Multiwave Solutions for the Toda Lattice Equation by Generalizing Exp-Function Method

Sheng Zhang, Ying-Ying Zhou

***Abstract*—In this paper, the exp-function method is generalized to construct multiwave solutions of a (2+1)-dimensional variable-coefficient Toda lattice equation. As a result, single-wave solution, double-wave solution and three-wave solution are obtained, from which the uniform formula of *N*-wave solution is derived. It is shown that the generalized exp-function method can be used for generating multiwave solutions of some other nonlinear differential-difference equations with variable coefficients.**

***Index Terms*—Multiwave solution, Toda lattice equation, exp-function method, nonlinear differential-difference equation.**

I. Introduction

It is the work of Fermi, Pasta and Ulam in the 1950s [1] that has attached much attention on exact solutions of nonlinear differential-difference equations (DDEs), which play a crucial role in modelling many phenomena in different fields like condensed matter physics, biophysics or mechanical engineering. In the numerical simulation of soliton dynamics in high energy physics, some DDEs often arise as approximations of continuum models. Unlike difference equations which are fully discretized, DDEs are semi-discretized with some (or all) of their spacial variables discretized while time is usually kept continuous. Among the existing DDEs, Toda lattice is a simple model for a nonlinear one-dimensional crystal. The equation of motion of such a lattice system is usually given by

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_n = V'(x_{n+1}-x_n) - V'(x_n - x_{n-1}), \quad (1)$$

where m denotes the mass of each particle, $x_n = x_n(t)$ is the displacement of the n-th particle from its equilibrium position, $V'(r) = \mathrm{d}V(r)/\mathrm{d}r$, $V(r)$ is the interaction potential. The Toda lattice equation (1) describes the motion of a chain of particles with nearest neighbor interaction [2], different versions of which are often used to construct the mathematical model, for example, the Toda lattice model of DNA in the field of biology [3]. One important property of such type of Toda lattice equations is the existence of so-called soliton solutions (stable waves) which spread in time without changing their size or shape and interact with each other in a particle-like way [4]. There is a close relation between the existence of soliton solutions and the integrability of equations, the known research results show that all the integrable systems exist soliton solutions [5]. Multiwave solutions are a kind of interaction solutions, which include not only classical multisoliton solutions (without singular points) but also singular multisoliton solutions. Usually, the interactions of singular solitons may show entirely different evolution characteristics from those of regular ones.

In the past several decades, there has been significant progression in the development of methods for solving nonlinear partial differential equations (PDEs), such as the inverse scattering method [6], Hirota's bilinear method [7], Bäcklund transformation [8], Painlevé expansion [9], homogeneous balance method [10], function expansion methods [11], [12], [13], [14], [15], and others [16], [17], [18], [19], [20]. With the development of soliton theory, finding multiwave solutions of nonlinear PDEs and DDEs has gradually developed into a significant direction in nonlinear science. Generally speaking, it is hard to generalize one method for nonlinear PDEs to solve DDEs because of the difficulty in searching for iterative relations from indices n to $n \pm 1$. Recently, the exp-function method [21] has been proposed and applied to many kinds of nonlinear PDEs [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33]. In 2008, Marinakis [34] generalized the exp-function method to obtain multisoliton solutions of the famous Korteweg-de Vries (KdV) equation. Later, Marinakis' work was improved for obtaining the uniform formula of *N*-soliton solution of a KdV equation with variable coefficients [35]. More recently, Zhang and Zhang [36] generalized the exp-function method to construct multiwave solutions of nonlinear DDEs by devising a rational ansätz of multiple exponential functions. More and more studies show that because of its more general ansätz with free parameters, the exp-function method can be used to construct multiple types of exact solutions of many nonlinear PDEs and DDEs.

In the present paper, we shall further generalize the exp-function method to construct multiwave solutions of nonlinear DDEs with variable coefficients. In order to illustrate the effectiveness and advantages of the generalized method, we would like to consider a (2+1)-dimensional variable-coefficient Toda lattice equation in the form [37]:

$$\frac{\partial^2 u_n}{\partial x \partial t} = \left[\frac{\partial u_n}{\partial t} + \alpha(t)\right](u_{n-1} - 2u_n + u_{n+1}), \quad (2)$$

where $u_n = u_n(x,t)$ and $\alpha(t)$ is an arbitrary function of t. Particularly, when $\alpha(t) = 1$, Eq. (2) becomes the (2+1)-dimensional constant-coefficient Toda lattice equation [38].

The rest of this paper is organized as follows. In Section 2, we generalize the exp-function method to construct multiwave solutions of nonlinear DDEs with variable coefficients. In Section 3, we apply the generalized method to Eq. (2). In Section 4, some conclusions are given.

Manuscript received March 28, 2014; revised August 30, 2014. This work was supported by the Natural Science Foundation of Liaoning Province of China (L2012404), the Liaoning BaiQianWan Talents Program (2013921055), the PhD Start-up Fund of Liaoning Province of China (20141137) and the Natural Science Foundation of China (11371071).

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II. Methodology

In this section, we describe the basic idea of the generalized exp-function method with a general ansätz for constructing multiwave solutions of variable-coefficient nonlinear DDEs, say, in three variables n, x and t:

$$\triangle(u_{nt}, u_{nx}, u_{ntt}, u_{nxt}, \dots u_{n-1}, u_n, u_{n+1}, \cdots,) = 0, \quad (3)$$

where $\triangle$ is a polynomial of u_n, $u_{n\pm s}(s = 1, 2, \cdots)$ and their derivatives, otherwise, a suitable transformation can transform Eq. (3) into such an equation.

