The Petrov-Galerkin Method for Numerical Solution of Stochastic Volterra Integral Equations

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Abstract—**In this paper, we introduce the Petrov-Galerkin method for solution of stochastic Volterra integral equations. Here, we use continues Lagrange-type k-0 elements, since these elements have simple structure and via them, the solution of stochastic Volterra integral equation is reduced to algebraic equations. Also the error analysis of this method is done. In Comparison with other methods, this method has less computation.**

Index Terms—**Petrov-Galerkin method; Continuous Lagrange-type k-0 elements; Stochastic Volterra integral equations; Itô integral; Brownian motion process.**

I. Introduction

STOCHASTIC Volterra integral equations (SVIEs) is a fast developing field, with applications in economics, sociology, biology, medical models and anthropology. Background material and countless references can be found in [1-9]. Stochastic Volterra integral equations arise when a random noise is introduced into Volterra integral equations. These systems are dependent on a noise source, which is a Gaussian white one. The Brownian motion process $B(t)$ serves as a basic model for the cumulative effect of pure noise. Generally, we are not able to find explicit formulae for the solutions of SVIEs and thus need to use a numerical method to approximate the solutions.

The Petrov-Galerkin method is a numerical method based on Galerkin method but with different trial and test spaces. This method has been used for approximation of the numerical solution of Fredholm second kind integral equations [10-11], Hammerstein integral equations [12-13], and integro-differential equations [14-15].

We consider the one-dimensional linear stochastic Volterra integral equation

$$u(t) = f(t) + \int_0^t k_1(s,t)u(s)ds + \int_0^t k_2(s,t)u(s)dB(s) \tag{1}$$

$$t, s \in [0, T] = I,$$

where, $u(t)$, $f(t)$, $k_1(s,t)$ and $k_2(s,t)$, for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, P)$, and $u(t)$ is unknown random function, $B(t)$ is a Brownian motion process and $\int_0^t k_2(s,t)u(s)dB(s)$ is the Itô integral.

We rewrite this equation in operator form as

$$(I - K_1 - K_2)u(t) = f(t), \tag{2}$$

where

$$(K_1 u)(t) := \int_0^t k_1(s,t)u(s)ds,$$

$$(K_2 u)(t) := \int_0^t k_2(s,t)u(s)dB(s) \quad t \in I.$$

In this article, we are going to use continuous Lagrange-type k-0 elements of Petrov-Galerkin system to approximate the numerical solution of stochastic Volterra integral equation.

The content of this paper is arranged in five sections. In Section II, we introduce some general concepts concerning the Petrov-Galerkin method and stochastic concepts. Section III, presents error analysis. In Section IV, we show numerical results. Finally, Section V provides the conclusion.

II. Preliminaries

The Petrov-Galerkin method uses regular pairs $\{X_n, Y_n\}$ of piecewise polynomial spaces that are called Petrov-Galerkin elements. In this section, we summarize the key concepts and results of Petrov-Galerkin method and stochastic calculus.

A. Petrov-Galerkin method

Let X be a Banach space and X^* be its dual space of continuous linear functionals, for each positive integer n, we assume that $X_n \subset X$,$Y_n \subset X^*$, and X_n and Y_n are finite dimensional vector spaces with dim X_n =dim Y_n.
In addition, we assume the following property:
(H) If $x \in X$ and $y \in Y$, then there are sequences x_n and y_n, with $x_n \in X$, $y_n \in Y$ for all n such that $x_n \to x$, $y_n \to y$.

Definition 1. For $x \in X$, an element $P_n x \in X_n$ is called the generalized best approximation from X_n to x with respect to Y_n by the equation

$$< x - P_n x, y_n >= 0, \quad \forall y_n \in Y_n. \tag{3}$$

Lemma 1. [10] For each $x \in X$, the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if

$$X_n^{\perp} \cap Y_n = \{0\}, \tag{4}$$

where, $X_n^{\perp} = \{x^* \in X; < x, x^* >= 0, \forall x \in X_n\}$.

When condition (4) satisfied, P_n defines a projection; that $P_n^2 = P_n$.
In addition, we must add some properties to (4) in order to bring a situation that P_n converges point-wise to the identity operator.

