On Permanence and Asymptotically Periodic Solution of A Delayed Three-level Food Chain Model with Beddington-DeAngelis Functional Response

Changjin Xu, Yusen Wu, and Lin Lu

***Abstract*—This paper deals with a delayed three-level food chain model with Beddington-DeAngelis functional response. By using the differential inequality theory, a set of sufficient conditions are obtained for the permanence of the system. By constructing a suitable Liapunov function, we derive that the system has a unique asymptotically periodic solution which is globally asymptotically stable. An example is given to illustrate the effectiveness of the results. The paper ends with a brief conclusion.**

***Index Terms*—food chain model, permanence, Beddington-DeAngelis functional response, asymptotically periodic solution, Liapunov function.**

I. Introduction

In recent years, predator-prey systems which model some biological phenomena and relationship between predators and preys in the real world play a crucial role in mathematics. They have been extensively investigated in many ways by numerous researches [1]. For example, Dai et al. [2] analyzed the multiple periodic solutions for impulsive Gause-type ratio-dependent predator-prey systems with non-monotonic numerical responses. Wang and Fan [3] made a discussion about the multiple periodic solutions for a non-autonomous delayed predator-prey model with harvesting terms. Li and Ye [4] investigated the multiple positive almost periodic solutions to an impulsive non-autonomous Lotka-Volterra predator-prey system with harvesting terms. Zhang and Luo [5] focused on the multiple periodic solutions of a delayed predator-prey system with stage structure for the predator. Zhang et al. [6] considered the multiplicity of positive periodic solutions to a generalized delayed predator-prey system with stocking. For more work about predator-prey models, one can see [7-15]. It shall be pointed out that all the papers mentioned above are concerned with the predator-prey models with periodic coefficients. However, the asymptotically periodic systems describe our real word more realistic and more accurate than the periodic ones, but the research work about asymptotically periodic predator-prey models is scare at present(see e.g.,[16-17,23-25]).

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In this paper, we will consider the following delayed predator-prey system that is given by Li and Wang [18]:

$$\begin{cases} \dfrac{du_1}{dt} = u_1(t)\Big[a_1 - b_1u_1(t) \\ \qquad - \dfrac{s_1u_2(t)}{A_1+u_1(t)+B_1u_2(t)}\Big], \\ \dfrac{du_2}{dt} = u_2(t)\Big[-a_2 - b_2u_2(t) \\ \qquad + \dfrac{s_3u_1(t-\tau)}{A_1+u_1(t-\tau)+B_1u_2(t-\tau)} \\ \qquad - \dfrac{s_2u_3(t)}{A_2+u_2(t)+B_2u_3(t)}\Big], \\ \dfrac{du_3}{dt} = u_3(t)\Big[-a_3 - b_3u_3(t) \\ \qquad + \dfrac{s_4u_2(t-\tau)}{A_2+u_2(t-\tau)+B_2u_3(t-\tau)}\Big], \end{cases} \tag{1}$$

with the initial condition $u_0(0)\ge 0, u_2(0)\ge 0, u_3(0)\ge 0$, where $u_1(t), u_2(t)$ and $u_3(t)$ denote the population density of prey, predator and top-predator at time t, respectively. $a_i, b_i, s_j (i=1,2,3; j=1,2,3,4)$ are positive constants. In detailed biologically meaning, one can see [18]. By choosing the time delay as bifurcation parameter, Li and Wang [18] found that Hopf bifurcation occurs as the delay passes through a sequence of critical values.

In real word, any biological or environmental parameters are naturally subject to fluctuation in time [19]. In 1977, Cushing [20] pointed out that it is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed (for example, those due to season effects of weather, food supply, mating habits, hunting or harvesting seasons, etc.). Thus the assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment. Further, to describe the real word more accurate, the assumption of asymptotically periodic parameters shall be considered. Motivated by the discussion above, we modify system (1) as

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follows:

$$
\begin{cases}
\dfrac{du_1}{dt} = u_1(t)\Big[a_1(t) - b_1(t)u_1(t) \\
\qquad - \dfrac{s_1(t)u_2(t)}{A_1(t) + u_1(t) + B_1(t)u_2(t)}\Big], \\
\dfrac{du_2}{dt} = u_2(t)\Big[- a_2(t) - b_2(t)u_2(t) \\
\qquad + \dfrac{s_3(t)u_1(t-\tau)}{A_1(t) + u_1(t-\tau) + B_1(t)u_2(t-\tau)} \\
\qquad - \dfrac{s_2(t)u_3(t)}{A_2(t) + u_2(t) + B_2(t)u_3(t)}\Big], \\
\dfrac{du_3}{dt} = u_3(t)\Big[- a_3(t) - b_3(t)u_3(t) \\
\qquad + \dfrac{s_4(t)u_2(t-\tau)}{A_2(t) + u_2(t-\tau) + B_2(t)u_3(t-\tau)}\Big].
\end{cases} \tag{2}
$$

Now we define $R_+ = [0, +\infty)$ and introduce the concept of the asymptotically function.

Definition 1.1. *If $f \in C(R_+, R)$, where $f(t) = g(t) + \alpha(t)$, $g(t)$ is continuous T-periodic function and $\lim_{t\to+\infty} \alpha(t) = 0$, then $f(t)$ is called asymptotically T-periodic function.*

Throughout this paper, we always assume that

(H1) $a_i(t), d_i(t), \alpha_i(t), \beta_i(t)(i = 1, 2), r(t), \gamma(t), K(t)$ and $\delta(t)$ are all continuous positive, bounded asymptotically periodic functions.