The exp-function method generalized in this paper for single-wave solution is based on the assumption that the solutions of Eq. (3) can be expressed as follows:

$$u_n = \frac{\sum_{i_1=0}^{p_1} a_{i_1} \mathrm{e}^{i_1\xi_1}}{\sum_{j_1=0}^{q_1} b_{j_1} \mathrm{e}^{j_1\xi_1}}, \quad (4)$$

$$u_{n-s} = \frac{\sum_{i_1=0}^{p_1} a_{i_1} \mathrm{e}^{i_1(\xi_1 - sk_1)}}{\sum_{j_1=0}^{q_1} b_{j_1} \mathrm{e}^{j_1(\xi_1 - sk_1)}}, \quad (5)$$

$$u_{n+s} = \frac{\sum_{i_1=0}^{p_1} a_{i_1} \mathrm{e}^{i_1(\xi_1 + sk_1)}}{\sum_{j_1=0}^{q_1} b_{j_1} \mathrm{e}^{j_1(\xi_1 + sk_1)}}, \quad (6)$$

where $\xi_1 = k_1 n + c_1(x,t) + \omega_1$, $c_1(x,t)$ is an unknown function of x and t, a_{i_1}, b_{j_1} and k_1 are constants to determine later, ω_1 is an arbitrary constant, the values of p_1 and q_1 can be determined by balancing the linear term of highest order in Eq. (3) with the highest order nonlinear term.

In order to seek *N*-wave solutions for any integer $N > 1$, we generalize Eqs. (4)–(6) as follows:

$$u_n = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \cdots \sum_{i_N=0}^{p_N} a_{i_1 i_2 \cdots i_N} \mathrm{e}^{\sum_{g=1}^{N} i_g \xi_g}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \cdots \sum_{j_N=0}^{q_N} b_{j_1 j_2 \cdots j_N} \mathrm{e}^{\sum_{g=1}^{N} j_g \xi_g}}, \quad (7)$$

$$u_{n-s} = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \cdots \sum_{i_N=0}^{p_N} a_{i_1 i_2 \cdots i_N} \mathrm{e}^{\sum_{g=1}^{N} i_g(\xi_g - sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \cdots \sum_{j_N=0}^{q_N} b_{j_1 j_2 \cdots j_N} \mathrm{e}^{\sum_{g=1}^{N} j_g(\xi_g - sk_g)}}, \quad (8)$$

$$u_{n+s} = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \cdots \sum_{i_N=0}^{p_N} a_{i_1 i_2 \cdots i_N} \mathrm{e}^{\sum_{g=1}^{N} i_g(\xi_g + sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \cdots \sum_{j_N=0}^{q_N} b_{j_1 j_2 \cdots j_N} \mathrm{e}^{\sum_{g=1}^{N} j_g(\xi_g + sk_g)}}, \quad (9)$$

where $\xi_g = k_g n + c_g(x,t) + \omega_g$. When $N = 2$, Eqs. (7)–(9) give:

$$u_n = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} a_{i_1 i_2} \mathrm{e}^{\sum_{g=1}^{2} i_g \xi_g}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} b_{j_1 j_2} \mathrm{e}^{\sum_{g=1}^{2} j_g \xi_g}}, \quad (10)$$

$$u_{n-s} = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} a_{i_1 i_2} \mathrm{e}^{\sum_{g=1}^{2} i_g(\xi_g - sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} b_{j_1 j_2} \mathrm{e}^{\sum_{g=1}^{2} j_g(\xi_g - sk_g)}}, \quad (11)$$

$$u_{n+s} = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} a_{i_1 i_2} \mathrm{e}^{\sum_{g=1}^{2} i_g(\xi_g + sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} b_{j_1 j_2} \mathrm{e}^{\sum_{g=1}^{2} j_g(\xi_g + sk_g)}}, \quad (12)$$

which can be used to construct double-wave solution of Eq. (3).

When $N = 3$, Eqs. (7)–(9) give:

$$u_n = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \sum_{i_3=0}^{p_3} a_{i_1 i_2 i_3} \mathrm{e}^{\sum_{g=1}^{3} i_g \xi_g}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \sum_{j_3=0}^{q_3} b_{j_1 j_2 j_3} \mathrm{e}^{\sum_{g=1}^{3} j_g \xi_g}}, \quad (13)$$

$$u_{n-s} = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \sum_{i_3=0}^{p_3} a_{i_1 i_2 i_3} \mathrm{e}^{\sum_{g=1}^{3} i_g(\xi_g - sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \sum_{j_3=0}^{q_3} b_{j_1 j_2 j_3} \mathrm{e}^{\sum_{g=1}^{3} j_g(\xi_g - sk_g)}}, \quad (14)$$

$$u_{n+s} = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \sum_{i_3=0}^{p_3} a_{i_1 i_2 i_3} \mathrm{e}^{\sum_{g=1}^{3} i_g(\xi_g + sk_g)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \sum_{j_3=0}^{q_3} b_{j_1 j_2 j_3} \mathrm{e}^{\sum_{g=1}^{3} j_g(\xi_g + sk_g)}}, \quad (15)$$

which can be used to obtain three-wave solution of Eq. (3).

Substituting Eqs. (10)–(12) into Eq. (3), and using *Mathematica*, then equating each coefficient of the same order power of exponential functions to zero yields a set of differential equations. Solving the set of differential equations, we can determine the double-wave solution, and the following three-wave solution by the use of Eqs. (13)–(15), provided they exist. If possible, we may conclude with the uniform formula of *N*-wave solution for any integer $N \geq 1$.

