General Parametrization of Stabilizing Controllers with Doubly Coprime Factorizations over Commutative Rings

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***Abstract*—In this paper, we consider the factorization approach to control systems with plants admitting coprime factorizations. Further, the set of stable causal transfer functions is a general commutative ring. The objective of this paper is to present that even in the case where the set of stable causal transfer functions is a general commutative ring, the parametrization of stabilizing controllers is achieved by the Youla-parametrization.**

***Index Terms*—Linear systems, Feedback stabilization, Coprime factorization over commutative rings Parametrization of stabilizinng controllers**

I. Introduction

In the factorization approach[1], [2], [3], [4], a transfer function is given as the ratio of *two stable causal transfer functions* and the set of stable causal transfer functions forms a *commutative ring*.

Since stabilizing controllers are not unique in general, the choice of stabilizing controllers is important for the resulting closed loop. In the classical case such as continuous-time LTI systems and discrete-time LTI systems, the stabilizing controllers can be parametrized by the method called “Youla-parametrization”[1], [2], [4], [5], [6] (also called Youla-Kučera-parametrization). We note that in these cases, the set of stable causal transfer functions over an integral domain such as a Euclidean domain and a unique factorization domain.

However, there exist models in which some stabilizable transfer matrices do not have their right-/left-coprime factorizations in general[7], [8]. In such models, we cannot employ the Youla-parametrization in general.

It has also investigated the parametrization of the case where plants admits either right- or left-coprime factorizations[9].

In this paper, the commutative ring, the set of stable causal transfer functions, includes a commutitative ring with zero divisors, that is, we consider *general* commutative rings.

The contribution of this paper is to present that in the factorization approach, if a plant admits both right-/left-coprime factorizations (even if some other stabilizable plants in the same model do not have right-/left-coprime factorizations), we can still employ the Youla-parametrization for the parametrization of stabilizing controllers of the plant.

II. Preliminaries

In the following we begin by introducing notations used in this paper. Then we give the formulation of the feedback stabilization problem.

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A. Notations

a) Commutative Rings: We will consider that *the set of all stable causal transfer functions* is a commutative ring, denoted by $\mathcal{A}$. Again, we are considering that $\mathcal{A}$ may include zero divisors. The total ring of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \,|\, n, d \in \mathcal{A},\ d$ is a nonzerodivisor$\}$. This will be considered to be *the set of all possible transfer functions*. If the commutative ring $\mathcal{A}$ is an integral domain, $\mathcal{F}$ becomes a field of fractions of $\mathcal{A}$. However, if $\mathcal{A}$ is not an integral domain, then $\mathcal{F}$ is not a field, because any nonzero zerodivisor of $\mathcal{F}$ is not a unit.

b) Matrices: Suppose that x and y denote sizes of matrices.

The set of matrices over $\mathcal{A}$ of size $x \times y$ is denoted by $\mathcal{A}^{x\times y}$. A square matrix is called *singular* over $\mathcal{A}$ if its determinant is a zerodivisor of $\mathcal{A}$, and *nonsingular* otherwise. The identity and the zero matrices are denoted by I_x and $O_{x\times y}$, respectively, if the sizes are required, otherwise they are denoted simply by I and O.