This paper is organized as follows. In Section 2, the permanence of system (2) is investigated by using the differential inequality theory. In Section 3, the existence and uniqueness of asymptotically periodic solution are analyzed by constructing a suitable Liapunov function. An example is given to illustrate the effectiveness of the results is Section 4. A brief conclusion is given in Section 5.

II. Permanence

For convenience in the following discussion, we always use the notations:

$$f^l = \inf_{t\in R} f(t), \quad f^u = \sup_{t\in R} f(t),$$

where $f(t)$ is a continuous function. The initial value condition of system (2) is $u_1(0) = \phi_1(0) > 0, u_2(0) = \phi_2(0) > 0, u_3(0) = \phi_3(0) > 0$. In order to obtain the main results of this paper, we shall first state some definitions and several lemmas which will be useful in proving the main results.

Definition 2.1. *We say that system (2) is permanence if there are positive constants $m_i, M_i(i = 1, 2, 3)$ such that for each positive solution $(u_1(t), u_2(t), u_3(t))$ of system (2) satisfies*

$$m_i \le \lim_{t\to+\infty} \inf u_i(t) \le \lim_{t\to+\infty} \sup u_i(t) \le M_i (i = 1, 2, 3).$$

Definition 2.2. *The solution $X(t, t_0, \phi)$ is called ultimately bounded. If there exists $B > 0$ such that for any $t_0 \ge 0, \phi \in C$, there exists $T = T(t_0, \phi) > 0$ when $t \ge t_0 + T, |X(t, t_0, \phi)| \le B$.*

Lemma 2.1. [21] *If $a > 0, b > 0$ and $\dot{x} \ge x(b - ax)$, when $t \ge 0$ and $x(0) > 0$, we have*

$$\lim_{t\to+\infty} \inf x(t) \ge \frac{b}{a}.$$

If $a > 0, b > 0$ and $\dot{x} \le x(b - ax)$, when $t \ge 0$ and $x(0) > 0$, we have

$$\lim_{t\to+\infty} \sup x(t) \le \frac{b}{a}.$$

Now we state our permanence result for system (2).

Lemma 2.2. *The set $R^3_+ = \{(u_1, u_2, u_3)|u_1 > 0, u_2 > 0, u_3 > 0\}$ is the positively invariant set of system (2).*

Proof It follows from the initial value condition $u_1(0) = \phi_1(0 > 0, u_2(0) = \phi_2(0) > 0$ and $u_3(0) = \phi_3(0) > 0$ that

$$
\begin{cases}
u_1(t) = u_1(0)\exp\Big\{\displaystyle\int_0^t \Big[a_1(s) - b_1(s)u_1(s) \\
\qquad - \dfrac{s_1(s)u_2(s)}{A_1(s) + u_1(s) + B_1(s)u_2(s)}\Big] ds\Big\}, \\
u_2(t) = u_2(0)\exp\Big\{\displaystyle\int_0^t \Big[- a_2(s) - b_2(s)u_2(s) \\
\qquad + \dfrac{s_3(s)u_1(s-\tau)}{A_1(s) + u_1(s-\tau) + B_1(s)u_2(s-\tau)} \\
\qquad - \dfrac{s_2(s)u_3(s)}{A_2(s) + u_2(s) + B_2(t)u_3(s)}\Big] ds\Big\}, \\
u_3(t) = u_3(0)\exp\Big\{\displaystyle\int_0^t \Big[- a_3(s) - b_3(s)u_3(s) \\
\qquad + \dfrac{s_4(s)u_2(s-\tau)}{A_2(s) + u_2(s-\tau) + B_2(s)u_3(s-\tau)}\Big] ds\Big\}.
\end{cases} \tag{3}
$$

The proof of Lemma 2.2 is complete.

Theorem 2.1. *Let M_1, M_2, M_3, m_1, m_2 and m_3 be defined by (4), (6), (8), (10), (15) and (19), respectively. In addition to the condition (H1), suppose that the following conditions*

$$
\begin{aligned}
&(H2)\ s^u_3 > a^l_2, s^u_4 > a^l_3, a^l_1 B^l_1 > s^u_1, \\
&\qquad s^l_4 m_2 > a^u_3(A^u_2 + M_2 + B^u_2 M_3); \\
&(H3)\ s^l_3 m_1 B^l_2 > s^u_2(A^u_1 + M_1 + B^u_1 M_2) \\
&\qquad + a^u_2 B^l_2(A^u_1 + M_1 + B^u_1 M_2)
\end{aligned}
$$

hold, then system (2) is permanent, that is, there exist positive constants $m_i, M_i(i = 1, 2, 3)$ which are independent of the solution of system (2), such that for any positive solution $(u_1(t), u_2(t), u_3(t))$ of system (2) with the initial condition

$$u_1(0) \ge 0, u_2(0) \ge 0, u_3(0) \ge 0,$$

one has

$$m_i \le \lim_{t\to+\infty} \inf u_i(t) \le \lim_{t\to+\infty} \sup u_i(t) \le M_i (i = 1, 2, 3).$$