III. Multiwave solutions

In this section, let us apply the generalized exp-function method described in Section 2 to Eq. (2). To seek single-wave solution, we suppose that:

$$u_n(x,t) = \frac{a_1 \mathrm{e}^{\xi_1}}{1 + b_1 \mathrm{e}^{\xi_1}}, \quad (16)$$

$$u_{n-1}(x,t) = \frac{a_1 \mathrm{e}^{\xi_1 - k_1}}{1 + b_1 \mathrm{e}^{\xi_1 - k_1}}, \quad (17)$$

$$u_{n+1}(x,t) = \frac{a_1 \mathrm{e}^{\xi_1 + k_1}}{1 + b_1 \mathrm{e}^{\xi_1 + k_1}}, \quad (18)$$

where $\xi_1 = k_1 n + c_1(x,t) + \omega_1$. Obviously, Eqs. (16)–(18) are embedded in the same form as Eqs. (4)–(6). Substituting Eqs. (16)–(18) into Eq. (2), and using *Mathematica*, then equating each coefficient of the same order power of exponential functions to zero yields a set of PDEs as follows:

$$\begin{aligned}
&-a_1\alpha(t) + 2a_1\alpha(t)\mathrm{e}^{k_1} - a_1\alpha(t)\mathrm{e}^{2k_1} \\
&\quad + a_1 c_{1x}(x,t)c_{1t}(x,t)\mathrm{e}^{k_1} + a_1 c_{1xt}(x,t)\mathrm{e}^{k_1} = 0, \\
&-a_1 b_1\alpha(t) + 2a_1 b_1\alpha(t)\mathrm{e}^{k_1} - a_1 b_1\alpha(t)\mathrm{e}^{2k_1} - a_1^2 c_{1t}(x,t) \\
&\quad + 2a_1^2 c_{1t}(x,t)\mathrm{e}^{k_1} - a_1^2 c_{1t}(x,t)\mathrm{e}^{k_1} \\
&\quad + a_1 b_1 c_{1x}(x,t)c_{1t}(x,t) - a_1 b_1 c_{1x}(x,t)c_{1t}(x,t)\mathrm{e}^{k_1} \\
&\quad + a_1 b_1 c_{1x}(x,t)c_{1t}(x,t)\mathrm{e}^{2k_1} + a_1 b_1 c_{1xt}(x,t) \\
&\quad + a_1 b_1 c_{1xt}(x,t)\mathrm{e}^{k_1} + a_1 b_1 c_{1xt}(x,t)\mathrm{e}^{2k_1} = 0, \\
&a_1 b_1^2\alpha(t) - 2a_1 b_1^2\alpha(t)\mathrm{e}^{k_1} + a_1 b_1^2\alpha(t)\mathrm{e}^{2k_1} + a_1^2 b_1 c_{1t}(x,t) \\
&\quad - 2a_1^2 b_1 c_{1t}(x,t)\mathrm{e}^{k_1} + a_1^2 b_1 c_{1t}(x,t)\mathrm{e}^{2k_1} \\
&\quad - a_1 b_1^2 c_{1x}(x,t)c_{1t}(x,t) + a_1 b_1^2 c_{1x}(x,t)c_{1t}(x,t)\mathrm{e}^{k_1} \\
&\quad - a_1 b_1^2 c_{1x}(x,t)c_{1t}(x,t)\mathrm{e}^{2k_1} + a_1 b_1^2 c_{1xt}(x,t) \\
&\quad + a_1 b_1^2 c_{1xt}(x,t)\mathrm{e}^{k_1} + a_1 b_1^2 c_{1xt}(x,t)\mathrm{e}^{2k_1} = 0, \\
&a_1 b_1^3\alpha(t) - 2a_1 b_1^3\alpha(t)\mathrm{e}^{k_1} + a_1 b_1^3\alpha(t)\mathrm{e}^{2k_1} \\
&\quad - a_1 b_1^3 c_{1x}(x,t)c_{1t}(x,t)\mathrm{e}^{k_1} + a_1 b_1^3 c_{1xt}(x,t)\mathrm{e}^{k_1} = 0.
\end{aligned}$$

Solving the set of PDEs, we have

$$a_1 = b_1 d_1, \quad c_1(x,t) = d_1 x + \frac{4\sinh\frac{k_1}{2}}{d_1} \int \alpha(t)\mathrm{d}t. \quad (19)$$

We, therefore, obtain the single-wave solution of Eq. (2):

$$u_n = \frac{b_1 d_1 \mathrm{e}^{\xi_1}}{1 + b_1 \mathrm{e}^{\xi_1}} = [\ln(1 + b_1 \mathrm{e}^{\xi_1})]_x, \quad (20)$$

where $\xi_1 = k_1 n + d_1 x + \frac{4\sinh\frac{k_1}{2}}{d_1}\int \alpha(t)\mathrm{d}t + \omega_1$, b_1, d_1, k_1 and ω_1 are arbitrary constants.

In Fig. 1, the evolution characteristics of a single-kink soliton determined by solution (20) is shown, where we selected $k_1 = 0.6$, $b_1 = 2$, $d_1 = 1$, $\omega_1 = 0$, $\alpha(t) = 1+\mathrm{sech}t+\mathrm{sn}(t, 0.5)$. Fig. 1(c) shows the asymptotic property of amplitude u_0 at $x = 3$. Fig. 1(d) shows that the velocity of u_n periodically changes with time.

