# Modified Gradient-Projection Algorithm for Solving Convex Minimization Problem in Hilbert Spaces

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*Abstract*— **We proposed a new iterative scheme for finding a minimizer for a constrained convex minimization problem. We proved that the sequence generated by our new iterative scheme converges strongly to solution of the constrained convex minimization problem in real Hilbert spaces.**

***Index Terms*** **—Average Mappings, Constrained Convex Minimization problem, Fixed point, Gradient Projection Algorithm, Nonexpansive mappings.**

## I. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|.\|$.

*Definition 1.1* Let C be a nonempty closed and convex subset of a real Hilbert space H. A map T:C→H is said to be nonexpansive if for all x,z∈C we have

$$\|Tx - Tz\| \leq \|x - z\|.$$

We will denote the fixed point set of *T* by Fix(T).

*Definition 1.2* For any x∈H, we define the map $P_C: H \to C$ satisfying $\|x - P_C x\| \leq \|x - z\| \quad \forall\, z \in C$.

$P_C$ is called the *metric projection* of H onto C. it is well known that $P_C$ is nonexpansive and

$$\langle x - P_C x, P_C x - z \rangle \geq 0 \quad \forall\, z \in C. \tag{1}$$

Clearly, (1) is equivalent to

$$\|x - z\|^2 \geq \|x - P_C x\|^2 + \|z - P_C x\|^2 \; \forall x \in H \; and \; \forall z \in C$$

Furthermore, $P_C x$ is characterized by the property that $P_C x \in C$ and $\langle x - z, P_C x - P_C z \rangle \geq 0 \quad \forall\, x \in C.$ (2)

*Definition 1.3* A mapping $A$ of $C$ into $H$ is called monotone if $\langle Ax - Az, x - z \rangle \geq 0 \quad \forall\, x, z \in C$,

$A$ is called $\beta - strongly$ monotone if there exists $\beta > 0$ such that $\langle x - z, A\,x - Az \rangle \geq \beta\|x - z\|^2 \quad \forall\, x, z \in C$,

$A$ is called $\alpha - inverse - strongly$ monotone if there exists $\alpha > 0$ such that

$$\langle x - z, A\,x - Az \rangle \geq \alpha\|Ax - Az\|^2 \quad \forall\, x, z \in C,$$

also $A$ is $L - Lipschitz - continuous$ if there exists $L > 0$ such that for all $x, z \in C \quad \|Ax - Az\| \leq L\|x - z\|$.

*Definition 1.4* A mapping $T: H \to H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently $\langle x - z, T\,x - Tz \rangle \geq \|Tx - Tz\|^2 \; \forall\, x, z \in H.$

Alternatively, $T$ is *firmly nonexpansive* if $T$ can be expressed as $T = \frac{1}{2}(I + S)$, for some $S: H \to H$ nonexpansive . As an example, the *projection mappings* are firmly nonexpansive.

*Definition 1.5* A mapping $T: H \to H$ is said to be an *averaged mapping* if it can be written as the average of the identity mapping *I* and a nonexpansive mappings; i.e.

$$T = (1 - \alpha)I + \alpha S, \tag{3}$$

where α∈(0,1) and S:H→H is non expansive. More precisely, when (3) holds, we say that T is α-averaged. Therefore, firmly nonexpansive mappings (e.g. projections) are $\frac{1}{2}$-averaged mappings.

The proposition below gives some properties of averaged mappings.

*Proposition 1.1 (Bryne [3], Combettes [5]) For given operators S,T,V:H→H:*

*(a) If T=(1−α)S+αV for some α∈(0,1) and if S is averaged and V is nonexpansive, then T is averaged.*

*(b) T is firmly nonexpansive if and only if the complement I−T is nonexpansive.*

*(c) If T=(1−α)S+αV for some α∈(0,1) and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