Definition 2. $\{X_n, Y_n\}$ is called regular pair if a linear operator $\prod_n : X_n \to Y_n$ with $\prod_n X_n = Y_n$ exists and

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satisfy the following two conditions
(H-1) $\|x_n\| \leq C_1 < x_n, \prod x_n >^{1/2}, \forall x_n \in X_n$.
(H-2) $\| \prod_n x_n\| \leq C_2 \parallel x_n \parallel, \forall x_n \in X_n$.
$C_1$and C_2 are constants independent of n and $\parallel . \parallel$ is L^2 norm.

Theorem 1. Let pair$\{X_n.Y_n\}$ satisfies $dimX_n = dimY_n$ and condition (H) then the corresponding generalized projection P_n satisfies:
(P1) $\parallel P_nx - x \parallel \rightarrow 0$ as $n \rightarrow \infty$.
(P2) $\parallel P_n \parallel \leq C, n = 1, 2, 3, ...$ for some constants C.
(P3) $\parallel P_nx - x \parallel \leq C \parallel Q_nx - x \parallel, n = 1, 2, 3, ...$ for some constants C,
where Q_n, is the best approximation from X_n to x.

Proof. See [10].

B. Stochastic concepts of Itô integral

Definition 3.[16] (Brownian motion process). A real-valued stochastic process $B(t)$, $t \in [0, T]$ is called Brownian motion, if it satisfies the following properties:
(i) (Independence of increments) $B(t) - B(s)$, $t > s$, is independent of the past, that is, of $B(u), 0 \leq u \leq s$, or of $\mathcal{F}_s$, the σ-field generated by $B(u), u \leq s$.
(ii) (Normal increments) $B(t) - B(s)$ has Normal distribution with mean 0 and variance $t - s$.
(iii) (Continuity of paths) $B(t), t \geq 0$ are continuous functions of t.

Definition 4. [16] Let $\{N(t)\}_{t\geq 0}$ be an increasing family of σ-algebras of sub-sets of Ω. A process $g(t, \omega)$ from $[0, \infty) \times \Omega$ to R^n is called $N(t)$-adapted if for each $t \geq 0$ the function $\omega \longrightarrow g(t, \omega)$ is $N(t)$-measurable.

Definition 5. [16] Let $\nu = \nu(S, T)$ be the class of functions $f(t, \omega) : [0, \infty) \times \Omega \longrightarrow R$ such that,
(i) $(t, \omega) \longrightarrow f(t, \omega)$, is $B \times \mathcal{F}$-measurable, where B denotes the Borel σ-algebra on $[0, \infty)$ and $\mathcal{F}$ is the σ-algebra on Ω.
(ii) $f(t, \omega)$ is $\mathcal{F}_t$-adapted, where $\mathcal{F}_t$ is the σ-algebra generated by the random variables $B(s)$; $s \leq t$.
(iii) $E\big[\int_S^T f^2(t, \omega)dt\big] < \infty$.

Definition 6. [16] (The Itô integral). Let $f \in \nu(S, T)$, then the Itô integral of f (from S to T) is defined by

$$\int_S^T f(t, \omega)dB(t)(\omega) = \lim_{n\rightarrow\infty} \int_S^T \phi_n(t, \omega)dB(t)(\omega),$$
$$(limit \quad in \quad L^2(P))$$

where, ϕ_n is a sequence of elementary functions such that

$$E\big[\int_S^T (f(t, \omega) - \phi_n(t, \omega))^2 dt\big] \rightarrow 0, \quad as \quad n \rightarrow \infty.$$

Theorem 2. (The Itô isometry), Let $f \in \nu(S, T)$, then

$$E\big[(\int_S^T (f(t, \omega)dB(t)(w))^2\big] = E\big[\int_S^T f^2(t, \omega)d(t)\big].$$

Proof. See [16].