To construct double-wave solution, we suppose that:

$$u_n = \frac{a_{10}\mathrm{e}^{\xi_1} + a_{01}\mathrm{e}^{\xi_2} + a_{11}\mathrm{e}^{\xi_1+\xi_2}}{1 + b_1\mathrm{e}^{\xi_1} + b_2\mathrm{e}^{\xi_2} + b_3\mathrm{e}^{\xi_1+\xi_2}}, \quad (21)$$

$$u_{n-1} = \frac{a_{10}\mathrm{e}^{\xi_1-k_1} + a_{01}\mathrm{e}^{\xi_2-k_2} + a_{11}\mathrm{e}^{\xi_1+\xi_2-k_1-k_2}}{1 + b_1\mathrm{e}^{\xi_1-k_1} + b_2\mathrm{e}^{\xi_2-k_2} + b_3\mathrm{e}^{\xi_1+\xi_2-k_1-k_2}}, \quad (22)$$

$$u_{n+1} = \frac{a_{10}\mathrm{e}^{\xi_1+k_1} + a_{01}\mathrm{e}^{\xi_2+k_2} + a_{11}\mathrm{e}^{\xi_1+\xi_2+k_1+k_2}}{1 + b_1\mathrm{e}^{\xi_1+k_1} + b_2\mathrm{e}^{\xi_2+k_2} + b_3\mathrm{e}^{\xi_1+\xi_2+k_1+k_2}}, \quad (23)$$

where $\xi_i = k_i n + c_i(x, t) + \omega_i$ $(i = 1, 2)$. Clearly, Eqs. (21)–(23) possess the same form as Eqs. (10)–(12). Substituting Eqs. (21)–(23) into Eq. (2), and using the similar manipulations as illustrated above, we get a set of PDEs. Solving the set of PDEs, we have

$$a_{10} = b_1 d_1, \quad a_{01} = b_2 d_2, \quad (24)$$

$$a_{11} = b_1 b_2 (d_1 + d_2)\mathrm{e}^{B_{12}}, \quad b_3 = b_1 b_2 \mathrm{e}^{B_{12}}, \quad (25)$$

$$c_i(x, t) = d_i x + \frac{4\sinh\frac{k_i}{2}}{d_i}\int \alpha(t)\mathrm{d}t \quad (i = 1, 2), \quad (26)$$

$$\mathrm{e}^{B_{12}} = \frac{d_1^2\Omega_2^2 + d_2^2\Omega_1^2 - 2d_1 d_2\Omega_1\Omega_2\cosh(\frac{k_1}{2} - \frac{k_2}{2})}{d_1^2\Omega_2^2 + d_2^2\Omega_1^2 - 2d_1 d_2\Omega_1\Omega_2\cosh(\frac{k_1}{2} + \frac{k_2}{2})}, \quad (27)$$

$$\Omega_i = \sinh^2\frac{k_i}{2} \quad (i = 1, 2). \quad (28)$$

Thus, we obtain the double-wave solution of Eq. (2):

$$\begin{aligned} u_n &= \frac{b_1 d_1\mathrm{e}^{\xi_1} + b_2 d_2\mathrm{e}^{\xi_2} + b_1 b_2(d_1 + d_2)\mathrm{e}^{\xi_1+\xi_2+B_{12}}}{1 + b_1\mathrm{e}^{\xi_1} + b_2\mathrm{e}^{\xi_2} + b_1 b_2\mathrm{e}^{\xi_1+\xi_2+B_{12}}} \\ &= \left[\ln(1 + b_1\mathrm{e}^{\xi_1} + b_2\mathrm{e}^{\xi_2} + b_1 b_2\mathrm{e}^{\xi_1+\xi_2+B_{12}})\right]_x, \end{aligned} \quad (29)$$

where $\xi_i = k_i n + d_i x + \frac{4\sinh\frac{k_i}{2}}{d_i}\int \alpha(t)\mathrm{d}t + \omega_i$ $(i = 1, 2)$, b_1, b_2, d_1, d_2, k_1, k_2, ω_1 and ω_2 are free constants, $\mathrm{e}^{B_{12}}$ is defined by Eqs. (27) and (28).

In Fig. 2, the evolution characteristics of a double-kink soliton determined by solution (29) is shown, where $k_1 = 1$, $k_2 = 0.3$, $b_1 = 1$, $b_2 = 2$, $d_1 = 1$, $d_2 = 1$, $\omega_1 = 0$, $\omega_2 = 0$, $\alpha(t) = 1+\mathrm{sech}t+\mathrm{sn}(t, 0.5)$. Fig. 3 shows a singular double-kink soliton determined by solution (29), all the parameters of which are same as those of Fig. 2 except $b_1 = -1$. It is easy to see from Fig. 3 that u_0 increases to infinite rapidly as $t \to -5$ and u_n has a jump when $n = 10$, $x = -8$, $t = 0$.

We now construct three-wave solution, for this purpose, we suppose that:

$$u_n = \frac{f_{1,n}(\xi_1, \xi_2, \xi_3)}{f_{2,n}^2(\xi_1, \xi_2, \xi_3)}, \quad (30)$$

$$u_{n-1} = \frac{f_{1,n-1}(\xi_1, \xi_2, \xi_3)}{f_{2,n-1}^2(\xi_1, \xi_2, \xi_3)}, \quad (31)$$

$$u_{n+1} = \frac{f_{1,n+1}(\xi_1, \xi_2, \xi_3)}{f_{2,n+1}^2(\xi_1, \xi_2, \xi_3)}, \quad (32)$$

