A Note on the Beta Function And Some Properties of Its Partial Derivatives

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***Abstract*—In this paper, the partial derivatives $B_{p,q}(x,y) = \frac{\partial^{q+p}}{\partial x^p \partial y^q} B(x,y)$ of the Beta function $B(x,y)$ are expressed in terms of a finite number of the Polygamma function, where p and q are non-negative integers, x and y are complex numbers. In particular, $B_{p,q}(x,y)$ can be expressed by the Riemann zeta function if x is equal to n or $n+\frac{1}{2}$ and y is equal to m or $m+\frac{1}{2}$, where n and m are integers. Furthermore, many integral functions associated with $B(x,y)$ and $B_{p,q}(x,y)$ can be expressed as the closed forms.**

***Index Terms*—Riemann zeta function, Beta function, Gamma function, Polygamma function, Digamma function, closed form.**

I. Introduction

In mathematics, the Beta function was studied by Euler and Legendre as a special function. It is usually defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \tag{1}$$

for $Rex > 0$ and $Rey > 0$. It is often applied in many fields such as mathematical equations and probability theory. Its definition was extended to complex numbers values of x and y by using the neutrix limit in [1]. Furthermore, the partial derivatives of the Beta function on the complex numbers x and y exist a close relationship with many special functions and special integrals. For example, the following relation was proved in [2]

$$\int_0^1 t^{x-1}(1-t)^{y-1} \ln^p t \ln^q (1-t) dt = B_{p,q}(x,y) \tag{2}$$

for integers $p, q > 0$ and $q + Rex, p + Rey > 0$, where $B_{p,q}(x,y) = \frac{\partial^{p+q}}{\partial x^p \partial y^q} B(x,y)$. Moreover, K. S. KÖlbig gave the closed expressions of (2) for $x = 0$ and $y = 1$ in [3]. Putting $t = \sin^2 u$ in (2), we have

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \sin^{2x-1} u \cos^{2y-1} u \ln^p \sin u \ln^q \cos u du \\ &= 2^{-p-q-1} B_{p,q}(x,y). \end{aligned} \tag{3}$$

K. S. KÖlbig also gave the closed expression of (3) for $x = y = \frac{1}{2}$ in [4]. Note that the most effective way of computing (1) and (2) is based on power series expansion, even for integral equations[5]. Moreover, many scholars have studied different Integro-differential equation by different methods in [6] and [7].

In this paper, we concern about the recurrence formulas and the closed forms of $B_{p,q}(x,y)$ in (1) and (2). Also, we consider the existence condition of the closed forms and the representations of $B_{p,q}(x,y)$.

From

$$\begin{aligned} &\frac{1}{B(x,y)} \\ &= \frac{2^{x+y-1}(x+y-1)}{\pi} \int_0^{\pi/2} \cos(x-y)t \cos^{x+y-2} t dt, \end{aligned}$$

(see [8]) and $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, where $\Gamma(z)$ is the Gamma function, we see that the following formulas

$$\begin{aligned} &\int_0^{\pi} \left(2\cos\frac{\theta}{2}\right)^{2x} \cos y\theta d\theta \\ &= \begin{cases} B(2x+1, y-x) \sin \pi(y-x), \\ \quad x - y < 0, Rex > -\frac{1}{2}, \\ -B(2x+1, -x-y) \sin \pi(x+y), \\ \quad x + y < 0, Rex > -\frac{1}{2}, \end{cases} \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\int_0^{\pi} t^q \left(2\cos\frac{t}{2}\right)^{2x} \cos\left(yt + \frac{q\pi}{2}\right) \ln^p \left(2\cos\frac{t}{2}\right) dt \\ &= \begin{cases} \frac{(-1)^p}{2^p} \sum\limits_{j=0}^{p} \pi^{p-j} C_p^j \sum\limits_{k=0}^{q} \pi^k C_q^k \sum\limits_{u=0}^{j} (-1)^u 2^u C_j^u \cdot \\ \quad B_{u,q+j-u-k}(2x+1, y-x) \sin a\pi, \\ \quad a = \frac{2y-2x+p+k-j}{2}, Re(x-y) < 0, \\ -\frac{(-1)^q}{2^p} \sum\limits_{j=0}^{p} \pi^{p-j} C_p^j \sum\limits_{k=0}^{q} (-\pi)^k C_q^k \sum\limits_{u=0}^{j} (-1)^{j-u} 2^u C_j^u \cdot \\ \quad B_{u,q+j-u-k}(2x+1, -x-y) \sin b\pi, \\ \quad b = \frac{2x+2y+p+k-j}{2}, Re(x+y) < 0, \end{cases} \end{aligned} \tag{5}$$

exist for non-negative integers p, q and $Rex > -\frac{1}{2}$.

If we can establish the closed forms of $B_{p,q}(x,y)$, then the above integrals can be expressed as the associated closed forms. It is well known that an expression is said to be the closed form expression if it can be expressed analytically in terms of a finite number of the Riemann zeta function and some special constants γ, π, where γ denotes Euler constant.