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*(d) The composition of finitely many averaged mappings is averaged. That is, if each of the mappings* $\{T_i\}_{i=1}^{N}$ *is averaged, so is the composition* $T_1T_2\cdots T_N$. *In particular, if* $T_1$ *is* $\alpha_1$ *averaged and* $T_2$ *is* $\alpha_2$ *averaged, where* $\alpha_1,\alpha_2 \in (0,1)$, *then the composition* $T_1\ T_2$ *is α-averaged, where* $\alpha = \alpha_1 + \alpha_2 - \alpha_1\ \alpha_2$ *.*

The following are immediately noticeable: (i) If $T$ is nonexpansive, then $I - T$ is monotone; (ii) the projection mapping $P_C$ is a 1-*ism*. The *inverse strongly monotone (also known as co-coercive) operators* have being widely used in solving practical problems in diverse fields, e.g., the traffic assignment problems; see, [1, 6] and the references therein. The proposition that follows gives some important relationships between *average mappings and inverse strongly monotone operators*.

*Proposition 1.2 (Bryne [3], Combettes [5])*

*(a) T is nonexpansive if and only if the complement* $I-T$ *is* $\frac{1}{2}$*-ism.*

*(b) If T is v-ism, then for γ>0, γT is* $\frac{v}{\gamma}$*-ism.*

*(c) T is averaged if and only if the complement* $I-T$ *is* $\tau$ *−ism for some* $\tau > \frac{1}{2}$*. Indeed, for α∈(0,1), T is α-averaged if and only if* $I-T$ *is* $\frac{1}{2}$*-ism.*

Given the following constrained convex minimization problem:

$$\text{minimize } \{f(x): x \in C\}, \quad (4)$$

where *f:C→R* is a real-valued convex function. The minimization problem (4) is said to be *consistent* if it has a solution. In what follows, we shall denote the solution set of problem (4) by *S*. It is well known that if *f* is (*Fréchet*) differentiable, then the gradient-projection method (i.e., GPM) generates a sequence $\{x_n\}$ by using the following recursive formula:

$$x_{n+1} = P_C\big(x_n - \lambda\nabla f(x_n)\big), \forall n \geq 1, \quad (5)$$

or in a more general form;

$$x_{n+1} = P_C\big(x_n - \lambda_n\nabla f(x_n)\big), \forall n \geq 1, \quad (6)$$

where $x_0 \in C$ is an arbitrary initial guess in both (5) and (6), $\lambda$ or $\lambda_n$ are positive real numbers. It is known that if $\nabla f$ is α-strongly monotone and *L*-Lipschitzian with constants α, *L*>0, then the operator

$$T = P_C(I - \lambda\nabla f) \quad (7)$$

is a contraction; thus the sequence $\{x_n\}$ in (5) converges in norm to the unique minimizer of (4). More generally if the sequence $\{\lambda_n\}$ is such that

$$0 < \liminf \lambda_n \leq \limsup \lambda_n < \frac{2\alpha}{L^2} \quad (8)$$

then the sequence $\{x_n\}$ generated by (6) converges in norm to the unique minimizer of (4). However in a situation where the $\nabla f$ fails to be *strongly monotone*, the operator *T* defined by (7) would fail to be a contraction, consequently, the sequence $\{x_n\}$ generated by (6) may fail to converge in norm. (see [12], Sect. 4). If $\nabla f$ is Lipchitzian, then the schemes (5) and (6) can still converge weakly under certain assumptions.

The GPM for finding the approximate solutions of the constrained convex minimization problem has been studied by several authors; see, for example [9] and the references therein. The convergence of the sequence generated by this method depends largely on the behavior of the gradient of the objective function. If the gradient fails to be strongly monotone, then the sequence generated by the GPM may fail to converge strongly. Recently alternative operator-oriented approach to algorithm (6); namely an average mapping approach; see, for example Xu [12]. Xu [12] also presented two modifications of the gradient-projection algorithms which are shown to be strongly convergent.

Very recently, based on Yamada hybrid steepest descent method, Tian and Huang [10] proposed respectively the following implicit and explicit iterative scheme:

$$x_s = P_C(I - s_n\mu F)T_{\lambda_s}(x_s)$$

and

$$x_{n+1} = P_C(I - s_n\mu F)T_{\lambda_n}(x_n).$$

They proved that the sequence generated by their implicit and explicit schemes converge strongly to a solution of the constrained convex minimization problem, which also solves a certain variational inequality problem.

