Cube-Connected Complete Graphs

Juan Liu and Xindong Zhang

***Abstract*—The n-dimensional cube-connected complete graph, denoted by $CCCP(n)$, is constructed from the n-dimensional hypercube Q_n by replacing each vertex of Q_n with a complete graph of order n. In this paper, we prove that $CCCP(n)$ is Cayley graph, and study the basic properties of $CCCP(n)$, including spectra, connectivity, Hamiltonian, diameter etc.**

***Index Terms*—n-dimensional cube-connected complete graph, Cayley graph, Vertex-transitive.**

I. Introduction

THROUGHOUT this article, a graph $G = (V, E)$ always means a finite undirected connected graph without loops and multiple edges, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. The hypercube, suggested by Sullivan and Bashkow[1], is one of the most popular, versatile and efficient topological structures of interconnection networks. The hypercube Q_n has many excellent features, and thus becomes the first choice for the topological structure of parallel processing and computing systems. From hypercube, one of the most popular derivative networks is a cube-connected cycle. The n-dimensional cube-connected cycle, denoted by $CCC(n)$, is constructed from the n-dimensional hypercube Q_n by replacing each vertex of Q_n with an undirected cycle of length n. The ith dimensional edge incident to a vertex of Q_n is then connected to the ith vertex of the corresponding cycle of $CCC(n)$. In this paper, we define a new topological structure of interconnection networks from Q_n.

Definition 1.1: The n-dimensional cube-connected complete graph, denoted by $CCCP(n)$, is constructed from the n-dimensional hypercube Q_n by replacing each vertex of Q_n with a complete graph of order n.

By modifying the labeling scheme of Q_n, we can represent each vertex of $CCCP(n)$ by a pair $(\mathbf{x}; i)$ where $i (1 \le i \le n)$ is a position of the vertex within its complete graph and $\mathbf{x}$ (any n-bit binary string) is the label of the vertex in Q_n that corresponds to the complete graph. Precisely, the vertex set of $CCCP(n)$ is

$$V = \{(\mathbf{x}; i) : \mathbf{x} \in V(Q_n), 1 \le i \le n\}.$$

Two vertices $(\mathbf{x}; i)$ and $(\mathbf{y}; j)$ are linked by an edge in the $CCCP(n)$ if and only if either

(i). $\mathbf{x} = \mathbf{y}$ and $|i - j| \equiv s(mod\ n)$, $s \in \{1, 2, \cdots \lfloor \frac{1}{2}n \rfloor\}$, or

(ii). $i = j$ and $\mathbf{x}$ differs from $\mathbf{y}$ in precisely the ith bit.

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Edges of the first type are called $complete\ edges$, while edges of the second type are referred to as $hypercube\ edges$.

For a vertex $v \in V(G)$, $N(v)$ denotes the set of vertices adjacent to v, $d_G(v) = |N(v)|$ is the degree of v in G. An *edge-cut* in graph G is a set S of edges of G such that $G - S$ is disconnected. The *edge-connectivity* $\lambda(G)$ of a graph is the minimum cardinality of all edge-cuts of G. Obviously, $\lambda(G) \le \delta(G)$. A connected graph G is said to be *maximal edge connected*, for short *max*-λ, if $\lambda(G) = \delta(G)$. A graph G is said to be *super-edge-connected*, for short *super*-λ, if every minimum edge-cut of G isolates a single vertex. An edge-cut F of G is called a *restricted edge-cut* if $G - F$ contains no isolated vertices. The minimum cardinality of all restricted edge-cuts denoted by $\lambda'(G)$, is called the *restricted edge-connectivity* of G. Similarly, a *vertex-cut* in graph G is a set U of vertices of G such that $G - U$ is disconnected. The *vertex-connectivity* $\kappa(G)$ of a graph is the minimum cardinality of all vertex-cuts of G. Obviously, $\kappa(G) \le \delta(G)$. A connected graph G is said to be *maximal vertex-connected*, for short *max*-κ, if $\kappa(G) = \delta(G)$. A graph G is said to be *super-connected*, for short *super*-κ, if every minimum vertex-cut of G isolates a single vertex. A vertex-cut U of G is called a *restricted vertex-cut* if $G - U$ contains no isolated vertices. The minimum cardinality of all restricted vertex-cuts denoted by $\kappa'(G)$, is called the *restricted vertex-connectivity* of G. An independent vertex set of a graph is a subset of the vertices such that no two vertices in the subset represent an edge of G. The vertex independence number $\alpha(G)$ of a graph G, is the cardinality of the largest (vertex) independent set. Let Γ be a non-trivial finite group, and let S be a subset of Γ such that closed under taking inverses and does not contain the identity. Then the *Cayley graph* $C_\Gamma(S)$ is the graph with vertex set Γ and edge set $E(C_\Gamma(S)) = \{gh : hg^{-1} \in S\}$. A graph G is *vertex-transitive* if its automorphism group acts transitively on $V(G)$. It is wellknown that the Cayley graph is vertex-transitive. There are some authors studied the properties of Cayley graph [2-4]. In this paper, we show that $CCCP(n)$ is Cayley graph, and study the basic properties of $CCCP(n)$, including spectra, connectivity, Eulerian, Hamiltonian, diameter, eta.