II. The representation of $B_{p,q}(x,y)$ for $x, y, x+y \neq 0, -1, -2, \cdots$

Theorem 2.1 Let p and q be non-negative integers, x and y be complex numbers satisfying $x, y, x+y \neq 0, -1, -2, \cdots$. Then the following recurrence relations hold

$$B_{p,0}(x,y) = B(x,y) H_p^{\psi}(x, x+y), \tag{6}$$

$$B_{0,q}(x,y) = B(x,y) H_q^{\psi}(y, x+y), \tag{7}$$

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where

$$
\begin{gathered}
H_p^{\psi}(x,y)=\sum_{l=0}^{p-1}H_{p,l}^{\psi}(x,y),\\
H_{p,0}^{\psi}(x,y)=\psi^{(p-1)}(x)-\psi^{(p-1)}(y),\\
H_{p,1}^{\psi}(x,y)=\sum_{j=1}^{p-1}\left(\begin{array}{c}p-1\\ j\end{array}\right)\cdot\\
\left(\psi^{(p-1-j)}(x)-\psi^{(p-1-j)}(y)\right)\left(\psi^{(j-1)}(x)-\psi^{(j-1)}(y)\right),\\
H_{p,l}^{\psi}(x,y)=\sum_{j=1}^{p-1}\left(\begin{array}{c}p-1\\ j\end{array}\right)\cdot\\
\left(\psi^{(p-1-j)}(x)-\psi^{(p-1-j)}(y)\right)H_{j,l-1}(x,y),
\end{gathered}
\tag{8}
$$

and $\psi(x)$ denotes the Digamma function which is defined by

$$
\psi(x)=\frac{d}{dx}\ln\Gamma(x)=-\gamma-\frac{1}{x}+\sum_{l=1}^{\infty}\left(\frac{1}{l}-\frac{1}{l+x}\right), \tag{9}
$$

and $\psi^{(p)}(x)=\frac{d^p}{dx^p}\psi(x)(p=0,1,2,\cdots)$ denotes the Polygamma function.

Proof. It is well known that $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. Calculating p-order partial derivatives on x of $B(x,y)$ by using the Leibniz rule, we have

$$
B_{1,0}(x,y)=\left(\psi(x)-\psi(x+y)\right)B(x,y),
$$

and

$$
\begin{gathered}
B_{p,0}(x,y)=\sum_{k=0}^{p-1}\left(\begin{array}{c}p-1\\ k\end{array}\right)\cdot\\
\left(\psi^{(k)}(x)-\psi^{(k)}(x+y)\right)B_{p-1-k,0}(x,y).
\end{gathered}
\tag{10}
$$

By repeating use of (10), the formula (6) can be obtained. By $B_{0,q}(x,y)=B_{q,0}(y,x)$, the formula (7) can also be obtained.

Theorem 2.2 Let p and q be non-negative integers, x and y be complex numbers satisfying $x,y,x+y\neq 0,-1,-2,\cdots$. Then the partial derivatives of $B(x,y)$ is given by

$$
\begin{gathered}
B_{p,q}(x,y)=B(x,y)\tfrac{\partial^p}{\partial x^p}H_q^{\psi}(y,x+y)\\
+B(x,y)\sum_{k=0}^{p-1}\left(\begin{array}{c}p\\ k\end{array}\right)H_{p-k}^{\psi}(x,x+y)\tfrac{\partial^k}{\partial x^k}H_q^{\psi}(y,x+y),
\end{gathered}
\tag{11}
$$

or

$$
\begin{gathered}
B_{p,q}(x,y)=B(x,y)\tfrac{\partial^q}{\partial y^q}H_p^{\psi}(x,x+y)\\
+B(x,y)\sum_{k=0}^{q-1}\left(\begin{array}{c}q\\ k\end{array}\right)H_{q-k}^{\psi}(y,x+y)\tfrac{\partial^k}{\partial y^k}H_p^{\psi}(x,x+y),
\end{gathered}
\tag{12}
$$

where

$$
\begin{gathered}
\tfrac{\partial^k}{\partial y^k}H_p^{\psi}(x,y)=\sum_{l=0}^{p-1}\tfrac{\partial^k}{\partial y^k}H_{p,l}^{\psi}(x,y),\\
\tfrac{\partial^k}{\partial y^k}H_{p,0}^{\psi}(x,y)=-\psi^{(p+k-1)}(y),\\
\tfrac{\partial^k}{\partial y^k}H_{p,1}^{\psi}(x,y)\\
=\sum_{j=1}^{p-1}\left(\begin{array}{c}p-1\\ j\end{array}\right)\sum_{l=1}^{k-1}\left(\begin{array}{c}k\\ l\end{array}\right)\psi^{(p+l-1-j)}(y)\psi^{(j+k-l-1)}(y)\\
-\sum_{j=1}^{p-1}\left(\begin{array}{c}p-1\\ j\end{array}\right)\psi^{(p+k-1-j)}(y)\left(\psi^{(j-1)}(x)-\psi^{(j-1)}(y)\right)\\
-\sum_{j=1}^{p-1}\left(\begin{array}{c}p-1\\ j\end{array}\right)\cdot\\
\left(\psi^{(p-1-j)}(x)-\psi^{(p-1-j)}(y)\right)\psi^{(j+k-1)}(y),\\
\tfrac{\partial^k}{\partial y^k}H_{p,q}^{\psi}(x,y)\\
=\sum_{j=1}^{p-1}\left(\begin{array}{c}q-1\\ j\end{array}\right)\left(\psi^{(p-1-j)}(x)-\psi^{(p-1-j)}(y)\right)\cdot\\
\tfrac{\partial^k}{\partial y^k}H_{j,q-1}(x,y)\\
-\sum_{l=0}^{k-1}\left(\begin{array}{c}k\\ l\end{array}\right)\sum_{j=1}^{n-1}\left(\begin{array}{c}n-1\\ j\end{array}\right)\psi^{(n+k-l-1-j)}(y)\cdot\\
\tfrac{\partial^l}{\partial y^l}H_{j,q-1}(x,y),
\end{gathered}
\tag{13}
$$

for $k=1,2,\cdots$.