Also, motivated by the work of Xu [12], Shehu et al [8] proposed the following iterative scheme :

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)x_n \\ x_{n+1} = P_C\big(y_n - \lambda_n\nabla f(y_n)\big), \ n \geq 1, \end{cases} \quad (9)$$

they proved that the sequence generated by their scheme converges strongly to a solution of the constrained convex minimization problem, which also solves a certain variational inequality problem (see [8] for details).

Motivated by the works of Xu [12], Tian and Huang [10] and Shehu et al [8], Enyi and Mukiawa [13], we propose a new iterative scheme for finding the approximate solution of a constrained convex minimization problem and we proved that the sequence generated by our scheme converges strongly to a solution of the constrained minimization problem.

## II PRELIMINARIES

In the sequel, we shall also make use of the following lemmas.

*Lemma 2.1* *Let H be a real Hilbert space. Then the following inequality holds;*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$$

*for all x,y∈H.*

*Lemma 2.2* *Let H be a real Hilbert space. Then the following inequality holds;*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

*for all x,y∈H, λ∈[0,1].*

*Lemma 2.3 (Browder [2])* *Let H be a real Hilbert space,C a closed convex subset of H and T:C→C a nonexpansive mapping with a fixed point. Assume that a sequence* $\{x_n\}$ *in C is such that* $x_n \rightharpoonup x$ *weakly and* $x_n - Tx_n \to y$*. Then x−Tx=y*

*Lemma 2.4 (Xu [11])* *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following*

$$a_{n+1} \leq (1+\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n \quad n \geq 0$$

*where*

*(i)* $\{\alpha_n\} \subset [0,1], \sum_{n=1}^{\infty}\alpha_n = \infty$*;*

*(ii)* $\limsup \sigma_n \leq 0$ *(iii)* $\gamma_n \geq 0, \sum_{n=1}^{\infty}\gamma_n < \infty$.

*Then* $a_n \to 0$ *as n→∞.*

We adopt the following notions :

- $x_n \to x$ means that $x_n \to x$ strongly;
- $x_n \rightharpoonup x$ means that $x_n \rightharpoonup x$ weakly;
- $W_w(x_n) = \left\{x : \exists\, x_{n_j} \rightharpoonup x\right\}$ is the weak *w*-limit set of the sequence $\{x_n\}_{n=1}^{\infty}$

## III MAIN RESULTS

In this section we present a modification of the gradient projection method and prove its strong convergence. Our result in this section complements the result of Xu [12]. Furthermore, using the technique in [7, 12], we obtain the following theorem.

*Theorem 3.1* *Let C be a non empty, closed and convex subset of a real Hilbert space H. Suppose that the minimization problem (4) is consistent and let S denote its solution set. Assume that the gradient ∇f is L-Lipschitzian with constant L>0. For any given u∈C, let the sequence* $\{x_n\}$ *be generated iteratively by,* $x_1 \in C$*,*

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_C\big(x_n - \lambda_n \nabla f(x_n)\big) + \alpha_n(u - x_n) \qquad (10)$$

*where* $\{\alpha_n\} \subset [0,1]$ *and* $\{\lambda_n\}$ *in (0,L/2) satisfying the following conditions:*

*(C₁)* $\lim_{n\to\infty}\alpha_n = 0$,

*(C₂)* $\sum_{n=1}^{\infty}\alpha_n = +\infty$,

*(C₃)* $0 < \liminf \lambda_n \leq \limsup \lambda_n < 2/L$

*(C₄)* $\beta_n \in [a,b] \subset (0,1), \ \forall\, n \geq 1$.

*Then the sequence* $\{x_n\}$ *converges strongly to a minimizer* $\bar{x}$ *of (4), which is closest to u from the solution set S. In other words* $\bar{x} = P_S u$.