Matrices A and B over $\mathcal{A}$ are *right-coprime over* $\mathcal{A}$ if there exist matrices $\widetilde{X}$ and $\widetilde{Y}$ over $\mathcal{A}$ such that $\widetilde{X}A+\widetilde{Y}B = I$. Analogously, matrices $\widetilde{A}$ and $\widetilde{B}$ over $\mathcal{A}$ are *left-coprime over* $\mathcal{A}$ if there exist matrices X and Y over $\mathcal{A}$ such that $\widetilde{A}X + \widetilde{B}Y = I$. Further, pair (N, D) of matrices N and D over $\mathcal{A}$ is said to be a *right-coprime factorization of P over $\mathcal{A}$* if (i) the matrix D is nonsingular over $\mathcal{A}$, (ii) $P = ND^{-1}$ over $\mathcal{F}$, and (iii) N and D are right-coprime over $\mathcal{A}$. Also, pair $(\widetilde{N}, \widetilde{D})$ of matrices $\widetilde{N}$ and $\widetilde{D}$ over $\mathcal{A}$ is said to be a *left-coprime factorization of P over $\mathcal{A}$* if (i) $\widetilde{D}$ is nonsingular over $\mathcal{A}$, (ii) $P = \widetilde{D}^{-1}\widetilde{N}$ over $\mathcal{F}$, and (iii) $\widetilde{N}$ and $\widetilde{D}$ are left-coprime over $\mathcal{A}$. As we have seen, in the case where a matrix is potentially used to express *left* fractional form and/or *left* coprimeness, we usually attach a tilde ‘$\widetilde{\ }$’ to a symbol; for example $\widetilde{N}$, $\widetilde{D}$ for $P = \widetilde{D}^{-1}\widetilde{N}$ and $\widetilde{Y}$, $\widetilde{X}$ for $\widetilde{Y}N+\widetilde{X}D = I$.

c) Causality: We also define the causality of transfer functions, which is an important physical constraint, used in this paper. We employ the definition of causality from Vidyasagar *et al.*[4, Definition 3.1] and Mori and Abe[10].

Definition 1: Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$, with $\mathcal{Z} \neq \mathcal{A}$, including all zerodivisors. Define the subsets $\mathcal{P}$ and $\mathcal{P}_{\rm S}$ of $\mathcal{F}$ as follows:

$$\begin{aligned} \mathcal{P} &= \{n/d \in \mathcal{F} \,|\, n \in \mathcal{A},\ d \in \mathcal{A}\backslash\mathcal{Z}\}, \\ \mathcal{P}_{\rm S} &= \{n/d \in \mathcal{F} \,|\, n \in \mathcal{Z},\ d \in \mathcal{A}\backslash\mathcal{Z}\}. \end{aligned}$$

A transfer function is called *causal* (*strictly causal*) if it is in $\mathcal{P}$ ($\mathcal{P}_{\rm S}$). Similarly, a transfer matrix over $\mathcal{F}$ is called *causal* (*strictly causal*) if all entries of the matrix in $\mathcal{P}$ ($\mathcal{P}_{\rm S}$).

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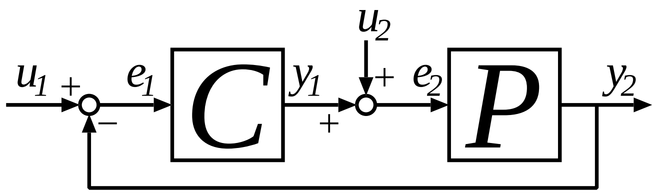

Fig. 1. Feedback system Σ.

■

B. Feedback Stabilization Problem

The stabilization problem considered in this paper follows that of Sule in [11] and Mori and Abe in [10] who consider the feedback system Σ [3, Ch.5, Figure 5.1] as in Figure 1. For further details the reader is referred to [3], [10]. Throughout this paper, the plant we consider has m inputs and n outputs, and its transfer matrix, which itself is also called simply a *plant*, is denoted by P and belongs to $\mathcal{P}^{n\times m}$.

Definition 2: Define $\widehat{F}_{\rm ad}$ by

$$\widehat{F}_{\rm ad} = \{(X,Y)\in\mathcal{F}^{x\times y}\times\mathcal{F}^{y\times x}\mid \det(I_x+XY)\text{ is a unit of }\mathcal{F},\ x\text{ and }y\text{ are positive integers}\}.$$