Proof. Calculating p-order partial derivatives on x for (7), we have

$$
B_{p,q}(x,y)=\sum_{k=0}^{p}\left(\begin{array}{c}p\\ k\end{array}\right)B_{p-k,0}(x,y)\frac{\partial^k}{\partial x^k}H_q^{\psi}(y,x+y). \tag{14}
$$

By (6) and (14), the formula (11) can be obtained. Similarly, we can get (12).

Now we consider the closed forms of $B_{p,q}(x,y)$. For the Polygamma function $\psi^{(k)}(x)(k=0,1,2,\cdots)$, the following relations were proved in [8]

$$
\begin{gathered}
\psi^{(k)}(n)=\left\{\begin{array}{c}-\gamma+H_{n-1},k=0,\\ k!(-1)^{k+1}\cdot\\ \left(\zeta(k+1)-H_{n-1}^{(k+1)}\right),k=1,2,\cdots,\end{array}\right.\\
\psi(\tfrac{1}{2}\pm n)=-\gamma-2\ln 2+2H_{2n-1}-H_{n-1},\\
\psi^{(k)}(n+\tfrac{1}{2})=k!(-1)^{k+1}\left(2^{k+1}-1\right)\zeta(k+1)\\
-k!(-1)^{k+1}(2^{k+1}H_{2n-1}^{(k+1)}+H_{n-1}^{(k+1)}),k=1,2,\cdots,\\
\psi^{(k)}(\tfrac{1}{2}-n)=k!(-1)^{k+1}\left(2^{k+1}-1\right)\zeta(k+1)\\
+k!(2^{k+1}H_{2n-1}^{(k+1)}-H_{n-1}^{(k+1)}),k=1,2,\cdots,
\end{gathered}
\tag{15}
$$

where $H_n^{(s)}=\sum_{l=1}^{n}\frac{1}{l^s}(s=1,2,\cdots)$, $H_n=H_n^{(1)}$, and $\zeta(s)=\sum_{l=1}^{\infty}\frac{1}{l^s}(s=1,2,\cdots)$ is the Riemann zeta function. Let p and q be non-negative integers and x,y be complex numbers satisfying one of the following conditions

1) $x=l,y=k,l,k=0,1,2,\cdots$;
2) $x=l+\frac{1}{2},y=k,\cdots,l=0,\pm1,\pm2,\cdots,k=0,1,2;$
3) $x=l,y=k+\frac{1}{2},l=0,1,2,\cdots,k=0,\pm1,\pm2,\cdots$;
4) $x=l+\frac{1}{2},y=k+\frac{1}{2},k,l=0,\pm1,\pm2,\cdots$.

Then it follows from (15) that $B_{p,q}(x,y)$ will be expressed as the closed forms.

Now we give two numerical examples.

Example 2.1 It follows from (5) and (11) that we give the following closed forms

$$\begin{gathered}\int_0^\pi t^2\cos^2\tfrac{t}{2}\cos\left(2t+\pi\right)\ln^3\left(2\cos\tfrac{t}{2}\right)dt\\ =\tfrac{601\pi}{5184}+\tfrac{77\pi^3}{864}-\tfrac{11\pi^5}{1440}-\tfrac{\pi}{6}\zeta(3)\\ =0.64184498762649453323900252412158858212469 9\cdots\end{gathered}$$

$$\begin{gathered}\int_0^\pi t^2\sqrt{2\cos\tfrac{t}{2}}\cos\left(\tfrac{3t}{4}+\pi\right)\ln^3\left(2\cos\tfrac{t}{2}\right)dt\\ =\tfrac{21\pi}{2}+\pi^3-\tfrac{151\pi^5}{120}+21\pi\ln 2+2\pi^3\ln 2+\tfrac{\pi^5}{12}\ln 2\\ +21\pi\ln^2 2+2\pi^3\ln^2 2+14\pi\ln^3 2-\tfrac{8\pi^3}{3}\ln^3 2\\ -7\pi\ln^4 2-21\pi\zeta(3)-\tfrac{13\pi^3\zeta(3)}{2}\\ -42\pi\zeta(3)\ln 2-30\pi\zeta(3)\ln^2 2+48\pi\zeta(5)\\ =-7.06326967556888082821974253984987641820 17\cdots\end{gathered}$$

$$\begin{gathered}\int_0^\pi t^3\cos^3\tfrac{t}{2}\cos\left(5t+\tfrac{3\pi}{2}\right)\ln^2\left(2\cos\tfrac{t}{2}\right)dt\\ =\tfrac{26973016606001262186415577 6}{185638065365206529228278125}-\tfrac{18436222012119568\pi^2}{27447014134377337 5}\\ -\tfrac{38000\pi^4}{27054027}+\tfrac{197859765184\pi^2\ln 2}{2031081077025}+\tfrac{4536864573440\ln^2 2}{731920376916729}\\ -\tfrac{121600\pi^2\ln^2 2}{9018009}-\tfrac{197859765184\zeta(3)}{290154439575}\\ -\tfrac{33902865076519117112 32\ln 2}{41211691722767572256 25}+\tfrac{243200\zeta(3)\ln 2}{1288287}\\ =0.02606937452491386271420865513133796157180 2\cdots\end{gathered}$$

However, the above closed forms can not be obtained by the symbolic computation systems in Mathematica.