*Proof.* Inspired by the method of proof of [8], it is well known that:

(a) $x^* \in C$ solves the minimization problem (4) if and only if $x^*$ solves the fixed point equation

$$x^* = P_C(I - \lambda\nabla f)\ x^*$$

where $\lambda > 0$ is any fixed positive number. For the sake of simplicity, we assume that (due to condition $C_3$)

$$0 < a \leq \lambda_n \leq b < \tfrac{2}{L},\ n \geq 1,$$

where *a* and *b* are constants.

(b) The gradient $\nabla f$ is $\frac{1}{L} - ism$.

(c) $P_C(I - \lambda\nabla f)$ is $\frac{2+\lambda L}{4}$ −average for $0 < \lambda < \frac{2}{L}$. Hence we have that, for each *n,* $P_C(I - \lambda\nabla f)$ is $\frac{2+\lambda L}{4}$ −average. Therefore we can write

$$P_C(I - \lambda\nabla f) = \frac{2+\lambda_n L}{4} I + \frac{2-\lambda_n L}{4} T_n = (1-\beta_n)I + \beta_n T_n, \qquad (11)$$

where $T_n$ is nonexpansive and $\beta_n = \frac{2+\lambda_n L}{4} \in [a_1, b_1]$ where $a_1 = \frac{2+aL}{4}$ and $b_1 = \frac{2+bL}{4} < 1$.

Then we can write (10) as

$$x_{n+1} = (1-\beta_n)x_n + \beta_n[(1-\beta_n)x_n + \beta_n T_n x_n] + \alpha_n(u - x_n) \qquad (12)$$

Let $P \in S$, observing that $T_n p = p$, we have

$$\|x_{n+1} - p\| = \|(1-\alpha_n - \beta_n)x_n + (1-\beta_n)\beta_n + \beta_n^2 T_n x_n + \alpha_n u - p\|$$
$$= \|(1-\alpha_n)(x_n - p) - \beta_n^2(x_n - p) + \beta_n^2(T_n x_n - p) + \alpha_n(u-p)\|$$
$$\leq (1-\alpha_n - \beta_n^2)\|x_n - p\| + \beta_n^2\|T_n x_n - p\| + \alpha_n\|u-p\|$$
$$\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|u - p\|$$
$$\leq \max\{\|x_n - p\|, \|u-p\|\}.$$

Therefore, by induction, we have

$$\|x_n - p\| \leq \max\{\|x_n - p\|, \|u-p\|\}, \ \forall n \geq 1.$$

Hence the sequence $\{x_n\}$ is bounded and so $\{T_n x_n\}$.

Now we have the following:

$$\|x_{n+1} - p\|^2 = \|(1-\alpha_n - \beta_n) + \beta_n[(1-\beta_n)x_n + \beta_n T_n x_n] + \|^2$$
$$= \|(x_n - p) - \beta_n^2(x_n - T_n x_n) + \alpha_n(u - x_n)\|^2. \quad (13)$$

We can observe that;

$$\|T_n x - p\|^2 \leq \|x_n - p\|^2$$
$$\Rightarrow \langle T_n x - p, T_n x - p\rangle \leq \langle x-p, x - T_n x\rangle + \langle x-p, T_n x - p\rangle$$
$$\Rightarrow \langle T_n x - p, T_n x - x\rangle \leq \langle x - p, x - T_n x\rangle$$
$$\Rightarrow \langle T_n x - x, T_n x - x\rangle + \langle x-p, T_n x - x\rangle \leq \langle x-p, x-T_n x\rangle,$$

which implies that

$$\|T_n x - p\|^2 \leq 2\langle x-p, x - T_n x\rangle. \quad (14)$$

Using Lemma (2.1), (13) and (14) we obtain that

$$\|x_{n+1} - p\|^2 = \|(x_n - p) - \beta_n^2(x_n - T_n x_n) + \alpha_n(u - x_n)\|^2$$
$$\leq \|x_n - p\|^2 + \beta_n^4\|x_n - T_n x_n\|^2 + \beta_n^2\|T_n x_n - x_n\|^2 - 2\langle x_n - p, \beta_n^2(x_n - Tx_n)\rangle + 2\langle \alpha_n(u - x_n), \ x_{n+1} - p\rangle$$
$$\leq \|x_n - p\|^2 + \beta_n^4\|x_n - T_n x_n\|^2 - \beta_n^2\|T_n x_n - x_n\|^2 - 2\alpha_n\langle x_n - u, \ x_{n+1} - p\rangle$$
$$= \|x_{n+1} - p\|^2 + \beta_n^2(\beta_n^2 - 1)\|x_n - T_n x_n\|^2 - 2\alpha_n\langle x_n - u, \ x_{n+1} - p\rangle.$$