For $P\in\mathcal{F}^{n\times m}$ and $C\in\mathcal{F}^{m\times n}$, the matrix $H(P,C)\in\mathcal{F}^{(m+n)\times(m+n)}$ is defined by

$$H(P,C)=\begin{bmatrix}(I_n+PC)^{-1} & -P(I_m+CP)^{-1}\\ C(I_n+PC)^{-1} & (I_m+CP)^{-1}\end{bmatrix} \quad (1)$$

provided $(P,C)\in\widehat{F}_{\rm ad}$. This $H(P,C)$ is the transfer matrix from $[\,u_1^t \quad u_2^t\,]^t$ to $[\,e_1^t \quad e_2^t\,]^t$ of the feedback system Σ. If (i) $(P,C)\in\widehat{F}_{\rm ad}$ and (ii) $H(P,C)\in\mathcal{A}^{(m+n)\times(m+n)}$, then we say that the plant P is *stabilizable*, C *stabilizes* P, P is *stabilized* by C, and C is a *stabilizing controller* of P. ■

In (1), there are two kind of inverse matrices, $(I_n+PC)^{-1}$, $(I_m+CP)^{-1}$. We can make them describe with only one inverse matrix as follows:

$$\begin{aligned}H(P,C)&=\begin{bmatrix}I-P(I_m+CP)^{-1}C & -P(I_m+CP)^{-1}\\ (I_m+CP)^{-1}C & (I_m+CP)^{-1}\end{bmatrix}\\ &=\begin{bmatrix}(I_n+PC)^{-1} & -(I_n+PC)^{-1}P\\ C(I_n+PC)^{-1} & I_m-C(I_n+PC)^{-1}P\end{bmatrix}.\end{aligned}$$

It is known that $W(P,C)$ defined below is over $\mathcal{A}$ if and only if $H(P,C)$ is over $\mathcal{A}$:

$$W(P,C):=\begin{bmatrix}C(I_n+PC)^{-1} & -CP(I_m+CP)^{-1}\\ PC(I_n+PC)^{-1} & P(I_m+CP)^{-1}\end{bmatrix}. \quad (2)$$

This $W(P,C)$ is the transfer matrix from u_1 and u_2 to y_1 and y_2. Then, we have

$$H(P,C)=I_{m+n}-FW(P,C),$$

where

$$F=\begin{bmatrix}O & I_n\\ -I_m & O\end{bmatrix}.$$

The matrix F is unimodular; in fact,

$$F^{-1}=\begin{bmatrix}O & -I_m\\ I_n & O\end{bmatrix},$$

which is over $\mathcal{A}$. Thus, $W(P,C)$ can be expressed in terms of F and $H(P,C)$:

$$W(P,C)=F^{-1}(I_{m+n}-H(P,C)).$$

Note 1: Instead of Definition 2, we can describe the definition of the stabilify by using $W(P,C)$ rather than $H(P,C)$ as follows: For $P\in\mathcal{F}^{n\times m}$ and $C\in\mathcal{F}^{m\times n}$, the matrix $W(P,C)\in\mathcal{F}^{(m+n)\times(m+n)}$ is as in (2) provided $(P,C)\in\widehat{F}_{\rm ad}$. If (i) $(P,C)\in\widehat{F}_{\rm ad}$ and (ii) $W(P,C)\in\mathcal{A}^{(m+n)\times(m+n)}$, then we say that the plant P is *stabilizable*, C *stabilizes* P, P is *stabilized* by C, and C is a *stabilizing controller* of P.

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It should be noted that when using "a stabilizing controller," we do not guarantee the causality. However, in the classical case of the factorization approach, once we restrict ourselves to strictly proper plants, it is known that any stabilizing controller of strictly causal plant is causal (cf. Corollary 5.2.20 of [3], Theorem 4.1 of [4], and Proposition 6.2 of [10]). One can see, in fact, that many practical systems are strictly causal.

III. Parametrization without Coprime Factorizability

Here we review the parametrization method without considering the coprime factorizability[12], [13]. Let $\mathcal{H}$ be the set of $H(P,C)$'s with all stabilizing controllers C of the plant P. This set $\mathcal{H}$ and all stabilizing controllers are obtained as in the following way.