Example 2.2 It follows from (5) and (11) that we can also obtain the following closed form

$$\begin{gathered}\int_{-\infty}^{\infty} t^q e^{2yt}\cosh^{-2x}(t-a)\ln^p\cosh(t-a)dt\\ =\tfrac{(-1)^p e^{2ay}}{2^{1-2x}}\sum_{j=0}^{p}C_p^j\tfrac{\ln^{p-j}2}{2^j}\sum_{u=0}^{j}C_j^u\sum_{k=0}^{q}C_q^k\tfrac{a^{q-k}}{2^k}\cdot\\ \sum_{v=0}^{k}(-1)^{k-v}C_k^v B_{u+v,j+k-u-v}(x+y,x-y).\end{gathered}\tag{16}$$

We note that if $2x+2y$ and $2x-2y$ are integers, then the integral (16) has the closed form. For example,

$$\begin{gathered}\int_{-\infty}^{\infty} t^3 e^{3t}\cosh^{-4}(t-a)\ln^2\cosh(t-a)dt=\tfrac{e^{3a}\pi}{288}\cdot\\ \left(\begin{array}{c}483\pi^4+2\pi^2(887+6\ln 2(145+69\ln 2))\\ +4a^2(69\pi^2+2(875+6\ln 2(145+69\ln 2)))\\ 12a^2(5\pi^2+6(3+10\ln^2 2+6\ln 2))\\ +8\left(\begin{array}{c}6\ln 2(22+6\ln 2+483\zeta(3))\\ +5(34+609\zeta(3))\end{array}\right)\\ +9a\left(\begin{array}{c}35\pi^4+6\pi^2\left(7+10\ln^2 2+6\ln 2\right)\\ +24\left(\begin{array}{c}42+21\zeta(3)\\ +2\ln 2(18+6\ln 2+35\zeta(3))\end{array}\right)\end{array}\right)\end{array}\right)\end{gathered}\tag{17}$$

III. THE REPRESENTATION OF $B_{p,q}(x,y)$ FOR $x=0,-1,-2,\cdots$ OR $y=0,-1,-2,\cdots$

More generally, it was proved in [1] that the neutrix limit

$$\begin{gathered}B_{p,q}(x,y)\\ =N-\lim_{\varepsilon\to 0+0}\int_\varepsilon^{1-\varepsilon}t^{x-1}(1-t)^{y-1}\ln^p t\ln^q(1-t)dt\end{gathered}\tag{18}$$

existed for all x,y and $p,q=0,1,2,\cdots$, where N denotes the neutrix[9]. To prove the next theorems, we need the following results

$$\begin{gathered}(1-t)^{-x}=\sum_{j=0}^{\infty}\tfrac{(x)_j}{j!}t^j,\\ \ln^k(1-t)=k!\sum_{i=k}^{\infty}\tfrac{(-1)^i s(i,k)t^i}{i!},\\ a_{n,q}(x)=\tfrac{d^q}{dx^q}(x)_n=\\ q!\sum_{l=0}^{j-q}\binom{j}{l}(-1)^{j-q-l}(x)_l\,s(j-l,q),\\ \tfrac{d^k}{dy^k}\tfrac{1}{(y)_l}=\tfrac{k!}{(l-1)!}\sum_{i=0}^{l-1}\binom{l-1}{i}\tfrac{(-1)^{k-i}}{(y+i)^{k+1}},\\ \tfrac{1}{t^{n+1}(1-t)^{m+1}}\\ =\sum_{j=0}^{n}\binom{m+j}{j}t^{j-n-1}+\sum_{i=0}^{m}\binom{n+i}{i}(1-t)^{i-m-1},\end{gathered}\tag{19}$$

where $(x)_n$ is a Pochhammer symbol, *i.e.*, $(x)_n=x(x+1)\cdots(x+n-1)$ and $s(i,k)$ is the Stirling number of the first kind.

Theorem 3.1 1) Let m be a non-negative integer and x be a complex number satisfying $x\neq m+1,m,\cdots,0,-1,-2,\cdots$. Then

$$B(x,-m)=\tfrac{(-1)^m(x-m)_m((x-m-1)B_{0,1}(x-m-1,1)-H_m)}{m!}.\tag{20}$$

2) Let m be a non-negative integer and n be a positive integer. Then

$$B(n,-m)=\sum_{i=0,i\neq m}^{n-1}\binom{n-1}{i}\tfrac{(-1)^i}{i-m}.\tag{21}$$

3) Let m and n be non-negative integers. Then

$$\begin{gathered}B(-n,-m)\\ =-\sum_{i=0}^{m-1}\binom{n+i}{i}\tfrac{1}{m-i}-\sum_{j=0}^{n-1}\binom{m+j}{j}\tfrac{1}{n-j}.\end{gathered}\tag{22}$$