For graph-theoretical terminology and notation not defined here we follow Bondy and Murty [5].

II. Main Result

The following proposition is quite apparent from the construction of $CCCP(n)$.

Proposition 2.1: The cube-connected complete graph $CCCP(n)$ is an n-regular graph with $n2^n$ vertices and $2^{n-1}(n + 2^n - 1)$ edges.

Theorem 2.2: The cube-connected complete graph $CCCP(n)$ is a Cayley graph, and hence is vertex-transitive.

Proof: In order to prove the theorem, we construct a Cayley graph firstly. Use $(Z_2)^n$ to denote $Z_2 \times Z_2 \times \cdots \times Z_2$, which is the Cartesian product of n sets $Z_2 = \{0, 1\}$. Let

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

be an n-square matrix. For any element **v** in $(Z_2)^n$, thinking of **v** as a column vector, we let M act on **v** in the normal manner except that all additions are computed modulo 2. Then $M\mathbf{v}$ is also an element of $(Z_2)^n$. We can define a new group $\Gamma = (Z_2)^n \times Z_n$. For any $(\mathbf{x}; i), (\mathbf{y}; j) \in (Z_2)^n \times Z_n$, the operation "$\circ$" of Γ is defined as follows:

$$(\mathbf{x}; i) \circ (\mathbf{y}; j) = (M^j\mathbf{x} + \mathbf{y}; i + j),$$

where the first addition is componentwise modulo 2(in $(Z_2)^n$) and the second is modulo n(in Z_n). It is a simple exercise to check that this new operation makes $\Gamma = (Z_2)^n \times Z_n$ a group. Its identity element of Γ is $(\mathbf{0}; 0)$ and the inverse

$$(\mathbf{x}; i)^{-1} = (-M^{n-i}\mathbf{x}; n - i).$$

Let

$$S = \{(10\cdots0; 0), (00\cdots0; 1), (00\cdots0; 2), \cdots, (00\cdots0; n-1)\},$$

where the first is self-inverse and the following are mutually inverse. Thus $S = S^{-1}$ and the Cayley graph $C_\Gamma(S)$ is an undirected graph.

Hence, in order to complete the proof, it is suffice to prove

$$CCCP(n) \cong C_\Gamma(S).$$

Consider **x** in the vertex $(\mathbf{x}; i)$ as a column vector **x**. Define a mapping

$$\phi : (Z_2)^n \times Z_n \to (Z_2)^n \times Z_n$$
$$(\mathbf{x}; i) \mapsto (M^{n-i+1}\mathbf{x}; n - i + 1)$$

It is easy to check that the mapping ϕ is bijective. We now prove that ϕ is preserves adjacency. Let $(\mathbf{x}; i)$ and $(\mathbf{y}; j)$ be any two distinct vertices of $CCCP(n)$. By the definition, $(\mathbf{x}; i)$ and $(\mathbf{y}; j)$ are adjacent in $CCCP(n)$ if and only if either

(i). $\mathbf{x} = \mathbf{y}$ and $|i - j| \equiv s(mod\ n)$, $s \in \{1, 2, \cdots \lfloor\frac{1}{2}n\rfloor\}$, or

(ii). $i = j$ and **x** differs from **y** in precisely the ith bit.

Noting that

$$\phi(\mathbf{x}; i) = (M^{n-i+1}\mathbf{x}; n - i + 1),$$
$$\phi(\mathbf{y}; j) = (M^{n-j+1}\mathbf{y}; n - j + 1),$$
$$\phi(\mathbf{x}; i)^{-1} = (M^{n-i+1}\mathbf{x}; n - i + 1)^{-1} = (-\mathbf{x}; i - 1),$$

we have that

$$\begin{aligned}\phi(\mathbf{x}; i)^{-1} \circ \phi(\mathbf{y}; j) &= (-\mathbf{x}; i - 1) \circ (M^{n-j+1}\mathbf{y}; n - j + 1) \\ &= (-M^{n-j+1}\mathbf{x} + M^{n-j+1}\mathbf{y}; n - j + i).\end{aligned}$$

If (i) occurs, then $\phi(\mathbf{x}; i)^{-1} \circ \phi(\mathbf{y}; j) = (\mathbf{0}; s) \in S(s \in \{1, 2, \cdots \lfloor\frac{1}{2}n\rfloor\})$ if and only if $(\mathbf{x}; i)$ and $(\mathbf{y}; j)$ are adjacent in $C_\Gamma(S)$.