Let H_0 be $H(P,C_0)$, where C_0 is a stabilizing controller of P. Let $\Omega(Q)$ be a matrix defined as follows:

$$\begin{aligned}\Omega(Q):=(H_0-\begin{bmatrix}I_n & O\\ O & O\end{bmatrix})Q \quad (3)\\ \times(H_0-\begin{bmatrix}O & O\\ O & I_m\end{bmatrix})+H_0\end{aligned}$$

with a stable causal and square matrix Q of size $(m+n)\times(m+n)$. Using this matrix Q, we have the following theorem, the controller parametrization, as follows.

Theorem 1 ([12], [13]): The set of all $H(P,C)$'s with all stabilizing controllers is given as follows

$$\mathcal{H}=\{\Omega(Q)\,|\,Q\text{ is stable causal and }\Omega(Q)\text{ is nonsingular}\} \quad (4)$$

Furthermore, any stabilizing controller has the following form:

$$-\,[\,O \quad I_m\,]\,\Omega(Q)^{-1}\begin{bmatrix}I_n\\ O\end{bmatrix}, \quad (5)$$

provided that $\Omega(Q)$ is nonsingular. ■

The parametrization above is given by a parameter matrix Q *without the coprime factorizability* of the plant. This parametrization method is applicable to the stabilizable plant with no coprime factorization (of course, any plant which admits coprime factorization can also applied to).

The parameter matrix Q is of size $(m+n)\times(m+n)$. That is, in order to archive the parametrization, we need $(m+n)^2$ parameters. On the other hand, the Youla-parametrization needs only mn parameters.

IV. Main Result

The following is the parametrization of stabilizing controllers presented as a Youla-parametrization.

Theorem 2: (cf. Theorems 5.2.1 and 8.3.12 of [3]) Suppose that the plant $P \in \mathcal{P}^{n\times m}$ is stabilizable. Suppose further that P admits right-/left-coprime factorizations over $\mathcal{A}$ of P. Let (N, D) and $(\widetilde{N}, \widetilde{D})$ be right-/left-coprime factorizations, respectively, over $\mathcal{A}$ of P and (Y_0, X_0) and $(\widetilde{Y}_0, \widetilde{X}_0)$ be right-/left-coprime factorizations, respectively, over $\mathcal{A}$ of C_0, a stabilizing controller of P, such that

$$\widetilde{Y}_0 N + \widetilde{X}_0 D = I_m, \quad \widetilde{N} Y_0 + \widetilde{D} X_0 = I_n.$$

Then all matrices X, Y, $\widetilde{X}$, $\widetilde{Y}$ over $\mathcal{A}$ satisfying

$$\widetilde{Y} N + \widetilde{X} D = I_m, \quad \widetilde{N} Y + \widetilde{D} X = I_n$$

are expressed as $X = X_0 - NS$, $Y = Y_0 + DS$, $\widetilde{X} = \widetilde{X}_0 - R\widetilde{N}$ and $\widetilde{Y} = \widetilde{Y}_0 + R\widetilde{D}$ for R and S in $\mathcal{A}^{m\times n}$.

Further the set of all stabilizing controllers, denoted by $\mathcal{S}(P)$, is given as

$$\begin{aligned}\mathcal{S}(P) &= \{(\widetilde{X}_0 - R\widetilde{N})^{-1}(\widetilde{Y}_0 + R\widetilde{D}) \mid \\ &\qquad R \in \mathcal{A}^{m\times n},\ \widetilde{X}_0 - R\widetilde{N} \text{ is nonsingular}\} \\ &= \{(Y_0 + DS)(X_0 - NS)^{-1} \mid \\ &\qquad S \in \mathcal{A}^{m\times n},\ X_0 - NS \text{ is nonsingular}\}.\end{aligned}$$

▪

The "integral domain version" of Theorem 2 was already shown in Section 8 of [3] without the proof. Nevertheless, considering general commutative rings as the set of stable causal transfer functions, we need to give the proof because the proofs have some differences.

To make this paper as self-contained as we can, we inlucde some lemmas which has even appeared in some literatures already.

Lemma 1: (cf. 8.3.12 of [3]) Suppose that $P \in \mathcal{F}^{n\times m}$ admits a right- (left-) coprime factorization and $C \in \mathcal{F}^{m\times n}$ is any stabilizing controller of P. Then the stabilizing controller C admits a left- (right-) coprime factorization.