Proof. 1) Integrating by parts, we have

$$\begin{gathered}B(x,y)=N-\lim_{\varepsilon\to 0+0}\int_\varepsilon^{1-\varepsilon}t^{x-1}(1-t)^{y-1}dt\\ =-\tfrac{1}{y}N-\lim_{\varepsilon\to 0+0}\left(\begin{array}{c}\varepsilon^y(1-\varepsilon)^{x-1}-\varepsilon^{x-1}(1-\varepsilon)^y\\ -(x-1)\int_\varepsilon^{1-\varepsilon}t^{x-2}(1-t)^y dt\end{array}\right)\\ =N-\lim_{\varepsilon\to 0+0}\left(\begin{array}{c}\tfrac{x-1}{y}\int_\varepsilon^{1-\varepsilon}t^{x-2}(1-t)^y dt\\ -\tfrac{\varepsilon^y(1-\varepsilon)^{x-1}-\varepsilon^{x-1}(1-\varepsilon)^y}{y}\end{array}\right)=\cdots\\ =\tfrac{(x-m)_m}{(y)_m}B(x-m,y+m)\\ -N-\lim_{\varepsilon\to 0+0}\sum_{l=1}^{m}\tfrac{(x+1-l)_{l-1}}{(y)_l}\cdot\\ \left(\varepsilon^{y-1+l}(1-\varepsilon)^{x-l}-\varepsilon^{x-l}(1-\varepsilon)^{y-1+l}\right),\end{gathered}\tag{23}$$

and thus

$$\begin{gathered}B(x,0)=N-\lim_{\varepsilon\to 0+0}\int_\varepsilon^{1-\varepsilon}t^{x-1}(1-t)^{-1}dt\\ =N-\lim_{\varepsilon\to 0+0}\int_\varepsilon^{1-\varepsilon}t^{x-1}d\ln(1-t)\\ =N-\lim_{\varepsilon\to 0+0}\left(\begin{array}{c}-(1-\varepsilon)^{x-1}\ln\varepsilon+\varepsilon^{x-1}\ln(1-\varepsilon)\\ +(x-1)\int_\varepsilon^{1-\varepsilon}t^{x-2}\ln(1-t)dt\end{array}\right)\\ =(x-1)B_{0,1}(x-1,1).\end{gathered}\tag{24}$$

Using (19), we have

$$\begin{aligned} & N-\lim_{\varepsilon\to 0+0}\sum_{l=1}^{m}(x+1-l)_{l-1}\cdot \\ & \frac{\left(\varepsilon^{-m-1+l}(1-\varepsilon)^{x-l}-\varepsilon^{x-l}(1-\varepsilon)^{-m-1+l}\right)}{(-n)_l} \\ & = N-\lim_{\varepsilon\to 0+0}\sum_{l=1}^{m}\frac{(x+1-l)_{l-1}}{(-m)_l}\cdot \\ & \left(\sum_{k=0}^{\infty}\frac{(l-x)_k}{k!}\varepsilon^{y+l+k-1}-\sum_{k=0}^{\infty}\frac{(m+1-l)_k}{k!}\varepsilon^{x+k-l}\right) \\ & = \sum_{l=1}^{m}\frac{(x+1-l)_{l-1}(l-x)_{m+1-l}}{(-m)_l(m+1-l)!}=-\frac{(-1)^m(x-m)_m}{m!}H_m. \end{aligned} \tag{25}$$

It follows from (23)-(25) that (20) holds.

2)

$$\begin{aligned} B(n,-m) &= N-\lim_{\varepsilon\to 0+0}\int_{\varepsilon}^{1-\varepsilon}t^{n-1}(1-t)^{-m-1}dt \\ &= N-\lim_{\varepsilon\to 0+0}\sum_{i=0}^{n-1}\begin{pmatrix} n-1 \\ i\end{pmatrix}(-1)^i\cdot \\ &\int_0^{1-\varepsilon}(1-t)^{i-m-1}dt \\ &= N-\lim_{\varepsilon\to 0+0}\left(\begin{array}{c}\sum_{i=0,i\neq m}^{n-1}\begin{pmatrix} n-1 \\ i\end{pmatrix}\frac{(-1)^i}{i-m}\left(1-\varepsilon^{i-m}\right) \\ -\begin{pmatrix} n-1 \\ m\end{pmatrix}(-1)^m\ln\varepsilon\end{array}\right) \\ &= \sum_{i=0,i\neq m}^{n-1}\begin{pmatrix} n-1 \\ i\end{pmatrix}\frac{(-1)^i}{i-m}. \end{aligned}$$

3) From (19), we have

$$\begin{aligned} & B(-n,-m) \\ & = N-\lim_{\varepsilon\to 0+0} \\ & \int_{\varepsilon}^{1-\varepsilon}\left(\begin{array}{c}\sum_{j=0}^{n}\begin{pmatrix} m+j \\ j\end{pmatrix}t^{j-n-1} \\ +\sum_{i=0}^{m}\begin{pmatrix} n+i \\ i\end{pmatrix}(1-t)^{i-m-1}\end{array}\right)dt \\ & = N-\lim_{\varepsilon\to 0+0}\left(\begin{array}{c}\sum_{j=0}^{n-1}\begin{pmatrix} m+j \\ j\end{pmatrix}\frac{(1-\varepsilon)^{j-n}-\varepsilon^{j-n}}{j-n} \\ -\sum_{i=0}^{m-1}\begin{pmatrix} n+i \\ i\end{pmatrix}\frac{(1-\varepsilon)^{i-m}-\varepsilon^{i-m}}{m-i}\end{array}\right) \\ & = -\sum_{i=0}^{m-1}\begin{pmatrix} n+i \\ i\end{pmatrix}\frac{1}{m-i}-\sum_{j=0}^{n-1}\begin{pmatrix} m+j \\ j\end{pmatrix}\frac{1}{n-j}. \end{aligned}$$