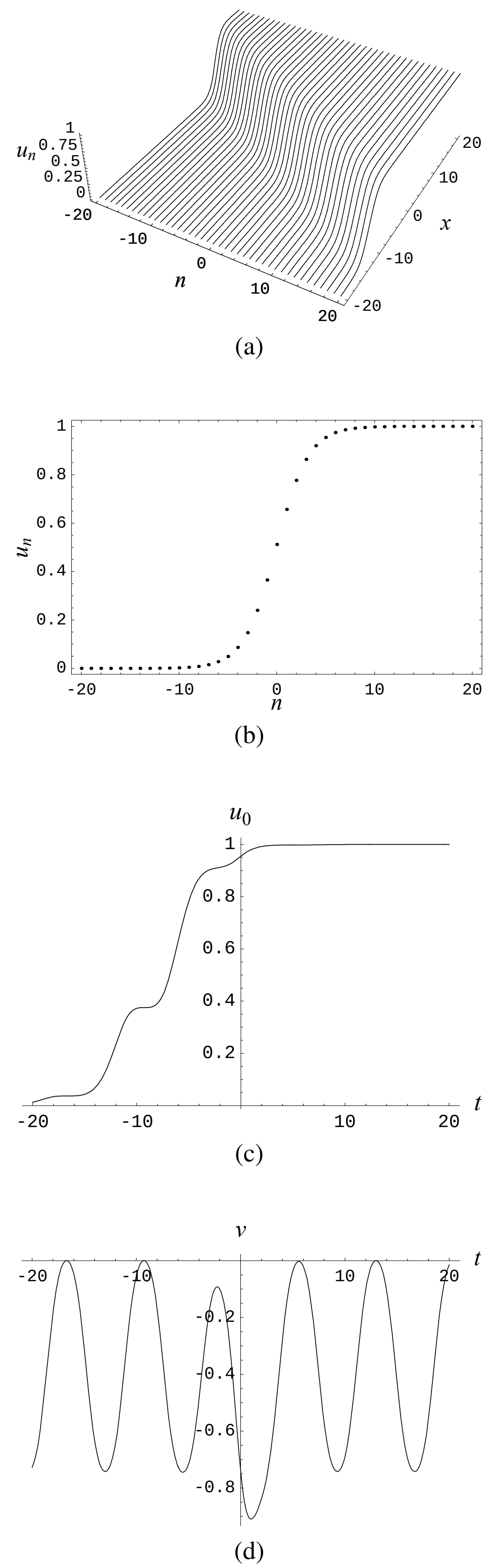

Fig. 1. Evolution plots of single-soliton determined by solution (20): (a) $t = 0$; (b) $x = 0$, $t = 0$; (c) $n = 0$, $x = 3$; (d) velocity curve.

where $\xi_i = k_i n + c_i(x, t) + \omega_i$ $(i = 1, 2, 3)$, and

$$\begin{aligned} f_{1,n}(\xi_1, \xi_2, \xi_3) &= a_{100}\mathrm{e}^{\xi_1} + a_{010}\mathrm{e}^{\xi_2} + a_{001}\mathrm{e}^{\xi_3} + a_{110}\mathrm{e}^{\xi_1+\xi_2} \\ &\quad + a_{101}\mathrm{e}^{\xi_1+\xi_3} + a_{011}\mathrm{e}^{\xi_2+\xi_3} + a_{111}\mathrm{e}^{\xi_1+\xi_2+\xi_3}, \end{aligned}$$

$$\begin{aligned} f_{2,n}(\xi_1, \xi_2, \xi_3) &= 1 + b_1\mathrm{e}^{\xi_1} + b_2\mathrm{e}^{\xi_2} + b_3\mathrm{e}^{\xi_3} + b_4\mathrm{e}^{\xi_1+\xi_2} \\ &\quad + b_5\mathrm{e}^{\xi_1+\xi_3} + b_6\mathrm{e}^{\xi_2+\xi_3} + b_7\mathrm{e}^{\xi_1+\xi_2+\xi_3}, \end{aligned}$$