Hence,

$$\beta_n^2(\beta_n^2 - 1)\|x_n - T_n x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - 2\alpha_n\langle x_n - u, \ x_{n+1} - p\rangle. \quad (15)$$

Since $\{x_n\}$ is bounded, then there exists a constant $M \geq 0$ such that $-2\langle x_n - u, \ x_{n+1} - p\rangle \leq M, \ \forall n \geq 1$. Therefore (15) gives

$$\|x_{n+1} - p\|^2 - \|x_n - p\|^2 + \beta_n^2(\beta_n^2 - 1)\|x_n - T_n x_n\|^2 \leq M\alpha_n. \quad (16)$$

The rest of the proof will be done considering 2 cases.

*Case 1:* Assume that the sequence $\{\|x_n - x^*\|\}$ is a monotonically decreasing sequence. Then $\{\|x_n - x^*\|\}$ converges.

Clearly, we have that

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \to 0.$$

Now by (16) we obtain that

$$\lim_{n\to\infty} \beta_n^4 \|x_n - T_n x_n\|^2 = 0.$$

Using the condition that $\beta_n \in [a,b] \subset (0,1)$, we have

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0. \quad (17)$$

Now we show that $W_\omega(x_n) \subset S$.

Let $x^* \in W_\omega(x_n)$ and $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$. We may assume that $\lambda_{n_j} \to \lambda$; then $0 < \lambda < \frac{2}{L}$. We set $T := P_C(I - \lambda\nabla f)$, hence $T$ is nonexpansive.

From (17), we have

$$\left\|x_{n_j} - Tx_{n_j}\right\| \leq \left\|x_{n_j} - T_{n_j}x_{n_j}\right\| + \left\|T_{n_j}x_{n_j} - Tx_{n_j}\right\|$$
$$= \left\|x_{n_j} - T_{n_j}x_{n_j}\right\| + \left\|P_C(x_{n_j} - \lambda_{n_j}\nabla f(x_{n_j})) - P_C(x_{n_j} - \lambda\nabla f(x_{n_j})\right\|$$
$$\leq \left\|x_{n_j} - T_{n_j}x_{n_j}\right\| + \left|\lambda_{n_j} - \lambda\right| \left\|\nabla f(x_{n_j})\right\|$$
$$\leq \left\|x_{n_j} - T_{n_j}x_{n_j}\right\| + \left|\lambda_{n_j} - \lambda\right| M_1 \to 0, n_j \to 0.$$

Therefore, by Lemma (2.3) we have that

$$W_\omega(x_n) \subset F(T) = S.$$

We next prove that the sequence $\{x_n\}$ converges strongly to $\bar{x} \in S$, where $\bar{x}$ is the solution of (4), which is the closest to $u$ from the solution set $S$.

Firstly, we show that $limsup_{n\to\infty}\langle x_n - \bar{x}, u - \bar{x}\rangle \leq 0$. Observe that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying

$$limsup_{n\to\infty}\langle x_n - \bar{x}, u - \bar{x}\rangle = limsup_{j\to\infty}\langle x_{n_j} - \bar{x}, u - \bar{x}\rangle.$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_i}} \rightharpoonup p \in F(T) = S$, without loss of generality, we assume that $x_{n_j} \rightharpoonup p \in F(T) = S$.