Theorem 3.2 Let m be a non-negative integer and x be a complex number satisfying $x \neq m+1, m, \cdots, 0, -1, -2, \cdots$. Then

$$\begin{aligned} & B_{p,q}(x,0) \\ & = \frac{(x-1)B_{p,q+1}(x-1,1)+pB_{p-1,q+1}(x-1,1)}{q+1}, \end{aligned} \tag{26}$$

and

$$\begin{aligned} & B_{p,q}(x,-m)=p!q!\sum_{k=0}^{p}\frac{1}{(p-k)!}\sum_{l=1}^{m}\cdot \\ & \frac{a_{l-1,p-k}(x+1-l)}{(l-1)!}\sum_{i=k}^{m+1-l}\frac{(-1)^i s(i,k)(l-x)_{m+1-l-i}}{i!(m+1-l-i)!}b_{l,m,q} \\ & -\frac{q!}{(m-1)!}\sum_{k=0}^{p}\begin{pmatrix} p \\ k\end{pmatrix}a_{m,p-k}(x-m)\cdot \\ & \sum_{j=0}^{q}\frac{(x-m-1)B_{k,j+1}(x-m-1,1)+kB_{k-1,j+1}(x-m-1,1)}{(j+1)!}b_{m,m,q-j}, \end{aligned} \tag{27}$$

where $b_{l,m,q}=\sum_{j=0}^{l-1}\begin{pmatrix} l-1 \\ j\end{pmatrix}\frac{(-1)^j}{(m-j)^{q+1}}$.

Proof. It can be easily seen that (26) holds for $m=0$. When $m>0$, calculating p-order partial derivatives on x and q-order partial derivatives on y for (23), respectively, we have

$$\begin{aligned} & B_{p,q}(x,y)=\sum_{k=0}^{p}\begin{pmatrix} p \\ k\end{pmatrix}\sum_{j=0}^{q}\begin{pmatrix} q \\ j\end{pmatrix}\cdot \\ & B_{k,j}(x-m,y+m)\frac{d^{p-k}}{dx^{p-k}}(x-m)_m\frac{d^{q-j}}{dy^{q-j}}\frac{1}{(y)_m} \\ & -\sum_{k=0}^{p}\begin{pmatrix} p \\ k\end{pmatrix}\sum_{j=0}^{q}\begin{pmatrix} q \\ j\end{pmatrix}\cdot \\ & N-\lim_{\varepsilon\to 0+0}\sum_{l=1}^{m}\varepsilon^{y-1+l}(1-\varepsilon)^{x-l}\cdot \\ & \ln^k(1-\varepsilon)\ln^j\varepsilon\frac{d^{p-k}}{dx^{p-k}}(x+1-l)_{l-1}\frac{d^{q-j}}{dy^{q-j}}\frac{1}{(y)_l} \\ & +\sum_{k=0}^{p}\begin{pmatrix} p \\ k\end{pmatrix}\sum_{j=0}^{q}\begin{pmatrix} q \\ j\end{pmatrix}\cdot \\ & N-\lim_{\varepsilon\to 0+0}\sum_{l=1}^{m}\varepsilon^{x-l}(1-\varepsilon)^{y-1+l}\cdot \\ & \ln^k\varepsilon\ln^j(1-\varepsilon)\frac{d^p}{dx^p}(x+1-l)_{l-1}\frac{d^{q-j}}{dy^{q-j}}\frac{1}{(y)_l}. \end{aligned} \tag{28}$$

From (19), we have

$$\begin{aligned} & N-\lim_{\varepsilon\to 0+0}\varepsilon^{-m-1+l}(1-\varepsilon)^{x-l}\ln^k(1-\varepsilon)\ln^j\varepsilon \\ & = k!\sum_{i=k}^{\infty}\frac{(-1)^i s(i,k)}{i!}\sum_{u=0}^{\infty}\frac{(l-x)_u}{u!}\cdot \\ & N-\lim_{\varepsilon\to 0+0}\varepsilon^{u+i+l-m-1}\ln^j\varepsilon \\ & = \begin{cases} k!\sum_{i=k}^{m+1-l}\frac{(-1)^i s(i,k)(l-x)_{m+1-l-i}}{i!(m+1-l-i)!}, & j=0, \\ 0, & j>0, \end{cases} \\ & N-\lim_{\varepsilon\to 0+0}\varepsilon^{x-l}(1-\varepsilon)^{-m-1+l}\ln^k\varepsilon\ln^j(1-\varepsilon)=0, \end{aligned} \tag{29}$$

and

$$\begin{aligned} & B_{k,j}(x,0)=N-\lim_{\varepsilon\to 0+0}\int_{\varepsilon}^{1-\varepsilon}\frac{t^{x-1}\ln^k t\ln^j(1-t)}{(1-t)^{-1}}dt \\ & = \frac{1}{j+1}N-\lim_{\varepsilon\to 0+0} \\ & \left(\begin{array}{c}\varepsilon^{x-1}\ln^k\varepsilon\ln^{j+1}(1-\varepsilon) \\ -(1-\varepsilon)^{x-1}\ln^k(1-\varepsilon)\ln^{j+1}\varepsilon \\ +(x-1)\int_{\varepsilon}^{1-\varepsilon}t^{x-2}\ln^k t\ln^{j+1}(1-t)dt \\ +k\int_{\varepsilon}^{1-\varepsilon}t^{x-2}\ln^{k-1}t\ln^{j+1}(1-t)dt\end{array}\right) \\ & = \frac{(x-1)B_{k,j+1}(x-1,1)+kB_{k-1,j+1}(x-1,1)}{j+1}. \end{aligned} \tag{30}$$

Combining (28), (29), (30) and (19), we can obtain (27).