Proof: Suppose that P admits a right-coprime factorization. Let (N, D) be a right-coprime factorization over $\mathcal{A}$ of P. Then exist $\widetilde{Y}$ and $\widetilde{X}$ over $\mathcal{A}$ such that $\widetilde{Y}N + \widetilde{X}D = I_m$.

Let C be a stabilizing controller of P. Then $H(P, C)$ is as follows:

$$\begin{aligned}& H(P,C) \\ = \ & H(ND^{-1}, C) \\ = \ & \begin{bmatrix} (I_n + ND^{-1}C)^{-1} & -ND^{-1}(I_m + CND^{-1})^{-1} \\ C(I_n + ND^{-1}C)^{-1} & (I_m + CND^{-1})^{-1}\end{bmatrix} \\ = \ & \begin{bmatrix} I_n - N(D+CN)^{-1}C & -N(D+CN)^{-1} \\ D(D+CN)^{-1}C & D(D+CN)^{-1}\end{bmatrix}. \quad (6)\end{aligned}$$

The matrix above is over $\mathcal{A}$ because C is a stabilizing controller. Now let S and T be

$$S = (D + CN)^{-1}, \quad T = (D + CN)^{-1}C,$$

respectively. Then, S holds

$$\begin{aligned} S &= (D + CN)^{-1} \\ &= \widetilde{Y}N(D + CN)^{-1} + \widetilde{X}D(D + CN)^{-1}.\end{aligned}$$

In the term $\widetilde{Y}N(D + CN)^{-1}$, $\widetilde{Y}$ is over $\mathcal{A}$, and $N(D + CN)^{-1}$ is the negative of the (1,2)-block of (6). Further, in the term $\widetilde{X}D(D+CN)^{-1}$, $\widetilde{X}$ is over $\mathcal{A}$, and $D(D+CN)^{-1}$ is the (2,2)-block of (6). Hence S is $\mathcal{A}$.

Analogously we show that T is of $\mathcal{A}$. We now have

$$\begin{aligned} T &= (D + CN)^{-1}C \\ &= \widetilde{Y}N(D + CN)^{-1}C + \widetilde{X}D(D + CN)^{-1}C.\end{aligned}$$

In the term $\widetilde{Y}N(D + CN)^{-1}C$, $\widetilde{Y}$ is over $\mathcal{A}$, and $N(D + CN)^{-1}C$ is a part of the (1,1)-block of (6). Further, in the term $\widetilde{X}D(D+CN)^{-1}C$, $\widetilde{X}$ is over $\mathcal{A}$, and $D(D+CN)^{-1}C$ is the (2,1)-block of (6). Hence T is also $\mathcal{A}$.

In the following, we show that (T, S) is a left-coprime factorization over $\mathcal{A}$ of C. It is obvious $C = ((D + CN)^{-1})^{-1}(D + CN)^{-1}C = S^{-1}T$. Now consider $SD + TN$, which is

$$\begin{aligned} SD + TN &= (D + CN)^{-1}D + (D + CN)^{-1}CN \\ &= (D + CN)^{-1}(D + CN) = I_m.\end{aligned}$$

Hence (T, S) is a left-coprime factorization over $\mathcal{A}$ of C.

The discussion of the case where P admits a left-coprime factorization over $\mathcal{A}$ can be achieved entirely analogously. ■

Theorem 3: (cf. Theorem 4.1.60 of [3]) Suppose that $P \in \mathcal{F}^{n\times m}$ and let (N, D) and $(\widetilde{N}, \widetilde{D})$ be a right- and a left-coprime factorizations, respectively, over $\mathcal{A}$. Suppose that matrices $\widetilde{Y}$ and $\widetilde{X}$ over $\mathcal{A}$ satisfy $\widetilde{Y}N + \widetilde{X}N = I_m$.