Definition 7. (1-dimensional Itô processes), [16]. Let $B(t)$ be 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. A 1-dimensional Itô process (stochastic integral) is a stochastic process $X(t)$ on $(\Omega, \mathcal{F}, P)$ of the form

$$X(t) = X(0) + \int_0^t u(s, \omega)ds + \int_0^t v(s, \omega)dB(s),$$

or

$$dX(t) = udt + vdB(t), \tag{5}$$

where

$$P\big[\int_0^t v^2(s, \omega)ds < \infty, \quad for \quad all \quad t \geq 0\big] = 1,$$

$$P\big[\int_0^t \mid u(s, \omega) \mid ds < \infty, \quad for \quad all \quad t \geq 0\big] = 1.$$

Theorem 3. (The 1-dimensional Itô formula). Let $X(t)$ be an Itô process given by (1) and $g(t, x) \in C^2([0, \infty) \times R)$, then

$$Y(t) = g\big(t, X(t)\big),$$

is again an Itô process, and

$$dY(t) = \frac{\partial g}{\partial t}\big(t, X(t)\big)dt + \frac{\partial g}{\partial x}\big(t, X(t)\big)dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\big(t, X(t)\big)\big(dX(t)\big)^2, \tag{6}$$

where $(dX(t))^2 = (dX(t))(dX(t))$ is computed according to the rules

$$dt.dt = dt.dB(t) = dB(t).dt = 0, \quad dB(t).dB(t) = dt. \tag{7}$$

Proof. See [16].

C. Applying the Petrov-Galerkin method to solve SVIEs

Consider

$$u(t) - \int_0^t k_1(s, t)u(s)ds + \int_0^t k_2(s, t)u(s)dB(s) := L(t, \omega, u(t)),$$

with random input parameters

$$L(t, \omega, u(t)) = f(t).$$

In the first step, Petrov-Galerkin method for Eq.(1) is a numerical method for finding $u_n = \sum_{i=0}^n c_i\phi_i(t) \in X_n$, such that c_i is unknown and must be determined. So

$$L(t, \omega, \sum_{i=0}^n c_i\phi_i(t)) = f(t).$$

Using inner product, we take a projection of the equation onto each basis polynomial $\psi_n \in Y_n$, i.e.

$$< L(t, \omega, \sum_{i=0}^n c_i\phi_i(t)), \psi_n > = < f(t), \psi_n > \quad \forall \quad \psi_n \in Y_n. \tag{8}$$

Eq. (8) can be derive from $P_nx = 0$ for any $x \in X$ if and only if $< x, \psi_n > = 0$ for all $\psi_n \in Y_n$. In this case our problem reduced to a linear system and we can use any appropriate method for finding coefficients.

D. Continuous Lagrange-type k-0 elements

Here we give a brief review of construction of continuous Lagrange-type $k-0$ elements [10]. We subdivide the interval $[0,1]$ into n subintervals. Suppose that X_n be the space of continuous piecewise polynomials of degree not exceeding k, and with knots at $t_i, i = 1,2,...,n-1$. It is clear that $dimX_n = nk+1$.

Let $0 = t_0 < t_1 < ... < t_n = 1$ and $I_i = [t_{i-1}, t_i]$; $h_i = t_i - t_{i-1}$ for $i = 1,...,n$. The basis of X_n is constructed by concept of Lagrange polynomials, for this purpose, let $\tau_j = \frac{j}{k}, j = 0,1,...,k$ and $t_j^{(i)} = t_{i-1} + \tau_j h_i, j = 0,1,...,k, i = 1,...,n$.Then, it is clear that $t_{i-1} = t_0^{(i)} < ... < t_k^{(i)} = t_i$.

Then $nk+1$ functions $\phi_j^{(i)}(t)$ as basis of X_n is given by

$$\phi_j^{(i)}(t) = \begin{cases} \prod_{l=0_{l\neq j}}^{k} \frac{t-t_l^{(i)}}{t_j^{(i)}-t_l^{(i)}} & t \in I_i, \\ 0 & t \notin I_i. \end{cases} \tag{9}$$

for $i = 1,2,...,n, j = 1,2,...,k-1; i = 1, j = 0; i = n, j = k$.