If (ii) occurs, then

$$\begin{aligned}\phi(\mathbf{x}; i)^{-1} \circ \phi(\mathbf{y}; j) &= M^{n-i+1}(-\mathbf{x} + \mathbf{y}; 0) \\ &= (10\cdots0; 0) \in S\end{aligned}$$

if and only if $(\mathbf{x}; i)$ and $(\mathbf{y}; j)$ are adjacent in $C_\Gamma(S)$. The proof is completed. ■

Next, we consider the spectra of $CCCP(n)$. Recall that a multigraph G is called semiregular of degrees r_1, r_2 if it is bipartite having a representation $G = (\mathcal{X}_1, \mathcal{X}_2; \mathcal{U})$ with $|\mathcal{X}_1| = n_1, |\mathcal{X}_2| = n_2$, $n_1 + n_2 = |V(G)|$, where each vertex $x \in \mathcal{X}_i$ has degree $r_i (i = 1, 2)$.

Lemma 2.3: [6] Let G be a regular graph of degree r with ν vertices and m edges, then

$$P_{S(G)}(\lambda) = \lambda^{m-\nu} P_G(\lambda^2 - r).$$

Lemma 2.4: [6] Let G be a semiregular multigraph with $n_1 \geq n_2$, then

$$P_{L(G)}(\lambda) = (\lambda+2)^\beta \sqrt{(-\frac{\alpha_1}{\alpha_2})^{n_1-n_2} P_G(\sqrt{\alpha_1\alpha_2}) P_G(-\sqrt{\alpha_1\alpha_2})}.$$

holds, where $\alpha_i = \lambda - r_i + 2 (i = 1, 2)$ and $\beta = n_1 r_1 - n_1 - n_2$.

Theorem 2.5: The cube-connected complete graph $CCCP(n)$,

$$P_{CCCp(n)}(\lambda) = [\lambda(\lambda + 2)]^{(n-2)2^{n-1}} P_{Q_n}[\lambda(\lambda - n + 2) - n].$$

Proof: From the definition of $CCCP(n)$, we can obtain that $CCCP(n) \cong L(S(Q_n))$. Let Q_n be the n-dimensional hypercube, we have known that Q_n is n-regular and $|V(Q_n)| = 2^n, |E(Q_n)| = n2^{n-1}$. By lemma 2.3, we have

$$P_{S(Q_n)}(\lambda) = \lambda^{n2^{n-1}-2^n} P_G(\lambda^2 - n) = \lambda^{(n-2)2^{n-1}} P_G(\lambda^2 - n).$$

Since $S(Q_n)$ is semiregular graph with $n_1 = n2^{n-1}, n_2 = 2^n, r_1 = 2, r_2 = n$. By Lemma 2.4, we have $\alpha_1 = \lambda - r_1 + 2 = \lambda - 2 + 2 = \lambda$, $\alpha_2 = \lambda - n + 2$ and $\beta = n_1 r_1 - n_1 - n_2 = n2^{n-1} \cdot 2 - n2^{n-1} - 2^n = (n-2)2^{n-1}$. Let $\alpha = \alpha_1\alpha_2 = \lambda(\lambda - n + 2)$, thus,

$$\begin{aligned} & P_{L(S(Q_n))}(\lambda) \\ = \; & (\lambda + 2)^{(n-2)2^{n-1}} \sqrt{(-\frac{\lambda}{\lambda - n + 2})^{n2^{n-1}-2^n} P_{S(Q_n)}(\sqrt{\alpha}) P_{S(Q_n)}(-\sqrt{\alpha})} \\ = \; & (\lambda + 2)^{(n-2)2^{n-1}} (\frac{\lambda}{\lambda - n + 2})^{(n-2)2^{n-2}} \sqrt{\alpha^{(n-2)2^{n-1}} (P_{Q_n}(\alpha - n))^2} \\ = \; & (\lambda + 2)^{(n-2)2^{n-1}} (\frac{\lambda}{\lambda - n + 2})^{(n-2)2^{n-2}} \alpha^{(n-2)2^{n-2}} P_{Q_n}(\alpha - n) \\ = \; & (\lambda + 2)^{(n-2)2^{n-1}} \lambda^{(n-2)2^{n-1}} P_{Q_n}[\lambda(\lambda - n + 2) - n] \\ = \; & [\lambda(\lambda + 2)]^{(n-2)2^{n-1}} P_{Q_n}[\lambda(\lambda - n + 2) - n]. \end{aligned}$$

■

In the following, we will consider the diameter and connectivity of $CCCP(n)$.