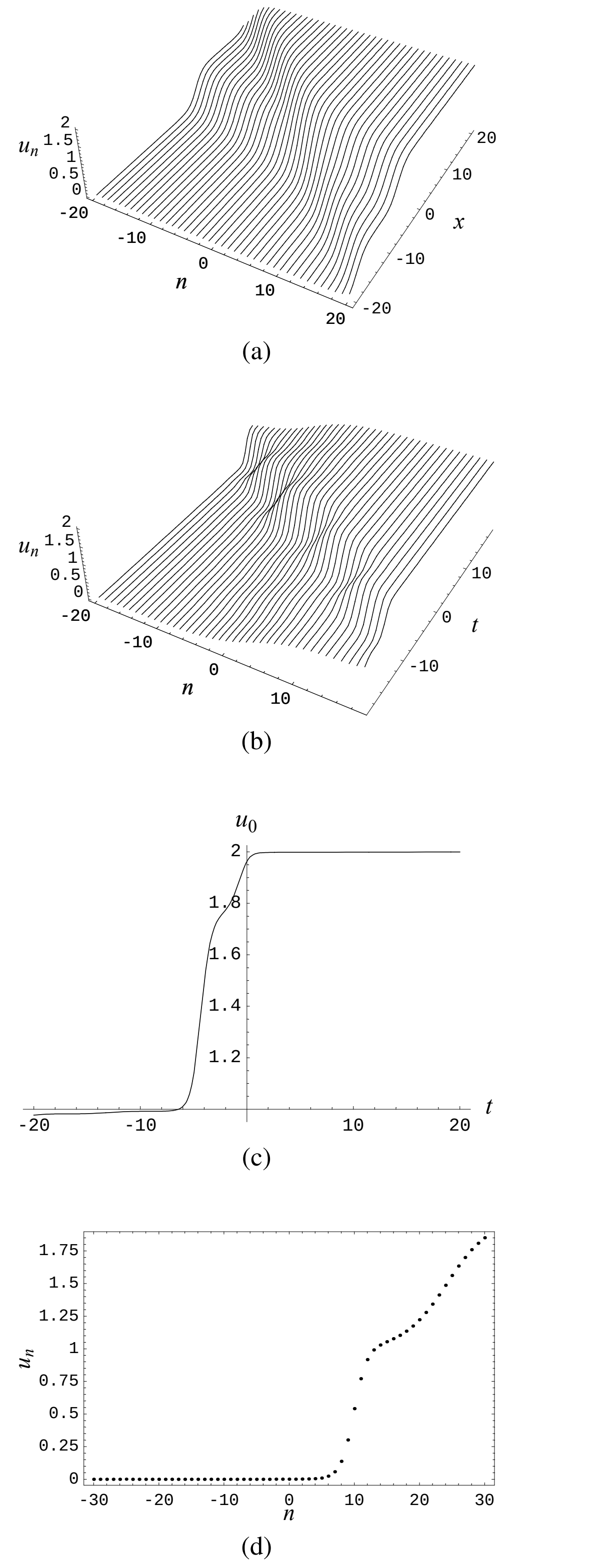

Fig. 2. Evolution plots of double-solution determined by solution (29): (a) $t=0$; (b) $x=0$; (c) $n=0$, $x=0$; (d) $x=-8$, $t=0$.

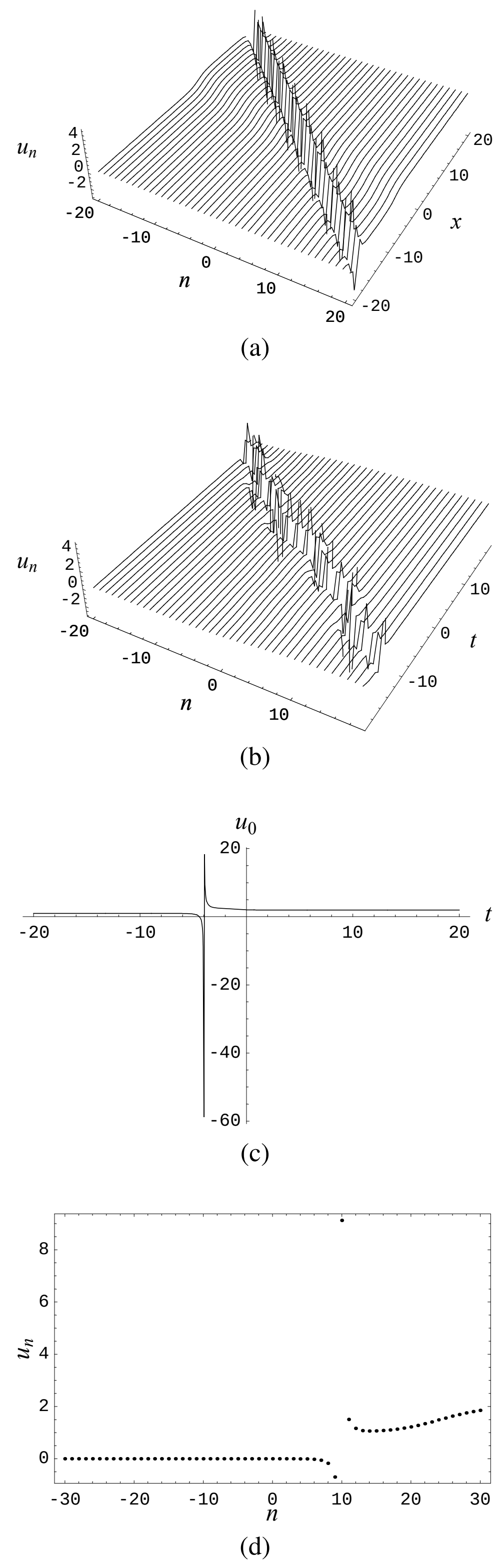

Fig. 3. Evolution plots of singular double-soliton determined by solution (29): (a) $t=0$; (b) $x=0$; (c) $n=0$, $x=0$; (d) $x=-8$, $t=0$.

$$\begin{aligned}
f_{1,n-1}(\xi_1,\xi_2,\xi_3) &= a_{100}\mathrm{e}^{\xi_1-k_1}+a_{010}\mathrm{e}^{\xi_2-k_2}+a_{001}\mathrm{e}^{\xi_3-k_3}\\
&\quad +a_{110}\mathrm{e}^{\xi_1+\xi_2-k_1-k_2}+a_{101}\mathrm{e}^{\xi_1+\xi_3-k_1-k_3}\\
&\quad +a_{011}\mathrm{e}^{\xi_2+\xi_3-k_2-k_3}+a_{111}\mathrm{e}^{\xi_1+\xi_2+\xi_3-k_1-k_2-k_3},\\
f_{2,n-1}(\xi_1,\xi_2,\xi_3) &= 1+b_1\mathrm{e}^{\xi_1-k_1}+b_2\mathrm{e}^{\xi_2-k_2}+b_3\mathrm{e}^{\xi_3-k_3}\\
&\quad +b_4\mathrm{e}^{\xi_1+\xi_2-k_1-k_2}+b_5\mathrm{e}^{\xi_1+\xi_3-k_1-k_3}+b_6\mathrm{e}^{\xi_2+\xi_3-k_2-k_3}\\
&\quad +b_7\mathrm{e}^{\xi_1+\xi_2+\xi_3-k_1-k_2-k_3},
\end{aligned}$$