$$\phi_k^{(i)}(t) = \begin{cases} \prod_{l=0}^{k-1} \frac{t-t_l^{(i)}}{t_k^{(i)}-t_l^{(i)}} & t \in I_i, \\ \prod_{l=1}^{k} \frac{t-t_l^{(i+1)}}{t_0^{(i+1)}-t_l^{(i+1)}} & t \in I_{i+1}, \\ 0 & t \notin I_{i+1} \cup I_i. \end{cases} \tag{10}$$

$i = 1,2,...,n-1$

To construct the test space Y_n, we define

$$\psi_0^{(1)} = \begin{cases} 1 & 0 \leq t < \frac{h_1}{2k}, \\ 0 & otherwise. \end{cases} \tag{11}$$

$$\psi_j^{(i)} = \begin{cases} 1 & t_{i-1} + \frac{2j-1}{2k}h_i \leq t < t_{i-1} + \frac{2j+1}{2k}h_i, \\ 0 & otherwise. \end{cases} \tag{12}$$

$i = 1,...,n, j = 1,2,...,k-1$,

$$\psi_k^{(i)} = \begin{cases} 1 & t_{i-1} + \frac{2k-1}{2k}h_i \leq t < t_i + \frac{1}{2k}h_{i+1}, \\ 0 & otherwise. \end{cases} \tag{13}$$

$$\psi_k^{(n)} = \begin{cases} 1 & t_{i-1} + \frac{2k-1}{2k}h_n \leq t < 1, \\ 0 & otherwise. \end{cases} \tag{14}$$

Furthermore, for any $x_n \in X_n$, we have

$$x_n = \sum_{j=0}^{k} x_n(t_j^{(i)})\phi_j^{(i)}(t), \quad t \in I_i, i = 1,...,n, \tag{15}$$

and it is proved in [10] that with linear operator

$$\prod_n X_n \to Y_n; \prod_n x_n(t) = \sum_{j=0}^{k} x_n(t_j^{(i)})\psi_j^{(i)}, \tag{16}$$

$t \in I_i, i = 1,...,n$,

$\{X_n, Y_n\}$ is a regular pair, therefore it can be used in Petrov-Galerkin system.

III. Error analysis

In this section, we investigate error analysis for our method. By Theorem 1 (P1)

$$\lim_{n\to\infty} \parallel P_n x - x \parallel = 0.$$

Furthermore, we assume that the speed of convergence is specified by

$$\parallel (P_n - I)x \parallel \leq Ch^m \parallel x \parallel . \tag{17}$$

We are interested to find $u_n(t) \in X_n$ such that for $t \in [0,T]$

$$u_n(t) = P_n f + P_n K_1 u_n(t) + P_n K_2 u_n(t).$$

Let $e(t) = u_n(t) - u(t)$ be the error of this method where, u_n is a PG solution and u is the exact solution of the stochastic Volterra integral Eq.(1).

Theorem 4. Assume that

1. $\parallel f(t) \parallel < \infty$,
2. $P(w \in \Omega : \parallel u(\omega,t) \parallel < \infty) = 1$.
3. $\parallel k_i(s,t) \parallel < \infty \quad i = 1,2$,

then

$$sup_{0\leq t\leq T}(E(\parallel (u(t) - u_n(t)) \parallel)^2)^{1/2} = O(h^m).$$

Proof: From

$$u_n(t) - u(t) = P_n f - f + P_n K_1 u_n - K_1 u + P_n K_2 u_n - K_2 u,$$

we get

$$e = (P_n - I)f + (P_n - I)K_1 u_n + K_1 e + (P_n - I)K_2 u_n + K_2 e,$$

so

$$E(\parallel u_n - u \parallel^2) \leq 3[E(\parallel (p_n - I)f) \parallel^2) + \tag{18}$$
$$E(\parallel (P_n - I)K_1 u_n + K_1 e \parallel^2) + E(\parallel (P_n - I)K_2 u_n + K_2 e \parallel^2)].$$