Proof Obviously, system (2) with the initial value condition $(u_1(0), u_2(0), u_3(0))$ has positive solution $(u_1(t), u_2(t), u_3(t))$ passing through $(u_1(0), u_2(0), u_3(0))$. Let $(u_1(t), u_2(t), u_3(t))$ be any positive solution of system (2) with the initial condition $(u_1(0), u_2(0), u_3(0))$. It follows from the first equation of system (2) that

$$
\begin{aligned}
\frac{du_1(t)}{dt} &= u_1(t)\Big[a_1(t) - b_1(t)u_1(t) \\
&\qquad - \frac{s_1(t)u_2(t)}{A_1(t) + u_1(t) + B_1(t)u_2(t)}\Big] \\
&\le u_1(t)\left[a^u_1 - b^l_1 u_1(t)\right].
\end{aligned}
$$

It follows from Lemma 2.1 that

$$\lim_{t\to+\infty}\sup u_1(t) \le \frac{a_1^u}{b_1^l} := M_1. \tag{4}$$

For any positive constant $\varepsilon_1 > 0$, it follows (4) that there exists a $T_1 > 0$ such that for all $t > T_1$,

$$u_1(t) \le M_1 + \varepsilon_1. \tag{5}$$

By the second equation of system (2), we have

$$\begin{aligned}\frac{du_2(t)}{dt} &= u_2(t)\Big[-a_2(t) - b_2(t)u_2(t) \\ &\quad +\frac{s_3(t)u_1(t-\tau)}{A_1(t)+u_1(t-\tau)+B_1(t)u_2(t-\tau)} \\ &\quad -\frac{s_2(t)u_3(t)}{A_2(t)+u_2(t)+B_2(t)u_3(t)}\Big] \\ &\le u_2(t)\left[-a_2^l - b_2^l u_2(t) + s_3^u\right].\end{aligned}$$

It follows from Lemma 2.1 that

$$\lim_{t\to+\infty}\sup u_2(t) \le \frac{s_3^u - a_2^l}{b_2^l} := M_2. \tag{6}$$

For any positive constant $\varepsilon_2 > 0$, it follows from (6) that there exists a $T_2 > 0$ such that for all $t > T_2$,

$$u_2(t) \le M_2 + \varepsilon_2. \tag{7}$$

By the third equation of system (2), we have

$$\begin{aligned}\frac{du_3(t)}{dt} &= u_3(t)\Big[-a_3(t) - b_3(t)u_3(t) \\ &\quad +\frac{s_4(t)u_2(t-\tau)}{A_2(t)+u_2(t-\tau)+B_2(t)u_3(t-\tau)}\Big] \\ &\le u_3(t)\left[-a_3^l - b_3^l u_3(t) + s_4^u\right].\end{aligned}$$

It follows from Lemma 2.1 that

$$\lim_{t\to+\infty}\sup u_3(t) \le \frac{s_4^u - a_3^l}{b_3^l} := M_3. \tag{8}$$

For any positive constant $\varepsilon_3 > 0$, it follows from (8) that there exists a $T_3 > 0$ such that for all $t > T_3$,

$$u_3(t) \le M_3 + \varepsilon_3. \tag{9}$$

In view of the first equation of system (2), we have

$$\begin{aligned}\frac{du_1(t)}{dt} &= u_1(t)\Big[a_1(t) - b_1(t)u_1(t) \\ &\quad -\frac{s_1(t)u_2(t)}{A_1(t)+u_1(t)+B_1(t)u_2(t)}\Big] \\ &\ge u_1(t)\left[a_1(t) - b_1(t)u_1(t) - \frac{s_1(t)}{B_1(t)}\right] \\ &\ge u_1(t)\left[a_1^l - b_1^u u_1(t) - \frac{s_1^u}{B_1^l}\right].\end{aligned}$$

Thus, as a direct corollary of Lemma 2.1, one has

$$\lim_{t\to+\infty}\inf u_1(t) \ge \frac{a_1^l B_1^l - s_1^u}{b_1^u B_1^l} := m_1. \tag{10}$$

For any positive constant $\varepsilon_4 > 0$, it follows from (10) that there exists a $T_4 > 0$ such that for all $t > T_4$,

$$u_1(t) \ge m_1 - \varepsilon_4. \tag{11}$$

For $t \ge \max\{T_1, T_2, T_4\}$, from (5),(7),(11) and the second equation of system (1.2), we have

$$\begin{aligned}\frac{du_2(t)}{dt} &= u_2(t)\Big[-a_2(t) - b_2(t)u_2(t) \\ &\quad +\frac{s_3(t)u_1(t-\tau)}{A_1(t)+u_1(t-\tau)+B_1(t)u_2(t-\tau)} \\ &\quad -\frac{s_2(t)u_3(t)}{A_2(t)+u_2(t)+B_2(t)u_3(t)}\Big] \\ &\ge u_2(t)\Big[-a_2^u - b_2^u u_2(t) \\ &\quad +\frac{s_3^l(m_1-\varepsilon_4)}{A_1^u+(M_1+\varepsilon_1)+B_1^u(M_2+\varepsilon_2)} \\ &\quad -\frac{s_2^u}{B_2^l}\Big].\end{aligned} \tag{12}$$