Lemma 2.6: [7] For any given vertex x of Q_n, there exists the unique vertex y such that the distance $d(Q_n; x, y) = n$.

Theorem 2.7: For any given vertex x of $CCCP(n)$, there exists the unique vertex y such that the distance $d(CCCP(n); x, y) = 2n$.

Proof: Since $CCCP(n)$ is vertex-transitive by Theorem 2.2, we can, without loss of generality, suppose that

$$x = (000\cdots0; 1),$$

if there exists y such that $d(CCCP(n); x, y) = 2n$, then $y = (111\cdots 11, i)$ for some $i \in \{1, 2, \ldots, n\}$ by Lemma 2.6. Thus, $x = (000\cdots 00; 1) \to (100\cdots 00; 1) \to (100\cdots 00; 2) \to (110\cdots 00; 2) \to (110\cdots 00; 3) \to (111\cdots 00; 3) \to \ldots \to (111\cdots 10; n-1) \to (111\cdots 10; n) \to (111\cdots 11; n) \to (111\cdots 11; 1)$, from the construction, we know that the path is shortest, and we can construct the paths of length $2n-1$ from $x = (000\cdots 00; 1)$ to any vertex $(111\cdots 111; i)$, $(i = 2, 3, \ldots, n)$. Therefore, the vertex $(111\cdots 11; 1)$ is the unique vertex such that the distance

$$d(CCCP(n); (000\cdots 00; 1), (111\cdots 11; 1)) = 2n.$$

■

By Theorem 2.7, we have the following corollary.

Corollary 2.8: The diameter of $CCCP(n)$ is 2n.

Theorem 2.9: The independent number of the cube-connected complete graph $CCCP(n)$ is 2^n.

Proof: Let S be an independent set of $CCCP(n)$. By Proposition 2.1, $CCCP(n)$ is an n-regular graph with $n2^n$ vertices, So, $\alpha(CCCP(n)) \leq \frac{n2^n}{n} = 2^n$. For every vertex $\mathbf{x} \in Q_n$, we can selecte a vertex in $\{(\mathbf{x}, i) | i = 1, 2, \ldots, n\}$ adding to S such that no vertex is adjacent in $CCCP(n)$, and $|S| = 2^n$. Thus, S is a maximum independent set of $CCCP(n)$. ■

Lemma 2.10: The cube-connected cycles $CCC(n)$ is 3-regular, has $n2^n$ vertices and $3n2^{n-1}$ edges, has connectivity 3 and contains Hamilton cycles.

Theorem 2.11: The cube-connected complete graph $CCCP(n)$ is Eulerian if n is even, $CCCP(n)$ is Hamiltonian if $n \geq 2$.

Proof: It is evident that the cube-connected complete graph $CCCP(n)$ is Eulerian if n is even. Since $CCC(n)$ is a spanning subgraph of $CCCP(n)$ and $CCC(n)$ is Hamiltonian if $n \geq 2$, Therefore $CCCP(n)$ is Hamiltonian if $n \geq 2$. ■

Theorem 2.12: The cube-connected complete graph $CCCP(n)$ is max-κ and max-λ. $\kappa'(CCCP(n)) = n$; $\lambda'(CCCP(n)) = n$. Thus, $CCCP(n)$ is not super-κ, and not super-λ.

Proof: By Theorem 2.2, $CCCP(n) \cong C_\Gamma(S)$, and $|S| = n$, thus $CCCP(n)$ has connectivity n. Hence, $CCCP(n)$ is max-κ and max-λ. Considering the complete subgraph with vertices $X = \{(\mathbf{x}, 1), (\mathbf{x}, 2), \ldots, (\mathbf{x}, n)\}$, the vertex set $\{(\mathbf{y}, i) | \mathbf{x}$ differs from $\mathbf{y}$ in precisely the ith bit$\}$ forms a restricted vertex-cut. Thus, $\kappa'(CCCP(n)) = n$. n hypercube edges which incident with X forms a restricted edge-cut, thus, $\lambda'(CCCP(n)) = n$. Therefore, $CCCP(n)$ is not super-κ, and not super-λ. ■

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