Furthermore, from (17) we have

$$(I-1) \quad E(\parallel (p_n - I)f) \parallel^2) \leq E(C^2 h^{2m} \parallel f \parallel^2) = Ch^{2m} E(\parallel f \parallel^2).$$

$$(I-2) \quad E(\parallel (P_n - I)K_1 u_n + K_1 e \parallel^2) \leq$$
$$3[E(\parallel (P_n - I)K_1 u \parallel^2) + E(\parallel (P_n - I)K_1(u_n - u) \parallel^2) + E(\parallel K_1(u_n - u) \parallel^2)].$$

$$(I-3) \quad E(\parallel (P_n - I)K_1 u \parallel^2) \leq C.h^{2m} E(\parallel u \parallel^2).$$

$$(I-4) \quad E(\parallel (P_n - I)K_1(u_n - u) \parallel^2) < Ch^{2m} E(\parallel (u_n - u) \parallel^2).$$

$$(I-5) \quad E(\parallel K_1(u_n - u) \parallel^2) \leq C. \int_0^t E(\parallel (u_n - u) \parallel^2) ds.$$

Then

$$E(\parallel (P_n - I)K_1 u_n + K_1 e \parallel^2) \leq$$
$$C.h^{2m} E(\parallel u \parallel^2) + Ch^{2m} E(\parallel (u_n - u) \parallel^2) +$$
$$C. \int_0^t E(\parallel (u_n - u) \parallel^2) ds.$$

For third term on the right-hand side of Eq. (18) we can write

$$E(\parallel (P_n - I)K_2 u_n + K_2 e \parallel^2)$$
$$\leq 3[E(\parallel \int_0^t (P_n - I)k_2 u dB(s) \parallel^2) +$$
$$E(\parallel \int_0^t (P_n - I)k_2(u_n - u) dB(s) \parallel^2) +$$
$$E(\parallel \int_0^t k_2(u_n - u) dB(s) \parallel^2)]$$

$$\leq 3[\int_0^t E(\| (P_n - I)k_2 u \|^2)ds + \int_0^t E(\| (P_n - I)k_2(u_n - u) \|^2)ds + \int_0^t E(\| k_2(u_n - u) \|^2)ds]$$

$$\leq Ch^{2m}E(\| u \|^2) + Ch^{2m}E(\| u_n - u \|^2) + C\int_0^t E(\| u_n - u \|^2)ds.$$

Finally,

$$E(\| (u - u_n)^2 \|) \leq Ch^{2m} + C_2 \int_0^t E(\| u - u_n \|^2)ds,$$

and by using Gronwall's inequality

$$E(\| (u - u_n)^2 \|) \leq Ch^{2m}.$$

Proposition 1. Suppose that $\{X_n, Y_n\}$ is a regular pair that satisfies $dimX_n = dimY_n$ and condition (H). Then, for any given $u \in X$, there exists a positive integer N such that, for all $n \geq N$, the Petrov-Galerkin equation, $< (I - K_1 u_n - K_2 u_n, y_n >=< f, y_n >, \forall y_n \in Y_n$, has a unique solution $u_n \in X_n$ that satisfies $E(\| u_n - u \|^2) \leq C \quad inf_{x_n \in X_n} \| u - x_n \|, n \geq N$.

Proof: By Theorem 1 (P1), P_n converges pointwise to the identity operator I in X. Hence, it follows from Theorem 4 that there exists an integer $N > 0$ for which

$$E(\| u_n - u \|^2) \leq C \| P_n u - u \|; n > N.$$

Using this estimate and Theorem 1 (P3), the proof is completed.

IV. Numerical examples

In this section, we present four different examples for performance of the Petrov-Galerkin method. In these examples the Error is defined as

$$\|e\|_\infty = \max |x_n(t_j^{(i)}) - x(t_j^{(i)})|.$$

Example 1. [5] Consider the linear stochastic Volterra integral equation,

$$u(t) = 1 + \int_0^t s^2 u(s)ds + \int_0^t s u(s)dB(s) \qquad s, t \in [0, 1), \tag{19}$$

with the exact solution $u(t) = e^{\frac{t^3}{6} + \int_0^t s dB(s)}$, for $0 \leq t < 1$.
The numerical results are shown in Tables I and II.
In theses tables, n is the number of iterations, $\overline{x}_E$ is error mean, and s_E is standard deviation of error.