It follows form Lemma 2.1 and (12) that

$$\begin{aligned}&\lim_{t\to+\infty}\inf u_2(t) \\ &\ge \frac{\Theta}{B_2^l b_2^u[A_1^u+(M_1+\varepsilon_1)+B_1^u(M_2+\varepsilon_2)]},\end{aligned} \tag{13}$$

where

$$\begin{aligned}\Theta &= s_3^l(m_1-\varepsilon_4)B_2^l \\ &\quad -s_2^u[A_1^u+(M_1+\varepsilon_1)+B_1^u(M_2+\varepsilon_2)] \\ &\quad -a_2^u B_2^l[A_1^u+(M_1+\varepsilon_1)+B_1^u(M_2+\varepsilon_2)].\end{aligned} \tag{14}$$

Setting $\varepsilon_i \to 0 (i = 1, 2, 4)$ in (13) leads to

$$\begin{aligned}\lim_{t\to+\infty}\inf u_2(t) &\ge \frac{s_3^l m_1 B_2^l}{B_2^l b_2^u(A_1^u+M_1+B_1^u M_2)} \\ &\quad -\frac{s_2^u(A_1^u+M_1+B_1^u M_2)}{B_2^l b_2^u(A_1^u+M_1+B_1^u M_2)} \\ &\quad -\frac{a_2^u B_2^l(A_1^u+M_1+B_1^u M_2)}{B_2^l b_2^u(A_1^u+M_1+B_1^u M_2)} := m_2.\end{aligned} \tag{15}$$

For any positive constant $\varepsilon_5 > 0$, it follows from (15) that there exists a $T_5 > \max\{T_1, T_2, T_4\} > 0$ such that for all $t > T_5$,

$$u_2(t) \ge m_2 - \varepsilon_5. \tag{16}$$

For $t \ge \max\{T_2, T_3, T_5\}$, from (7),(9),(16) and the third equation of system (2), we have

$$\begin{aligned}\frac{du_3(t)}{dt} &= u_3(t)\Big[-a_3(t) - b_3(t)u_3(t) \\ &\quad +\frac{s_4(t)u_2(t-\tau)}{A_2(t)+u_2(t-\tau)+B_2(t)u_3(t-\tau)}\Big] \\ &\ge u_3(t)\Big[-a_3^u - b_3^u u_3(t) \\ &\quad +\frac{s_4^l(m_2-\varepsilon_5)}{A_2^u+(M_2+\varepsilon_2)+B_2^u(M_3+\varepsilon_3)}\Big].\end{aligned} \tag{17}$$

It follows from Lemma 2.1 and (17) that

$$\begin{aligned}&\lim_{t\to+\infty}\inf u_3(t) \ge \\ &\frac{s_4^l(m_2-\varepsilon_5)}{b_3^u[A_2^u+(M_2+\varepsilon_2)+B_2^u(M_3+\varepsilon_3)]} \\ &-\frac{a_3^u[A_2^u+(M_2+\varepsilon_2)+B_2^u(M_3+\varepsilon_3)]}{b_3^u[A_2^u+(M_2+\varepsilon_2)+B_2^u(M_3+\varepsilon_3)]}\end{aligned} \tag{18}$$

Setting $\varepsilon_i \to 0 (i = 2, 3, 5)$ in (18) leads to

$$\lim_{t\to+\infty} \inf u_3(t) \geq \frac{s_4^l m_2 - a_3^u(A_2^u + M_2 + B_2^u M_3)}{b_3^u(A_2^u + M_2 + B_2^u M_3} := m_3. \quad (19)$$

By (4),(6),(8),(10),(15) and (19), we can conclude that system (2) is permanent. The proof of Theorem 2.1 is complete.

Denote

$$\Omega = \{(u_1, u_2, u_3)^T \in R_+ | m_i \leq u_i \leq M_i (i = 1, 2, 3)\}.$$

Corollary 2.1. *The set Ω is the ultimately bounded set of system (2).*

III. Existence and uniqueness of asymptotically periodic solution

Let us consider the asymptotically periodic system as follows

$$\frac{dx}{dt} = f(t, x_t), \quad (20)$$

where $f \in C([-r, 0], R^n)$ and for any $x_t \in C$. Define $x_t(\theta) = x(t + \theta), \theta \in [-r, 0]$. For any $x = (x_1, x_2, \cdots, x_n) \in R^n$, we define $|x| = \sum_{i=1}^n |x_i|$, from the above proof, we can see that there exists $H > 0$ such that $|x| < H$. For any $\phi \in C$, define $||\phi|| = \sup_{-r\leq\theta\leq 0} |\phi(\theta)|$. Let $C_H = \{\phi \in C, ||\phi|| < H\}$ and $S_H = \{x \in R^n, |x| < H\}$. In order to focus on the existence and uniqueness of asymptotically periodic solution of system (20), we consider the adjoint system

$$\begin{cases} \dfrac{dx}{dt} = f(t, x_t), \\ \dfrac{dy}{dt} = f(t, y_t). \end{cases} \quad (21)$$

Then we begin with our analysis with Lemma 3.1.