Then we obtain

$$limsup_{n\to\infty}\langle x_n - \bar{x}, u - \bar{x}\rangle = limsup_{j\to\infty}\langle x_{n_j} - \bar{x}, u - \bar{x}\rangle = \langle p - \bar{x}, u - \bar{x}\rangle \leq 0. \quad (18)$$

Now, we have

$$\|x_{n+1}-\bar{x}\|^2 = \|(1-\alpha_n-\beta_n)x_n+\beta_n[(1-\beta_n)x_n + \beta_n T_n x_n] + \alpha_n u - \bar{x}\|^2$$
$$= \|(1-\alpha_n)(x_n-\bar{x})+\alpha_n(u-\bar{x}) - \beta_n(x_n - T_n x_n)\|^2$$
$$\leq (1-\alpha_n)\|x_n-\bar{x}\|^2 + 2\alpha_n\langle u-\bar{x}, x_{n+1}-\bar{x}\rangle - 2\beta_n^2\langle x_n - T_n x_n,\ x_{n+1}-\bar{x}\rangle$$
$$\leq (1-\alpha_n)\|x_n-\bar{x}\|^2 + 2\alpha_n\langle u-\bar{x}, x_{n+1}-\bar{x}\rangle - 2\alpha_n\beta_n^2\langle x_n - T_n x_n,\ x_{n+1}-\bar{x}\rangle. \quad (19)$$

It is clear that $\lim_{n\to\infty}\beta^2{}_n\langle x_n - Tx_n, x_{n+1}-\tilde{x}\rangle = 0$.

Therefore, by Lemma (2.4) applied to (19),

we obtain that $\lim_{n\to\infty}\|x_n-\tilde{x}\| = 0$.

*Case 2:* Assume that $\{\|x_n - x^*\|\}$ is not a monotonocally decreasing sequence. Set

$\Gamma_n = \{\|x_n-x^*\|\}^2$ and let $\tau$ : N → N be a mapping for all $n \geq n_0$ (for some $n_0$ large enough )

By $\tau(n) = max\{k \in N: k \leq n,\ \Gamma_k \leq \Gamma_{k+1}\}.\}$.

Clearly, $\tau$ is a non-decreasing sequence such that

$\tau(n) \to \infty\ as\ n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$.

From (17) we can see that

$$\beta_{\tau(n)}^2(1-\beta_{\tau(n)}^2)\|x_{\tau(n)} - T_{\tau(n)}x_{\tau(n)}\| \leq 2\alpha_{\tau(n)}M \to 0$$

as $n\to\infty$. Further more ,we have that

$$\|x_{\tau(n)} - T_{\tau(n)}x_{\tau(n)}\|^2 \to 0 \text{ as } n\to\infty.$$

Using similar argument as in case 1, we can show that $x_{\tau(n)}$ converges weakly to $\tilde{x}$ as $\tau(n)\to\infty$ and

$\lim_{\tau(n)\to\infty} sup\langle u - \tilde{x}, x_{\tau(n)} - \tilde{x}\rangle \leq 0$. We know that for all

$$n \geq n_0, 0 \leq \|x_{\tau(n)+1}-\tilde{x}\|^2 - \|x_{\tau(n)}-\tilde{x}\|^2 \leq \alpha_n[2\langle u-\tilde{x}, x_{\tau(n)+1}-\tilde{x}\rangle - 2\beta^2{}_n\langle T_{\tau(n)}x_{\tau(n)} - x_{\tau(n)}x_{\tau(n)+1}, x_{\tau(n)+1}-\tilde{x}\rangle - \|x_{\tau(n)}-\tilde{x}\|^2]$$

which implies that

$$\|x_{\tau(n)}-\tilde{x}\|^2 \leq 2\langle u-\tilde{x}, x_{\tau(n)+1}-\tilde{x}\rangle - 2\beta^2{}_n\langle T_{\tau(n)}x_{\tau(n)} - x_{\tau(n)}x_{\tau(n)+1}, x_{\tau(n)+1}-\tilde{x}\rangle.$$

Therefore, we conclude that $\lim_{n\to\infty}\|x_{\tau(n)}-\tilde{x}\| = 0$

Hence, $\lim_{n\to\infty}\Gamma_{\tau(n)} = \Gamma_{\tau(n)+1} = 0$.

Furthermore, for $n \geq n_0$, one could observe easily that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$, that is $n > \tau(n)$, because $\Gamma_j \leq \Gamma_{i+1}$ for $\tau(n)+1 \leq j \leq n$.