$$\begin{aligned}
f_{1,n+1}(\xi_1,\xi_2,\xi_3) &= a_{100}\mathrm{e}^{\xi_1+k_1}+a_{010}\mathrm{e}^{\xi_2+k_2}+a_{001}\mathrm{e}^{\xi_3+k_3}\\
&\quad +a_{110}\mathrm{e}^{\xi_1+\xi_2+k_1+k_2}+a_{101}\mathrm{e}^{\xi_1+\xi_3+k_1+k_3}\\
&\quad +a_{011}\mathrm{e}^{\xi_2+\xi_3+k_2+k_3}+a_{111}\mathrm{e}^{\xi_1+\xi_2+\xi_3+k_1+k_2+k_3},\\
f_{2,n+1}(\xi_1,\xi_2,\xi_3) &= 1+b_1\mathrm{e}^{\xi_1+k_1}+b_2\mathrm{e}^{\xi_2+k_2}+b_3\mathrm{e}^{\xi_3+k_3}\\
&\quad +b_4\mathrm{e}^{\xi_1+\xi_2+k_1+k_2}+b_5\mathrm{e}^{\xi_1+\xi_3+k_1+k_3}+b_6\mathrm{e}^{\xi_2+\xi_3+k_2+k_3}\\
&\quad +b_7\mathrm{e}^{\xi_1+\xi_2+\xi_3+k_1+k_2+k_3}.
\end{aligned}$$

It is easy to see that Eqs. (30)–(32) have the same form as Eqs. (13)–(15). By the similar manipulations mentioned above, we have

$$a_{100} = b_1 d_1, \quad a_{010} = b_2 d_2, \quad a_{001} = b_3 d_3, \tag{33}$$

$$a_{110} = b_1 b_2 d_1 d_2 \mathrm{e}^{B_{12}}, \quad a_{101} = b_1 b_3 d_1 d_3 \mathrm{e}^{B_{13}}, \tag{34}$$

$$a_{011} = b_2 b_3 d_2 d_3 \mathrm{e}^{B_{23}}, \tag{35}$$

$$a_{111} = b_1 b_2 b_3 (d_1 + d_2 + d_3) \mathrm{e}^{B_{12}+B_{13}+B_{23}}, \tag{36}$$

$$b_4 = b_1 b_2 \mathrm{e}^{B_{12}}, \quad b_5 = b_1 b_3 \mathrm{e}^{B_{13}}, \quad b_6 = b_2 b_3 \mathrm{e}^{B_{23}}, \tag{37}$$

$$b_7 = b_1 b_2 b_3 \mathrm{e}^{B_{12}+B_{13}+B_{23}}, \tag{38}$$

$$c_i(x,t) = d_i x + \frac{4\sinh\frac{k_i}{2}}{d_i} \int \alpha(t)\mathrm{d}t \quad (i = 1,2,3), \tag{39}$$

$$\mathrm{e}^{B_{ij}} = \frac{d_i^2\Omega_j^2 + d_j^2\Omega_i^2 - 2d_i d_j \Omega_i \Omega_j \cosh(\frac{k_i}{2} - \frac{k_j}{2})}{d_i^2\Omega_j^2 + d_j^2\Omega_i^2 - 2d_i d_j \Omega_i \Omega_j \cosh(\frac{k_i}{2} + \frac{k_i}{2})}, \tag{40}$$

$$\Omega_i = \sinh^2\frac{k_i}{2}, \quad \Omega_j = \sinh^2\frac{k_j}{2} \quad (1 \le i < j \le 3). \tag{41}$$

Employing Eqs. (33)–(41), we obtain the three-wave solution of Eq. (2):

$$\begin{aligned} u_n = \big[&\ln(1 + b_1\mathrm{e}^{\xi_1} + b_2\mathrm{e}^{\xi_2} + b_3\mathrm{e}^{\xi_3} + b_1 b_2 \mathrm{e}^{\xi_1+\xi_2+B_{12}} \\ &+ b_1 b_3 \mathrm{e}^{\xi_1+\xi_3+B_{13}} + b_2 b_3 \mathrm{e}^{\xi_2+\xi_3+B_{23}} \\ &+ b_1 b_2 b_3 \mathrm{e}^{\xi_1+\xi_2+\xi_3+B_{12}+B_{13}+B_{23}})\big]_x, \end{aligned} \tag{42}$$

where $\xi_i = k_i n + d_i x + \frac{4\sinh\frac{k_i}{2}}{d_i}\int \alpha(t)\mathrm{d}t + \omega_i$ $(i = 1,2,3)$, b_1, b_2, b_3, d_1, d_2, d_3, k_1, k_2, k_3, ω_1, ω_2 and ω_3 are arbitrary constants, B_{12}, B_{13} and B_{23} are determined by Eqs. (40) and (41).

In Fig. 4, the evolution characteristics of a three-kink soliton determined by solution (42) is shown, the parameters of which are selected as $k_1 = 1$, $k_2 = -1$, $k_3 = 0.36$, $b_1 = 2$, $b_2 = 1$, $b_3 = 3$, $d_1 = 1$, $d_2 = 1$, $d_3 = 2$, $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = 0$, $\alpha(t) = 1 + \mathrm{sech}t + t^2$.