Lemma 3.1.(Yuan [22]) *Let $V \in C(R_+ \times S_H \times S_H, R_+)$ satisfy*
(i) $a(|x - y| \leq V(t, x, y) \leq b(|x - y|)$, where $a(r)$ and $b(r)$ are continuously positively increasing functions;
(ii) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \leq l(|x_1 - x_2| + |y_1 - y_2|)$, where l is a constant and satisfies $l > 0$;
(iii) there exists continuous non-increasing function $P(s)$, such that for $s > 0, P(s) > s$. And as $P(V(t, \phi(0), \phi(0))) > V(t + \theta, \phi(\theta), \phi(\theta)), \theta \in [-r, 0]$, it follows that

$$V'_{(3.2)}(t, \phi(0), \phi(0)) \leq -\delta V(t, \phi(0), \phi(0)),$$

where δ is a constant and satisfies $\delta > 0$. Furthermore, system (20) has a solution $\xi(t)$ for $t \geq t_0$ and satisfies $||\xi_t|| \leq H$. Then system (20) has a unique asymptotically periodic solution, which is uniformly asymptotically stable.

Theorem 3.1. *Let $\sigma_1, \sigma_2, \sigma_3$ and ρ be defined by (32), (33),(34) and (35), respectively. In addition to the conditions (H1)-(H3), assume further that $\mu > 0$ is fulfilled, then there exists a unique asymptotically periodic solution of system (2) wich is uniformly asymptotically stable.*

Proof. It follows from Theorem 2.1 (or Corollary 2.1) that the solution of system (2) is ultimately bounded. Ω is the region of ultimately bounded. We consider the adjoint system of system (2) as follows

$$\begin{cases} \dfrac{du_1}{dt} = u_1(t)\Big[a_1(t) - b_1(t)u_1(t) \\ \qquad - \dfrac{s_1(t)u_2(t)}{A_1(t) + u_1(t) + B_1(t)u_2(t)}\Big], \\ \dfrac{du_2}{dt} = u_2(t)\Big[- a_2(t) - b_2(t)u_2(t) \\ \qquad + \dfrac{s_3(t)u_1(t-\tau)}{A_1(t) + u_1(t-\tau) + B_1(t)u_2(t-\tau)} \\ \qquad - \dfrac{s_2(t)u_3(t)}{A_2(t) + u_2(t) + B_2(t)u_3(t)}\Big], \\ \dfrac{du_3}{dt} = u_3(t)\Big[- a_3(t) - b_3(t)u_3(t) \\ \qquad + \dfrac{s_4(t)u_2(t-\tau)}{A_2(t) + u_2(t-\tau) + B_2(t)u_3(t-\tau)}\Big], \\ \dfrac{dv_1}{dt} = v_1(t)\Big[a_1(t) - b_1(t)v_1(t) \\ \qquad - \dfrac{s_1(t)v_2(t)}{A_1(t) + v_1(t) + B_1(t)v_2(t)}\Big], \\ \dfrac{dv_2}{dt} = v_2(t)\Big[- a_2(t) - b_2(t)v_2(t) \\ \qquad + \dfrac{s_3(t)v_1(t-\tau)}{A_1(t) + v_1(t-\tau) + B_1(t)v_2(t-\tau)} \\ \qquad - \dfrac{s_2(t)v_3(t)}{A_2(t) + v_2(t) + B_2(t)v_3(t)}\Big], \\ \dfrac{dv_3}{dt} = v_3(t)\Big[- a_3(t) - b_3(t)v_3(t) \\ \qquad + \dfrac{s_4(t)v_2(t-\tau)}{A_2(t) + v_2(t-\tau) + B_2(t)v_3(t-\tau)}\Big]. \end{cases} \quad (22)$$

For

$$X(t) = (u_1(t), u_2(t), u_3(t))$$

and

$$U(t) = (v_1(t), v_2(t), v_3(t))$$

are the solutions of system (22) in $\Omega \times \Omega$. Let

$$u_i^*(t) = \ln u_i(t), v_i^*(t) = \ln v_i(t) (i = 1, 2, 3).$$

Now we construct a Lyapunov functional as follows

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) + V_6(t), \quad (23)$$

where

$$\begin{aligned} V_1(t) &= |u_1^*(t) - v_1^*(t)|, \\ V_2(t) &= |u_2^*(t) - v_2^*(t)|, \\ V_3(t) &= |u_3^*(t) - v_3^*(t)|, \\ V_4(t) &= \frac{s_3^u(A_1^u + B_1^u M_2)}{(A_1^l + m_1 + B_1^l m_2)^2} \\ &\quad \times \int_{t-\tau}^t |u_1(s) - v_1(s)| ds, \\ V_5(t) &= \Big[\frac{s_3^u B_1^u M_1}{(A_1^l + m_1 + B_1^l m_2)^2} \\ &\quad + \frac{s_4^u(A_2^u + B_2^u M_3)}{(A_2^l + m_2 + B_2^l m_3)^2}\Big] \\ &\quad \times \int_{t-\tau}^t |u_2(s) - v_2(s)| ds, \\ V_6(t) &= \frac{s_4^u B_2^u M_2}{(A_2^l + m_2 + B_2^l m_3)^2} \\ &\quad \times \int_{t-\tau}^t |u_3(s) - v_3(s)| ds. \end{aligned}$$