Example 2. [5] Consider the following linear stochastic Volterra integral equation,

$$u(t) = \frac{1}{12} + \int_0^t \cos(s)u(s)ds + \int_0^t \sin(s)u(s)dB(s) \qquad s, t \in [0, 1), \tag{20}$$

with the exact solution $u(t) = \frac{1}{12}e^{-\frac{t}{4} + \sin(t) + \frac{\sin(2t)}{8} + \int_0^t \sin(s)dB(s)}$, for $0 \leq t < 1$.
For this example, the numerical results are shown in Tables III and IV.

Example 3. [1] Consider the following linear stochastic Volterra integral equation,

$$u(t) = \frac{1}{3} + \int_0^t ln(s+1)u(s)ds + \int_0^t \sqrt{ln(s+1)}u(s)dB(s), \qquad s, t \in [0, 0.5), \tag{21}$$

with the exact solution

$$u(t) = \frac{1}{3}e^{-\frac{1}{2}t + \frac{1}{2}tln(t+1) + \frac{1}{2}ln(t+1) + \int_0^t \sqrt{ln(s+1)}dB(s)},$$

for $0 \leq t < 0.5$.
The numerical results are shown in Table V and Table VI.
As can be seen, by less computation, we get good accuracy.

Example 4. The Langevin model (Paul Langevin, 1908) has been quite successfully used to study rotational motion of molecules in gases, liquids, and solids. Suppose that a small macroscopic particle of mass m (such as a pollen grain) is immersed in a liquid at a temperature T. In addition to any macroscopic motion that the particle may have, its velocity fluctuates due to the random collisions of the particle with the molecules of the liquid. For simplicity, we confine ourselves to one-dimensional motion along the x-axis. Then the equation of motion of the particle may be written in the form of stochastic differential equation

$$m\frac{d^2x}{dt^2} = -\alpha\frac{dx}{dt} - \frac{dV}{dx} + F(t). \tag{22}$$

The first term on the right-hand side is due to the viscosity of the fluid and α is the friction constant. The second term, where $V(x)$ is a potential, represents the interaction of the particle with any external forces, such as gravity. The final term is the random force due to collisions with the molecules of the liquid. Clearly to complete the specification of the dynamics of the particle we need to give (i) the initial position and velocity of the particle, and (ii) the statistics of the random force *F(t)* (the noise term). Note that since *F(t)* is a random variable, solving the Langevin equation will give $x(t)$ as a random variable [17-18].
Eq. (21) may be equivalent written as

$$\frac{dx}{dt} = v,$$

$$m\frac{dv}{dt} = -\alpha v - \frac{dV}{dx} + F(t). \tag{23}$$

A particularly well-known case is when the Brownian particle is moving in the harmonic potential $V(x) = \frac{ax^2}{2}$ or double-well potential $V(x) = (x^2 - 1)^2$. Also, $F(t) = \sigma dB(t)$, where, $\sigma = \frac{2kT}{\tau m}$ is diffusion coefficient, and $B(t)$ is Brownian motion.
Here we apply this method for solving the following system

$$\begin{cases} x(t) = x_0(t) + \int_0^t v(s)ds, \\ mv(t) = v_0(t) - \int_0^t \alpha v(s)ds - \int_0^t \frac{dV}{dx}(s)ds + \int_0^t \sigma dB(s). \end{cases} \tag{24}$$

In order to conform the results above, initial value $x(0) = 0, v(0) = 5$ are chosen and parameters $\sigma = 0.1, m = 1, \alpha = 1$.