As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq max\{\Gamma_{\tau(n)},\ \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Thus $\lim_{n\to\infty}\Gamma_{\tau(n)} = 0$. that is $\{x_n\}$ converges strongly to $\tilde{x}$. This completes the proof.

*Corollary 3.2 Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Suppose that the minimization problem (5) is consistent, and let $S$ denote its solution set. Assume that the gradient $\nabla f$ is $L$ −Lipschitzian with constant $L > 0$. For any given $u \in C$, let the sequence $\{x_n\}$ be generated iteratively by $x_1 \in C$,*

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_C(x_n - \lambda\nabla f(x_n)) + \alpha_n(u - x_n), \quad (20)$$

*where $\lambda \in (0, L/2)$ and $\{\alpha_n\} \subset [0,1]$ satisfies the following conditions:*

*(C1)* $lim_{n\to\infty}\alpha_n = 0$

*(C2)* $\sum_{n=1}^{\infty}\alpha_n$.

*(C3)* $\beta_n \in [a,b] \subset (0,1),\ \forall\, n \geq 1$.

*Then the sequence $\{x_n\}$ converges strongly to a minimizer $\bar{x}$ of (4).*

## IV APPLICATION

Applications of Theorem 3.1 is given in this section.

### I. MINIMUM-NORM SOLUTION

We apply Theorem 3.1 to find the minimum-norm solution of problem (4). Suppose that the minimization problem (4) is consistent and let its solution set be denoted by S. A point $x^* \in S$ is said to be the minimum-norm solution of problem (4) if $\|x^*\| = min\{\|x\| : x \in S\}$. In other words, $x^*$ is the projection of the origin onto S.

*Theorem 4.1 Let C be a non empty, closed and convex subset of a real Hilbert space H, with $0 \in C$. Suppose that the minimization problem (4) is consistent and let S denote its solution set. Assume that the gradient $\nabla f$ is L-Lipschitzian with constant L>0. Let the sequence $\{x_n\}$ be generated iteratively by, $x_1 \in C$,*

$$x_{n+1} = (1-\beta_n-\alpha_n)x_n + \beta_n P_C(x_n - \lambda_n\nabla f(x_n)) \quad (21)$$

*where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\}$ in (0,L/2) satisfying the following conditions:*

*(C1)* $lim_{n\to\infty}\alpha_n = 0$,

*(C2)* $\sum_{n=1}^{\infty}\alpha_n = +\infty$,

*(C3)* $0 < liminf\,\lambda_n \leq limsup\,\lambda_n < 2/L$

*(C4)* $\beta_n \in [a,b] \subset (0,1),\ \forall\, n \geq 1$.

*Then the sequence* $\{x_n\}$ *converges strongly to a minimizer* $\bar{x}$ *of (4), which is closest to the origin from the solution set S. In other words* $\|\bar{x}\| = min\{\|x\| : x \in S\}$ .

*Proof.* In Theorem 3.1, we take $u \equiv 0$. Therefore by Theorem 3.1, we obtain that the sequence $\{x_n\}$ converges strongly to a minimizer $\bar{x}$ of (4), where $\bar{x} = P_S 0$, i.e., $\|\bar{x}\| = min\{\|x\| : x \in S\}$.

## II. SPLIT FEASIBILITY PROBLEM

We apply Theorem 3.1 to the split feasibility problem (denoted as *SPF*), which was introduced by Censor and Elfving [4]. *SFP* has received considerable attention of many authors because of its applications in image reconstruction, signal processing and intensity-modulation therapy (see [3, 8, 10] and the references therein).

*SFP* can be mathematically formulated as a problem of finding a point $x$ with the property that

$$x \in C \text{ and } Bx \in Q, \qquad (22)$$

Where $Q$ and $C$ are nonempty, closed and convex subset of Hilbert spaces $H_1$ and $H_2$ respectively and $B: H_1 \to H_2$ is a bounded linear operator.

Clearly, $x^*$ is a solution of *SFP* (22) if and only if $x^* \in C$ and $Bx^* - P_Q Bx^* = 0$.