Taking $a(r) = b(r) = V_1(t)+V_2(t)+V_3(t)+V_4(t)+V_5(t)+V_6(t)$ and using the inequality $||a| - |b|| \le |a-b|$, we can easily prove that (i) and (ii) in Lemma 3.1 hold true. In the sequel, we will investigate (iii) of Lemma 3.1. It follows from (20) that

$$\begin{aligned}
D^+V_1(t) &= \left(\frac{\dot{u}_1(t)}{u_1(t)} - \frac{\dot{v}_1(t)}{v_1(t)}\right)\operatorname{sign}(u_1(t)-v_1(t)) \\
&\le -b_1^l|u_1(t)-v_1(t)| \\
&\quad + \frac{s_1^u(A_1^u + B_1^u M_2)|u_1(t)-v_1(t)|}{(A_1^l + m_1 + B_1^l m_2)^2} \\
&\quad + \frac{s_1^u B_1^u M_1 |u_2(t)-v_2(t)|}{(A_1^l + m_1 + B_1^l m_2)^2}, \\
D^+V_2(t) &= \left(\frac{\dot{u}_2(t)}{u_2(t)} - \frac{\dot{v}_2(t)}{v_2(t)}\right)\operatorname{sign}(u_2(t)-v_2(t)) \\
&\le -b_2^l|u_2(t)-v_2(t)| \\
&\quad + \frac{s_3^u(A_1^u + B_1^u M_2)}{(A_1^l + m_1 + B_1^l m_2)^2} \\
&\quad \times |u_1(t-\tau)-v_1(t-\tau)| \\
&\quad + \frac{s_3^u B_1^u M_1}{(A_1^l + m_1 + B_1^l m_2)^2} \\
&\quad \times |u_2(t-\tau)-v_2(t-\tau)|, \\
D^+V_3(t) &= \left(\frac{\dot{u}_3(t)}{u_3(t)} - \frac{\dot{v}_3(t)}{v_3(t)}\right)\operatorname{sign}(u_3(t)-v_3(t)) \\
&\le -b_3^l|u_3(t)-v_3(t)| \\
&\quad + \frac{s_4^u(A_2^u + B_2^u M_3)}{(A_2^l + m_2 + B_2^l m_3)^2} \\
&\quad \times |u_2(t-\tau)-v_2(t-\tau)| \\
&\quad + \frac{s_4^u B_2^u M_2}{(A_2^l + m_2 + B_2^l m_3)^2} \\
&\quad \times |u_3(t-\tau)-v_3(t-\tau)|, \\
D^+V_4(t) &= \frac{s_3^u(A_1^u + B_1^u M_2)}{(A_1^l + m_1 + B_1^l m_2)^2} \\
&\quad \times |u_1(t)-v_1(t)| \\
&\quad - \frac{s_3^u(A_1^u + B_1^u M_2)}{(A_1^l + m_1 + B_1^l m_2)^2} \\
&\quad \times |u_1(t-\tau)-v_1(t-\tau)|, \\
D^+V_5(t) &= \left[\frac{s_3^u B_1^u M_1}{(A_1^l + m_1 + B_1^l m_2)^2} \right. \\
&\quad \left. + \frac{s_4^u(A_2^u + B_2^u M_3)}{(A_2^l + m_2 + B_2^l m_3)^2}\right] \\
&\quad \times |u_2(t)-v_2(t)| \\
&\quad - \left[\frac{s_3^u B_1^u M_1}{(A_1^l + m_1 + B_1^l m_2)^2} \right. \\
&\quad \left. + \frac{s_4^u(A_2^u + B_2^u M_3)}{(A_2^l + m_2 + B_2^l m_3)^2}\right] \\
&\quad \times |u_2(t-\tau)-v_2(t-\tau)|, \\
D^+V_6(t) &= \frac{s_4^u B_2^u M_2}{(A_2^l + m_2 + B_2^l m_3)^2} \\
&\quad \times |u_3(t)-v_3(t)| \\
&\quad - \frac{s_4^u B_2^u M_2}{(A_2^l + m_2 + B_2^l m_3)^2} \\
&\quad \times |u_3(t-\tau)-v_3(t-\tau)|.
\end{aligned}$$

Thus

$$\begin{aligned}
D^+V(t) &\le -b_1^l|u_1(t)-v_1(t)| \\
&\quad -b_2^l|u_2(t)-v_1(t)| \\
&\quad -b_3^l|u_3(t)-v_3(t)| \\
&\quad + \frac{2s_1^u(A_1^u + B_1^u M_2)}{(A_1^l + m_1 + B_1^l m_2)^2} \\
&\quad \times |u_1(t)-v_1(t)| \\
&\quad + \frac{s_4^u B_2^u M_2}{(A_2^l + m_2 + B_2^l m_3)^2} \\
&\quad \times |u_3(t)-v_3(t)| \\
&\quad + \left[\frac{(s_1^u + s_3^u)B_1^u M_1}{(A_1^l + m_1 + B_1^l m_2)^2} \right. \\
&\quad \left. + \frac{s_4^u(A_2^u + B_2^u M_3)}{(A_2^l + m_2 + B_2^l m_3)^2}\right] \\
&\quad \times |u_2(t)-v_2(t)|. \qquad (24)
\end{aligned}$$