If we continue to construct the N-wave solution for any $N \ge 4$, the following similar manipulations become rather complicated since equating the coefficients of the exponential functions to zero implies a highly nonlinear system as pointed out in [34]. Fortunately, by analyzing the obtained solutions (20), (29) and (42) we can obtain the uniform formula of N-wave solution as follows:

$$u_n = \left[\ln\left(\sum_{\mu=0,1}\prod_{i=1}^{N} b_i^{\mu_i}\mathrm{e}^{\sum_{i=1}^{N}\mu_i\xi_i + \sum_{1\le i<j\le N}\mu_i\mu_j B_{ij}}\right)\right]_x, \tag{43}$$

where the summation $\sum_{\mu=0,1}$ refers to all combinations of each $\mu_i = 0,1$ for $i = 1,2,\cdots,N$, $\xi_i = k_i n + d_i x + \frac{4\sinh\frac{k_i}{2}}{d_i}$, and

$$\mathrm{e}^{B_{ij}} = \frac{d_i^2\Omega_j^2 + d_j^2\Omega_i^2 - 2d_i d_j \Omega_i \Omega_j \cosh(\frac{k_i}{2} - \frac{k_j}{2})}{d_i^2\Omega_j^2 + d_j^2\Omega_i^2 - 2d_i d_j \Omega_i \Omega_j \cosh(\frac{k_i}{2} + \frac{k_i}{2})}, \tag{44}$$

$$\Omega_i = \sinh^2\frac{k_i}{2}, \quad \Omega_j = \sinh^2\frac{k_j}{2} \quad (i < j; i,j = 1,2,\cdots,N). \tag{45}$$

Remark 1. Solutions (20), (29) and (42) obtained above have been checked with *Mathematica* by putting them back into the original Eq. (2). To the best of our knowledge, solutions (20), (29), (42) and (43) with arbitrary function $\alpha(t)$ have not been reported in literatures.

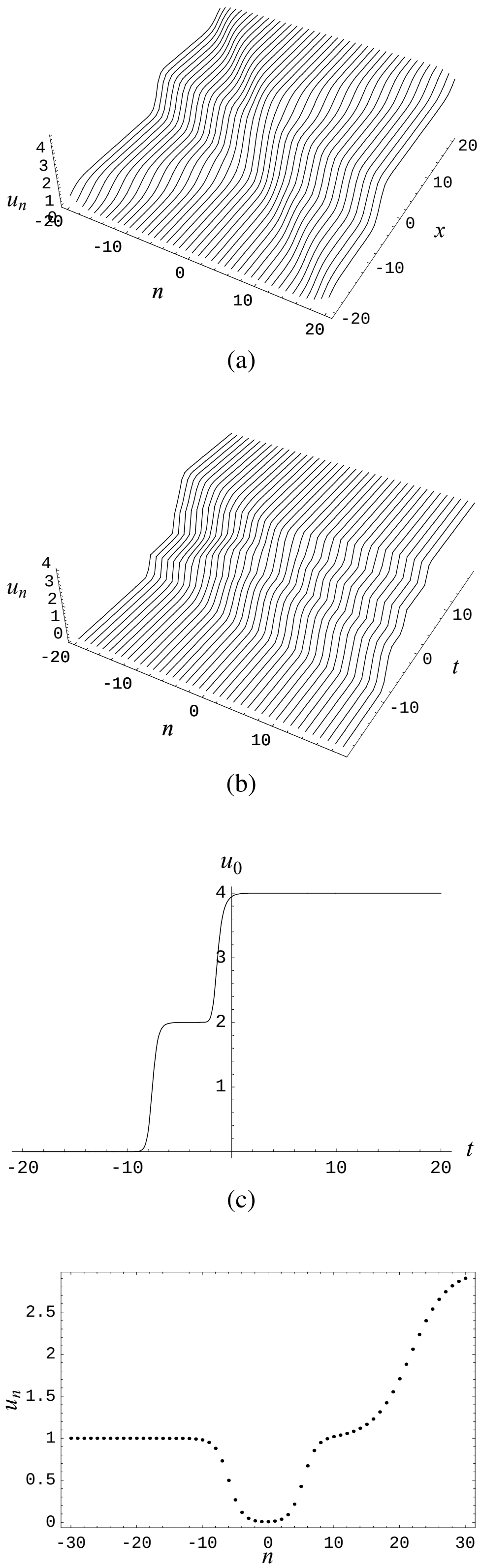

Fig. 4. Evolution plots of three-soliton determined by solution (42): (a) $t = 0$; (b) $x = 0$; (c) $n = 0$, $x = 0$; (d) $x = -6$, $t = 0$.

IV. CONCLUSION

In summary, single-wave solution (20), double-wave solution (29) and three-wave solution (42) of the (2+1)-dimensional variable-coefficient Toda lattice equation (2) have been obtained, from which the uniform formula of N-wave solution (43) is derived. This is due to the generalization of the exp-function method presented in this paper. Though these solutions can be constructed by some a future improvement of Hirota's bilinear method [7], the proposed method with the help of *Mathematica* for generating solutions (20), (29) and (42) is more simple and straightforward. Hirota's bilinear method has three steps [36], one of which is taking a transformation of new dependent variable(s) to reduce a given DDE to bilinear form(s). However, no general method has been found for such a transformation. Compared to Hirota's bilinear method, the method of this paper does not follow these steps. Besides, the multiwave solutions constructed by the generalized exp-function method contain free parameters b_1, b_2, $\cdots$, b_N so that they are more general than the ones $(b_i = 1, i = 1, 2, \cdots, N)$ obtained by Hirota's bilinear method. More importantly, these multiwave solutions with free parameters maybe possess new evolution characteristics. For example, when any one parameter is negative, the multiwave solutions can give singular multisoliton solutions like the one $(b_1 = -1)$ shown in Fig. 3. In this sense, we may conclude that the generalized exp-function method with the advantage of simplicity and effectiveness may provide us with a straightforward and applicable mathematical tool for generating multiwave solutions of some variable-coefficient DDEs or testing their existence.

Acknowledgment

The authors would like to express their sincere thanks to anonymous reviewers for the valuable suggestions and comments.

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