Remark From (6), (7) and (8), we can get all results in [3]. For example,

$$B(n+\tfrac{1}{2},-n)=\frac{(-1)^{n-1}}{n!}\left(\tfrac{1}{2}\right)_n\left(\tfrac{1}{2}B_{0,1}(\tfrac{1}{2},1)+H_n\right). \tag{31}$$

Substituting

$$\begin{aligned} & B_{0,1}(-\tfrac{1}{2},1)=B(-\tfrac{1}{2},1)H_1^{\psi}(1,\tfrac{1}{2})=-2H_1^{\psi}(1,\tfrac{1}{2}), \\ & H_1^{\psi}(1,\tfrac{1}{2})=H_{1,0}^{\psi}(1,\tfrac{1}{2})=\psi(1)-\psi(\tfrac{1}{2}) \\ & = -\gamma-(-\gamma-2\ln 2)=2\ln 2 \end{aligned}$$

to (31), we obtain Theorem 3 in [10]. Let $x=\frac{1}{2}\pm n$, $y=r+1$ and $q=1$ in (7), respectively. Then we obtain

$$\begin{aligned} & B_{0,1}(\tfrac{1}{2}\pm n,r+1) \\ & = B(\tfrac{1}{2}\pm n,r+1)H_1^{\psi}(r+1,\tfrac{3}{2}\pm n+r) \\ & = \frac{r!}{\left(\frac{1}{2}\pm n\right)_{r+1}}H_1^{\psi}(r+1,\tfrac{1}{2}\pm n+r+1), \end{aligned} \tag{32}$$

and

$$\begin{aligned} & H_1^{\psi}(r+1,\tfrac{3}{2}\pm n+r)=H_{1,0}^{\psi}(r+1,\tfrac{3}{2}\pm n+r) \\ & = \psi(r+1)-\psi(\tfrac{1}{2}\pm(n\pm(r+1))) \\ & = 2\ln 2+H_r-\left\{\begin{array}{c}2H_{2n+2r+1}-H_{n+r} \\ 2H_{2n-2r-3}-H_{n-r-2},\end{array}\right\}, \end{aligned} \tag{33}$$

so

$$B_{0,1}(\tfrac{1}{2}+n,r+1)=\frac{r!(2\ln 2+H_r-2H_{2n+2r+1}+H_{n+r})}{\left(\frac{1}{2}+n\right)_{r+1}}, \tag{34}$$

and

$$B_{0,1}(\tfrac{1}{2}-n,r+1)=\frac{r!(2\ln 2+H_r-2H_{2n-2r-3}+H_{n-r-2})}{\left(\frac{1}{2}-n\right)_{r+1}},\quad n\geq r+1 \tag{35}$$

We notice that the formulas (34) and (35) contain Theorem 4 in [10]. Furthermore, it can be shown that all Corollaries in [4] have been simplified here.

For $B_{p,q}(n,-m)(n=1,2,\cdots,m+1,m\geq 0)$ and $B_{p,q}(-n,-m)(n,m\geq 0)$, the condition of Theorem 3.2 is not satisfied. Therefore, we give the following Theorem.

Theorem 3.3 1)

$$B_{p,q}(n,-m)=\sum_{l=0}^{m-n+2}\binom{m-n+2}{l}\cdot B_{p,q}(m+2-l,l-m) \tag{36}$$

for the integer $m\geq 0, n=1,2,\cdots,m+1$.

2)

$$\begin{aligned}B_{p,q}(-n,-m)=&\sum_{j=0}^{n}\binom{m+j}{j}\sum_{l=0}^{n-j+1}\binom{n-j+1}{l}\cdot B_{q,p}(2+n-j-l,l+j-n)\\&+\sum_{i=0}^{m}\binom{n+i}{i}\sum_{l=0}^{m-i+1}\binom{m-i+1}{l}\cdot B_{p,q}(2+m-i-l,l+i-m)\end{aligned} \tag{37}$$

for the integer $m\geq 0, n=1,2,\cdots,m+1$.

Proof. 1) By the following relation

$$B_{p,q}(n,-m)=\sum_{l=0}^{k}\binom{k}{l}B_{p,q}(n+k-l,l-m),$$

we have

$$B_{p,q}(n,-m)=\sum_{l=0}^{m-n+2}\binom{m-n+2}{l}\cdot B_{p,q}(m+2-l,l-m).$$

2) By (36) and (19), we have

$$B_{p,q}(1,j-m)=\sum_{l=0}^{m-j+1}\binom{m-j+1}{l}\cdot B_{p,q}(m-j+2-l,l+j-m),$$

$$\begin{aligned}B_{p,q}(-n,-m)=&\sum_{j=0}^{n}\binom{m+j}{j}B_{p,q}(j-n,1)+\sum_{i=0}^{m}\binom{n+i}{i}B_{p,q}(1,i-m)\\=&\sum_{j=0}^{n}\binom{m+j}{j}\sum_{l=0}^{n-j+1}\binom{n-j+1}{l}\cdot B_{q,p}(2+n-j-l,l-n+j)\\&+\sum_{i=0}^{m}\binom{n+i}{i}\sum_{l=0}^{m-i+1}\binom{m-i+1}{l}\cdot B_{p,q}(2+m-i-l,l-m+i).\end{aligned}$$