TABLE I
MEAN, STANDARD DEVIATION AND CONFIDENCE INTERVAL FOR ERROR MEAN IN EXAMPLE 1 WITH $k = 4$.

n	$\overline{x}_E$	s_E	%95 Confidence Interval for mean of E	
			Lower	*Upper*
30	3.274×10^{-3}	6.793×10^{-3}	8.435×10^{-4}	5.705×10^{-3}
100	6.174×10^{-3}	3.264×10^{-3}	5.538×10^{-3}	6.810×10^{-3}
200	5.523×10^{-3}	2.399×10^{-3}	5.191×10^{-3}	5.856×10^{-3}
500	6.910×10^{-3}	1.510×10^{-3}	6.777×10^{-3}	7.042×10^{-3}
1000	7.014×10^{-3}	1.151×10^{-3}	6.943×10^{-3}	7.086×10^{-3}

TABLE II
MEAN, STANDARD DEVIATION AND CONFIDENCE INTERVAL FOR ERROR MEAN IN EXAMPLE 1 WITH $k = 6$.

n	$\overline{x}_E$	s_E	%95 Confidence Interval for mean of E	
			Lower	*Upper*
30	5.043×10^{-3}	2.731×10^{-3}	4.065×10^{-3}	6.020×10^{-3}
100	2.311×10^{-3}	2.072×10^{-3}	1.905×10^{-3}	2.718×10^{-3}
200	2.314×10^{-3}	1.409×10^{-3}	2.119×10^{-3}	2.510×10^{-3}
500	3.840×10^{-3}	$9,099 \times 10^{-4}$	3.760×10^{-3}	3.919×10^{-3}
1000	2.230×10^{-3}	6.477×10^{-4}	2.190×10^{-3}	2.270×10^{-3}

TABLE III
MEAN, STANDARD DEVIATION AND CONFIDENCE INTERVAL FOR ERROR MEAN IN EXAMPLE 2 WITH $k = 4$.

n	$\overline{x}_E$	s_E	%95 Confidence Interval for mean of E	
			Lower	*Upper*
30	1.198×10^{-3}	1.256×10^{-3}	7.485×10^{-4}	1.647×10^{-3}
100	1.758×10^{-3}	6.779×10^{-4}	1.625×10^{-3}	1.891×10^{-3}
200	3.770×10^{-3}	5.423×10^{-4}	3.694×10^{-3}	3.845×10^{-3}
500	3.298×10^{-3}	3.271×10^{-4}	3.269×10^{-3}	3.327×10^{-3}
1000	2.272×10^{-3}	2.424×10^{-4}	2.709×10^{-3}	2.739×10^{-3}

TABLE IV
MEAN, STANDARD DEVIATION AND CONFIDENCE INTERVAL FOR ERROR MEAN IN EXAMPLE 2 WITH $k = 6$.

n	$\overline{x}_E$	s_E	%95 Confidence Interval for mean of E	
			Lower	*Upper*
30	5.948×10^{-3}	3.243×10^{-4}	5.832×10^{-3}	6.064×10^{-3}
100	5.944×10^{-3}	2.025×10^{-4}	5.904×10^{-3}	5.983×10^{-3}
200	5.906×10^{-3}	1.535×10^{-4}	5.885×10^{-3}	5.927×10^{-3}
500	3.840×10^{-3}	$9,099 \times 10^{-4}$	3.760×10^{-3}	3.919×10^{-3}
1000	2.230×10^{-3}	6.477×10^{-4}	2.190×10^{-3}	2.270×10^{-3}

TABLE V
MEAN, STANDARD DEVIATION AND CONFIDENCE INTERVAL FOR ERROR MEAN IN EXAMPLE 3 WITH $k = 4$.

n	$\overline{x}_E$	s_E	%95 Confidence Interval for mean of E	
			Lower	*Upper*
30	2.490×10^{-3}	2.191×10^{-3}	1.706×10^{-3}	3.275×10^{-3}
100	6.210×10^{-3}	1.329×10^{-3}	5.949×10^{-3}	5.647×10^{-3}
200	5.263×10^{-3}	8.649×10^{-4}	5.143×10^{-3}	5.383×10^{-3}
500	5.824×10^{-3}	9.658×10^{-5}	5.815×10^{-3}	5.832×10^{-3}
1000	5.830×10^{-3}	6.881×10^{-4}	5.787×10^{-3}	5.872×10^{-3}

V. CONCLUSION

We introduced the Petrov-Galerkin method for numerical solution of stochastic Volterra integral equations. The advantages of this method is that it can be constructed simply, and by choosing the degree of test space lower than trial space's, we can achieve the same order of Galerkin method convergence with less computational cost. This method specific properties lead to numerical results with considerable accuracy and less computational effort.

