 5, pp. 2509–2521, 2009.

[4] A. Vahidian Kamyad, M. Mehrabinezhad, and J. Saberi-Nadjafi, "A numerical approach for solving linear and nonlinear volterra integral equations with controlled error," *IAENG: International Journal of Applied Mathematics*, vol. 40, no. 2, pp. 69–74, 2010.

[5] A. M. Bijura, "Systems of singularly perturbed fractional integral equations ii," *IAENG: International Journal of Applied Mathematics*, vol. 42, no. 4, pp. 198–203, 2012.

[6] K. Wang and Q. Wang, "Taylor collocation method and convergence analysis for the volterra-fredholm integral equations," *Journal of Computational and Applied Mathematics*, vol. 260, no. 1, pp. 294–300, 2014.

[7] S. Xiang, "Laplace transforms for approximation of highly oscillatory volterra integral equations of the first kind," *Applied Mathematics and Computation*, vol. 232, no. 1, pp. 944–954, 2014.

[8] A. Armand and Z. Gouyandeh, "Numerical solution of the system of volterra integral equations of the first kind," *International Journal of Industrial Mathematics*, vol. 6, no. 1, pp. 27–35, 2014.

[9] S. Mashayekhi, M. Razzaghi, and O. Tripak, "Solution of the nonlinear mixed volterra-fredholm integral equations by hybrid of block-pulse functions and bernoulli polynomials," *The Scientific World Journal*, vol. 2014, Article ID 413623, 8 pages, doi:10.1155/2014/413623, 2014.

[10] V. Balakumar and K. Murugesan, "Single-term walsh series method for systems of linear volterra integral equations of the second kind," *Applied Mathematics and Computation*, vol. 228, pp. 371–376, 2014.

[11] J. P. Berrut, S. A. Hosseini, and G. Klein, "The linear barycentric rational quadrature method for volterra integral equations," *SIAM Journal on Scientific Computing*, vol. 36, no. 1, pp. 105–123, 2014.

[12] H. Dym and J. P. Mckean, *Fourier series and integrals. Probability and Mathematical Statistics*. Academic Press, 1972.

[13] D. C. Champeney, *A handbook of Fourier theorems*. Cambridge University press, 1987.

[14] D. F. Walnut, *An introduction to wavelet analysis*, 2nd ed. Birkhauser, 2002.

[15] R. N. Bracewell, *The Fourier transform and its applications*, 3rd ed. McGraw-Hill, 2000.

[16] F. P. Miller, A. F. Vandome, and J. McBrewster, *Hermitian Matrix*. VDM Publishing House Ltd., 2011.

[17] A. Serre, *Matrices Theory and Applications*, 2nd ed. Springer-Verlag, 2010.

[18] A. F. Voronin, "A method for determining the partial indices of symmetric matrix functions," *Siberian Mathematical Journal*, vol. 52, no. 1, pp. 54–69, 2011a.

[19] ——, "Partial indices of unitary and hermitian matrix functions," *Siberian Mathematical Journal*, vol. 51, no. 5, pp. 805–809, 2011b.

[20] N. I. Muskhelishvili, *Singular integral equations: Translated from Russian. Preprint of the 1946 translation. Noordhoff international publishing Leyden*. Groningen-Holland, 1977.

[21] M. J. Ablowitz and A. S. Fokas, *Complex variable, introduction and application*. Springer-Verlag, 1990.
[22] D. S. Jones, "Commutative wiener-hopf factorization of a matrix," *Proc. R. Soc. A*, vol. 393, no. 1804, pp. 185–192, 1984a.
[23] ——, "Factorization of a wiener-hopf matrix," *IMA journal of applied mathematics*, vol. 32, no. 1, pp. 211–220, 1984b.
[24] I. V. Benjamin and I. D. Abrahams, "On the commutative factorization of $n \times n$ matrix wiener-hopf kernels with distinct eigenvalues," *Proc. R. Soc. A*, vol. 463, pp. 613–639, 2007.
[25] A. T. Payandeh and D. Kucerovsky, "Exact solutions for a class of matrix riemann-hilbert problems," *IMA Journal of Applied Mathematics*, vol. 79, no. 1, pp. 109–123, 2014.
[26] A. A. Polosin, "Asymptotic solution of a system of wiener-hopf equation with piecewise constant fourier transforms of the kernels," *Differential equations*, vol. 43, no. 9, pp. 1197–1205, 2007.
[27] D. Kucerovsky, E. Marchand, A. T. Payandeh, and W. Strawderman, "On the bayesianity of maximum likelihood estimators of restricted location parameters under absolute value error loss," *Statistics and Decisions*, vol. 27, pp. 145–168, 2009.

Amir T. Payandeh Najafabadi & Dan Kucerovsky Amir T. Payandeh Najafabadi is an Associate Professo in Department of Mathematics sciences at Shahid Behashti University, Tehran, Evin (Email address: amirtpayandeh@sbu.ac.ir). He was born on Sep 3, 1973. He received his PhD from University of New Brunswick, Canada in 2006. He has published 28 papers and was co-author of two books. Dan Kucerovsky was born on Oct 4, 1967. He received his D.Phil from Magdalen College, at the University of Oxford, in Dec. 1994. After time spent at the University of Paris, the University of Switzerland, and the University of Toronto, he became a professor at the University of New Brunswick, Canada. He has published 44 papers and one book chapter (Email address: dkucerov@unb.ca).