Example 3.1 It follows from (2) that we can obtain the following closed forms

$$\begin{aligned}&\int_0^1\frac{\ln^2 t\ln^4(1-t)}{t^3(1-t)^2}dt=B_{2,4}(-2,-1)\\&=8\pi^2+\frac{13\pi^3}{10}-54\zeta(3)+18\pi^2\zeta(3)-\frac{18\pi^4\zeta(3)}{5}\\&\quad-204\zeta^2(3)-72\pi^2\zeta(5)+1152\zeta(7)\\&=29.4677632578940455122127294733195227\cdots,\end{aligned} \tag{38}$$

and

$$\begin{aligned}&\int_0^1\frac{\ln^3 t\ln^2(1-t)}{t\sqrt{t}(1-t)^3}dt=B_{3,2}(-\tfrac{1}{2},-2)\\&=\frac{107\pi^2}{4}+\frac{151\pi^4}{16}-\frac{65\pi^6}{32}+87\pi^2\ln 2+\frac{71\pi^4}{4}\ln 2\\&\quad-384\ln^2 2-6\pi^2\ln^2 2-\frac{15\pi^4}{2}\ln^2 2-714\zeta(3)+\\&\quad\frac{1883\pi^2}{8}\zeta(3)-840\zeta(3)\ln 2-\frac{315\pi^2}{2}\zeta(3)\ln 2\\&\quad-336\zeta(3)\ln^2 2+\frac{2205\zeta^2(3)}{2}-\frac{7347\zeta(5)}{2}+2790\zeta(5)\ln 2\\&=-5.58205562690478141496012318428821 39\cdots.\end{aligned} \tag{39}$$

Example 3.2 From [8], we can obtain

$$\begin{aligned}&(-1)^{(p+q)}z^y(1+z)^x\int_0^1\frac{t^{x-1}(1-t)^{y-1}}{(t+z)^{x+y}}\ln^p\frac{t}{t+z}\ln^q\frac{1-t}{t+z}dt\\&=\sum_{j=0}^{p}(-1)^jC_p^j\ln^{p-j}(1+z)\sum_{k=0}^{q}(-1)^k\cdot C_q^k\ln^{q-k}zB_{j,k}(x,y),\end{aligned} \tag{40}$$

where p and q are non-negative integers, x and y are complex numbers satisfying $Rex>0, Rey>0$ and $Re(x+y)<1$, z is a complex number with $z\neq -1,0$.

By (40), we have

$$\begin{aligned}&\int_0^1\frac{(1-t)^2}{t^2(2t+1)^2}\ln^2\frac{2t}{2t+1}\ln^2\frac{2-2t}{2t+1}dt=\frac{512\sqrt{2}}{12762815625\sqrt{3}}\cdot\\&\left(\begin{array}{c}25363225696-40516875\pi^4\\-17150\pi^2(71918-38340\ln 2)\\-257250\pi^2\ln\frac{3}{2}(1451+105\ln 3-1995\ln 2)\\+10080\ln 2(-1644827+181965\ln 2)\\+7350\ln^2\frac{3}{2}(75713+315\ln 2(-389+315\ln 2))\\+2520\ln\frac{3}{2}(792151+35\ln(-79283+22365\ln 2))\\+10804500(-673+840\ln 2-210\ln 3)\end{array}\right)\\&=2.78386839392356590317908504866491626196489\cdots\end{aligned} \tag{41}$$

IV. Conclusion

In this paper, the partial derivatives $B_{p,q}(x,y)$ of the Beta function $B(x,y)$ are expressed in terms of a finite number of the Polygamma functions $\psi^{(k)}(x),\psi^{(k)}(y)$ and $\psi^{(k)}(x+y)$, where p,q and k are positive integers, x and y are complex numbers. Moreover, the evaluation of many integrals associated with $B(x,y)$ and $B_{p,q}(x,y)$ are transformed into the calculation of $B(x,y)$ and $B_{p,q}(x,y)$. For the numerical calculations, the algorithm given in this paper not only improves the accuracy but also accelerates the speed of calculation. In Mathematica, $\frac{T_{NIntegrate}}{T_{B_{p,q}}}$ and P^2 are proportional, where P denotes specified precision, $T_{NIntegrate}$ denotes time-consuming of the numerical integration (NIntegrate) in Mathematica as $T_{B_{p,q}}$ denotes time-consuming of the algorithm given in this paper. For the symbolic computation in Mathematica, the algorithm given in this paper has obvious advantages. In Mathematica(Maple), $\frac{T_{Integrate}}{T_{B_{p,q}}}$ will be increased from tens to hundreds, thousands or even tens of thousands with the increase of p or q. Although integral is not the closed form sometimes, such as Example 3.2, the algorithm given in this paper can still get the closed form. In short, for the integrals associated with $B(x, y)$ and $B_{p,q}(x, y)$, the algorithm given in this paper has obvious advantage in the numerical or symbolic computation.

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