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TABLE VI
MEAN, STANDARD DEVIATION AND CONFIDENCE INTERVAL FOR ERROR MEAN IN EXAMPLE 3 WITH $k = 6$.

n	$\overline{x}_E$	s_E	%95 Confidence Interval for mean of E	
			Lower	*Upper*
30	3.768×10^{-3}	6.665×10^{-3}	1.383×10^{-3}	6.153×10^{-3}
100	2.272×10^{-3}	2.657×10^{-3}	1.752×10^{-3}	2.793×10^{-3}
200	4.908×10^{-3}	1.982×10^{-3}	4.633×10^{-3}	5.182×10^{-3}
500	2.998×10^{-3}	1.153×10^{-3}	2.897×10^{-3}	3.100×10^{-3}
1000	3.797×10^{-3}	1.104×10^{-3}	3.728×10^{-3}	3.865×10^{-3}

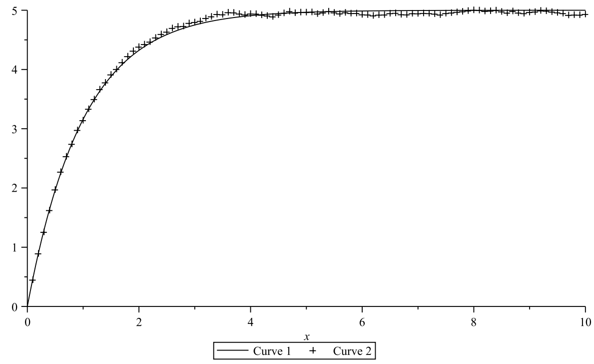

Fig. 1. Curve1: The exact solution $x(t)$ of example 4 with $V(x) = 0, \sigma = 0$. Curve2: The trajectory of the approximate solution $x(t)$ of example 4 with $V(x) = 0, \sigma = 0.1$.

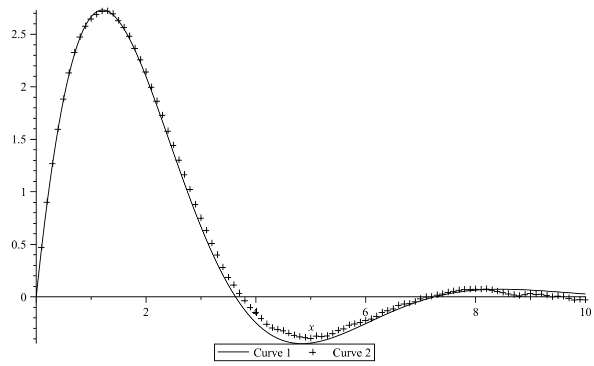

Fig. 2. Curve1: The exact solution $x(t)$ of example 4 with $V(x) = \frac{x^2}{2}, \sigma = 0$. Curve2: The trajectory of the approximate solution $x(t)$ of example 4 with $V(x) = \frac{x^2}{2}, \sigma = 0.1$.

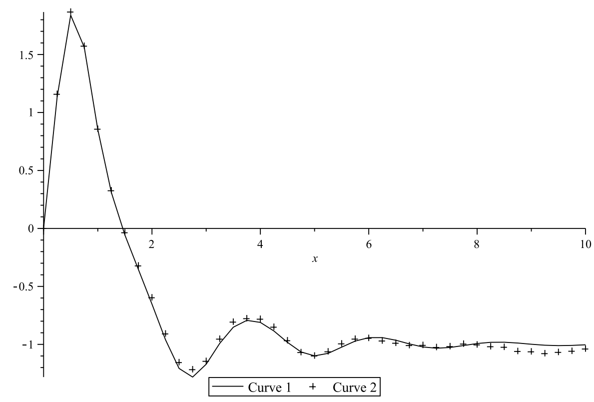

Fig. 3. Curve1: The exact solution $x(t)$ of example 4 with $V(x) = (x^2 - 1)^2, \sigma = 0$. Curve2: The trajectory of the approximate solution $x(t)$ of example 4 with $V(x) = (x^2 - 1)^2, \sigma = 0.1$.

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