Nothing that

$$\begin{aligned}
|u_1(t)-v_1(t)| &= |\exp(u_1^*(t)) - \exp(v_1^*(t))| \\
&= \exp(\xi(t))|u_1^*(t) - v_1^*(t)|, \qquad (25)
\end{aligned}$$

$$\begin{aligned}
|u_2(t)-v_2(t)| &= |\exp(u_2^*(t)) - \exp(v_2^*(t))| \\
&= \exp(\eta(t))|u_2^*(t) - v_2^*(t)|, \qquad (26)
\end{aligned}$$

$$\begin{aligned}
|u_3(t)-v_3(t)| &= |\exp(u_3^*(t)) - \exp(v_3^*(t))| \\
&= \exp(\zeta(t))|u_3^*(t) - v_3^*(t)|, \qquad (27)
\end{aligned}$$

where $\xi(t)$ lies between $u_1^*(t)$ and $v_1^*(t)$, $\eta(t)$ lies between $u_2^*(t)$ and $v_2^*(t)$ and $\zeta(t)$ lies between u_3^* and v_3^*, we have

$$\begin{aligned}
m_1|u_1^*(t)-v_1^*(t)| &\le |u_1(t)-v_1(t)| \\
&\le M_1|u_1^*(t)-v_1^*(t)|, \qquad (28)
\end{aligned}$$

$$\begin{aligned}
m_2|u_2^*(t)-v_2^*(t)| &\le |u_2(t)-v_2(t)| \\
&\le M_2|u_2^*(t)-v_2^*(t)|, \qquad (29)
\end{aligned}$$

$$\begin{aligned}
m_3|u_3^*(t)-v_3^*(t)| &\le |u_3(t)-v_3(t)| \\
&\le M_3|u_3^*(t)-v_3^*(t)|. \qquad (30)
\end{aligned}$$

By (24), (28), (29) and (30), we have

$$\begin{aligned}
D^+V(t) &\le -b_1^l m_1|u_1^*(t)-v_1^*(t)| \\
&\quad -b_2^l m_2|u_2^*(t)-v_1^*(t)| \\
&\quad -b_3^l m_3|u_3^*(t)-v_3^*(t)| \\
&\quad + \frac{2s_1^u M_1(A_1^u + B_1^u M_2)}{(A_1^l + m_1 + B_1^l m_2)^2} \\
&\quad \times |u_1^*(t)-v_1^*(t)| \\
&\quad + \frac{s_4^u B_2^u M_2 M_3}{(A_2^l + m_2 + B_2^l m_3)^2} \\
&\quad \times |u_3^*(t)-v_3^*(t)| \\
&\quad + M_2\left[\frac{(s_1^u + s_3^u)B_1^u M_1}{(A_1^l + m_1 + B_1^l m_2)^2} \right. \\
&\quad \left. + \frac{s_4^u(A_2^u + B_2^u M_3)}{(A_2^l + m_2 + B_2^l m_3)^2}\right] \\
&\quad \times |u_2^*(t)-v_2^*(t)| \\
&= -\sigma_1|u_1*(t)-v_1^*(t)| \\
&\quad -\sigma_2|u_2^*(t)-v_2^*(t)| \\
&\quad -\sigma_3|u_3^*(t)-v_3^*(t)|, \qquad (31)
\end{aligned}$$

where

$$\sigma_1 = b_1^l m_1 - \frac{2s_1^u M_1(A_1^u + B_1^u M_2)}{(A_1^l + m_1 + B_1^l m_2)^2}, \tag{32}$$

$$\sigma_2 = b_2^l m_2 - M_2 \left[\frac{(s_1^u + s_3^u)B_1^u M_1}{(A_1^l + m_1 + B_1^l m_2)^2} + \frac{s_4^u (A_2^u + B_2^u M_3)}{(A_2^l + m_2 + B_2^l m_3)^2} \right], \tag{33}$$

$$\sigma_3 = b_3^l m_3 - \frac{s_4^u B_2^u M_2 M_3}{(A_2^l + m_2 + B_2^l m_3)^2}. \tag{34}$$

Let

$$\mu = \min\{\sigma_1, \sigma_2, \sigma_3\}. \tag{35}$$

It follows from (31) and (35) that

$$D^+ V(t) \le -\mu V(t). \tag{36}$$

Then (iii) of Lemma 3.1 is fulfilled. Therefore system (2) has a unique positive asymptotically periodic solution in domain Ω, which is uniformly asymptotically stable. The proof is complete.

IV. An example

In this section, we give an example to illustrate our main results obtained in previous sections. We consider the following delayed three-level food chain model with Beddington-DeAngelis functional response

$$\begin{cases} \dfrac{du_1}{dt} = u_1(t)\Big[a_1(t) - b_1(t)u_1(t) \\ \qquad - \dfrac{s_1(t)u_2(t)}{A_1(t) + u_1(t) + B_1(t)u_2(t)}\Big], \\ \dfrac{du_2}{dt} = u_2(t)\Big[-a_2(t) - b_2(t)u_2(t) \\ \qquad + \dfrac{s_3(t)u_1(t-\tau)}{A_1(t) + u_1(t-\tau) + B_1(t)u_2(t-\tau)} \\ \qquad - \dfrac{s_2(t)u_3(t)}{A_2(t) + u_2(t) + B_2(t)u_3(t)}\Big], \\ \dfrac{du_3}{dt} = u_3(t)\Big[-a_3(t) - b_3(t)u_3(t) \\ \qquad + \dfrac{s_4(t)u_2(t-\tau)}{A_2(t) + u_2(t-\tau) + B_2(t)u_3(t-\tau)}\Big], \end{cases} \tag{37}$$

where

$$\begin{cases} a_1(t) = 7 + \cos t, b_1(t) = 17 + \cos t \\ s_1(t) = 2 + \sin t, A_1(t) = 0.05 + 0.01\cos t, \\ B_1(t) = 18 + \cos t, a_2(t) = 1 + 0.2\cos t, \\ s_2(t) = 0.5 + 0.2\sin t, s_3(t) = 1 + 0.2\sin t, \\ s_4(t) = 1 + 0.2\cos t, A_2(t) = 0.01 + 0.04\sin t, \\ B_2(t) = 2 + \sin t, b_2(t) = 111 + \sin t, \\ b_3(t) = 111 + \sin t. \end{cases}$$