The proximity function $f$ is defined by

$$f(x) = \frac{1}{2}\|Bx - PQBx\|^2 \qquad (23)$$

and consider the constrained convex minimization problem

$$\min_{x\in C} f(x) = \min_{x\in C} \frac{1}{2}\|Bx^* - PQBx^*\|^2 \qquad (24)$$

Then $x^*$ solves the minimization problem (24). In [3] CQ algorithm was introduced to solve *SFP*.

$$x_{n+1} = P_C\left(I - \lambda B^*\left(I - P_Q\right)B\right)x_n,\ n \geq n_0, \qquad (25)$$

where $0 < \lambda \frac{2}{\|B\|^2}$. It was proved that the sequence generated by (25) converges weekly to a solution of the *SFP*.

Now we introduce the following algorithm to obtain a strong convergence iterative sequence to solve the *SFP*. Let $u \in C$ be given and the sequence $\{x_n\}$ be generated by $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= (1-\beta_n)x_n \\ &+\beta_n P_C(I - \lambda_n(B^*(I - P_Q)B + \gamma I))x_n \\ &+\alpha_n(u - x_n), n \geq 1 \end{aligned} \qquad (26)$$

where $\{\alpha_n\} \subset [0,1], \gamma > 0$ and $\{\lambda_n\}$ *in* $(0,2/L)$ satisfy the following conditions:

$(C1)\lim_{n\to\infty} \alpha_n = 0$.

$(C2)\ \sum_{n=1}^{\infty} \alpha_n = \infty$.

$(C_3{}^*)\ 0 < liminf \lambda_n \leq limsup \lambda_n < \frac{2}{\|B\|^2+\gamma}$.

*(C₄)* $\beta_n \in [a,b] \subset (0,1),\ \forall n \geq 1$.

Applying Theorem 3.1, we obtain the following convergence result for solving the *SFP* (22).

*Theorem 4.2 Assume that the split feasibility problem (SFP) (22) is consistent. Let the sequence* $\{x_n\}$ *be generated by (26), where the sequence* $\{\alpha_n\} \subset [0,1]$ *and* $\{\lambda_n\}$ *in* $\left(0,\frac{2}{L}\right)$ *satisfy conditions* $(C_1 - C_3^*)$. *Then the sequence* $\{x_n\}$ *converges strongly to a solution of the split feasibility problem (22).*

*Proof* From the definition of the proximity function $f$, we have

$$\nabla f(x) = B^*\left(I - P_Q\right)Bx \qquad (27)$$

and $\nabla f$ is Lipchitz continuous, i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \leq \boldsymbol{L}\|\boldsymbol{x} - \boldsymbol{y}\| \qquad (28)$$

where $L = \|B\|^2$.

Set $f_\gamma(x) = f(x) + \frac{\gamma}{2}\|x\|^2$.

Consequently,

$$\begin{aligned} \nabla f_\gamma(x) &= \nabla f(x) + \gamma x \\ &= B^*\left(I - P_Q\right)Bx + \gamma x \end{aligned}$$

and $\nabla f_\gamma$ is Lipchitzian with Lipchitz constant $\|B\|^2 + \gamma$. Therefore the iterative scheme (26) is equivalent to

$$\begin{aligned} x_{n+1} &= (1-\beta_n)x_n + \beta_n P_C\left(x_n - \lambda\nabla f_\gamma(x_n)\right) \\ &+\alpha_n(u - x_n), \end{aligned} \qquad (29)$$

where $\{\alpha_n\} \subset [0,1], \gamma > 0$ and $\{\lambda_n\}$ in $\left(0,\frac{2}{L^*}\right)$ satisfy the following conditions:

$(C1)\lim_{n\to\infty} \alpha_n = 0$

$(C2)\ \sum_{n=1}^{\infty} \alpha_n = \infty$

$(C_3{}^*)\ 0 < liminf \lambda_n \leq limsup \lambda_n < \frac{2}{\|B\|^2+\gamma}$

Where $L^* = \|B\|^2 + \gamma$

The conclusion follows from Theorem 3.1.

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