Then we get $a_1^u = 8, a_1^l = 6, b_1^u = 18, b_1^l = 16, s_1^u = 3, s_1^l = 1, A_1^u = 0.06, B_1^u = 19, a_2^u = 1.2, a_2^l = 0.8, s_2^u = 0.7, s_2^l = 0.3, s_3^u = 1.2, s_2^l = 0.3, s_3^u = 1.2, s_3^l = 0.8, s_4^u = 1.2, s_4^l = 0.8, A_2^u = 0.05, B_2^u = 3, b_2^l = 110, b_3^l = 110$. Hence we have $M_1 \approx 0.5, M_2 \approx 0.003, M_3 \approx 0.009, m_1 \approx 0.05, m_2 \approx 0.00001, m_3 \approx 0.0009, \sigma_1 \approx 0.04728, \sigma_2 \approx 0.03402, \sigma_3 \approx 0.03866, \mu \approx 0.03402$. Thus we can easily check that all the conditions of Theorem 2.1 and Theorem 3.1 are fulfilled. Therefore system (37) has a unique positive asymptotically periodic solution, which is uniformly asymptotically stable(see Fig.1-3).

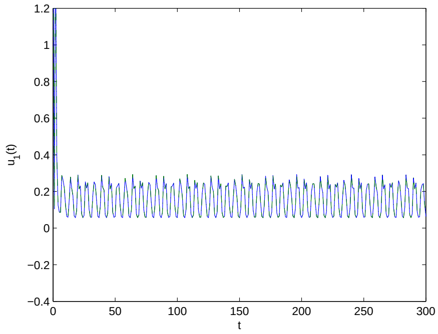

Fig. 1. Dynamical behavior of system (37): time series of u_1.

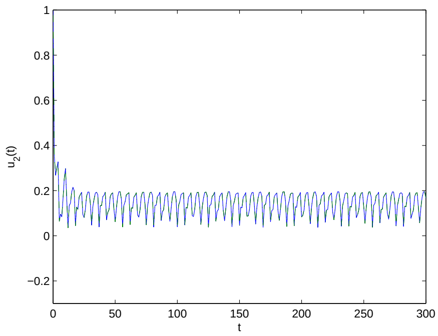

Fig. 2. Dynamical behavior of system (37): time series of u_2.

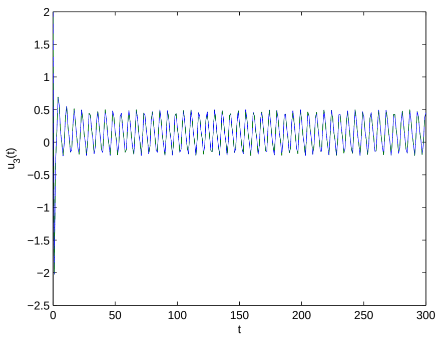

Fig. 3. Dynamical behavior of system (37): time series of u_3.

V. Conclusions

In this paper, we have investigated a delayed predator-prey system with Holling-type II functional response. Applying the differential inequality theory, some sufficient conditions for the permanence of the system are established. By constructing a suitable Liapunov function, we find that under some suitable conditions, the system has a unique asymptotically periodic solution which is globally asymptotically stable. Here we would like to point out that the discrete time delayed predator-prey system with Holling-type II functional response is more appropriate to describe the dynamics relationship among populations than continuous ones when the populations have non-overlapping generations. Moreover, discrete time models can also provide efficient models of continuous ones for numerical simulations. Thus it is reasonable and interesting to investigate the discrete time delayed predator-prey system with Holling-type II functional response. Following the idea and method in [26], one can easily derive the following discrete analogue of system (2), which takes the form of

$$\begin{cases} u_1(k+1) = u_1(k)\exp\Big\{a_1(k) - b_1(k)u_1(k) \\ \qquad - \dfrac{s_1(k)u_2(k)}{A_1(k)+u_1(k)+B_1(t)u_2(k)}\Big\}, \\ u_2(k+1) = u_2(k)\exp\Big\{-a_2(k) - b_2(k)u_2(k) \\ \qquad + \dfrac{s_3(k)u_1(k-\tau)}{A_1(k)+u_1(k-\tau)+B_1(k)u_2(t-\tau)} \\ \qquad - \dfrac{s_2(k)u_3(k)}{A_2(k)+u_2(k)+B_2(k)u_3(k)}\Big\}, \\ u_3(k+1) = u_3(k)\exp\Big\{-a_3(k) - b_3(t)u_3(k) \\ \qquad + \dfrac{s_4(k)u_2(k-\tau)}{A_2(k)+u_2(k-\tau)+B_2(k)u_3(k-\tau)}\Big\}. \end{cases} \tag{38}$$

It is interesting for us to establish the sufficient criteria for the existence of positive periodic solutions of system (38). This topic will be our future work.

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