Then there exist matrices Y and X over $\mathcal{A}$ such that

$$\begin{bmatrix} \widetilde{X} & \widetilde{Y} \\ -\widetilde{N} & \widetilde{D}\end{bmatrix}\begin{bmatrix} D & -Y \\ N & X\end{bmatrix} = I_{m+n}. \quad (7)$$

Proof: Because $\widetilde{N}$ and $\widetilde{D}$ are left-coprime factorization over $\mathcal{A}$ of P, there exist matrices Y_1 and X_1 over $\mathcal{A}$ such that $\widetilde{N}Y_1 + \widetilde{D}X_1 = I_n$. Define

$$E = \begin{bmatrix} \widetilde{X} & \widetilde{Y} \\ -\widetilde{N} & \widetilde{D}\end{bmatrix}.$$

Then

$$E\begin{bmatrix} D & -Y_1 \\ N & X_1\end{bmatrix} = \begin{bmatrix} I_m & \Delta \\ O & I_n\end{bmatrix}, \quad (8)$$

where $\Delta = -\widetilde{X}Y_1 + \widetilde{Y}X_1 \in \mathcal{A}^{m\times n}$. Since the right hand side of (8) is unimodular, so is E, whose inverse is

$$\begin{aligned} E^{-1} &= \begin{bmatrix} D & -Y_1 \\ N & X_1\end{bmatrix}\begin{bmatrix} I_m & \Delta \\ O & I_n\end{bmatrix}^{-1} \\ &= \begin{bmatrix} D & -Y_1 \\ N & X_1\end{bmatrix}\begin{bmatrix} I_m & -\Delta \\ O & I_n\end{bmatrix} \\ &= \begin{bmatrix} D & -(Y_1 + D\Delta) \\ N & X_1 - N\Delta\end{bmatrix}.\end{aligned}$$

Then (7) is satisfied with $Y = Y_1 + D\Delta$ and $X = X_1 - N\Delta$. ■

The following corollary is the parallel result of Theorem 3.

Corollary 1: Suppose that $P \in \mathcal{F}^{n\times m}$ and let (N, D) and $(\widetilde{N}, \widetilde{D})$ be a right- and a left-coprime factorizations, respectively, over $\mathcal{A}$. Suppose that matrices Y and X over $\mathcal{A}$ satisfy $\widetilde{N}Y + \widetilde{D}X = I_n$.

Then there exist matrices $\widetilde{X}$ and $\widetilde{Y}$ over $\mathcal{A}$ such that (7) holds. ▪

Theorem 4: (cf. Corollary 4.1.67 of [3]) Suppose that $P \in \mathcal{F}^{n\times m}$ and (N, D) and $(\widetilde{N}, \widetilde{D})$ be a right- and a left-coprime factorizations, respectively, over $\mathcal{A}$ of P. Then for every matrices $\widetilde{Y}$, $\widetilde{X}$, Y, and X over $\mathcal{A}$ such that $\widetilde{Y}N+\widetilde{X}D=I_m$, $\widetilde{N}Y+\widetilde{D}X=I_n$, the matrices

$$U_1=\begin{bmatrix}\widetilde{X} & \widetilde{Y}\\ -\widetilde{N} & \widetilde{D}\end{bmatrix},\quad U_2=\begin{bmatrix}D & -Y\\ N & X\end{bmatrix} \tag{9}$$

are unimodular. Moreover U_1^{-1} is a complementation of $[\,D^t \quad N^t\,]^t$, in that U_1^{-1} is of the form

$$U_1^{-1}=\begin{bmatrix}D & \\ & G\\ N & \end{bmatrix} \tag{10}$$

for some matrix G over $\mathcal{A}$. Analogously, U_2^{-1} is a complementation of $[\,-\widetilde{N} \quad \widetilde{D}\,]$, in that U_2^{-1} is of the form

$$U_2^{-1}=\begin{bmatrix} & F & \\ -\widetilde{N} & & \widetilde{D}\end{bmatrix} \tag{11}$$

for some matrix F over $\mathcal{A}$.

Proof: Using the right-coprimeness of (N,D) over over $\mathcal{A}$, consider matrices $\widetilde{Y}$ and $\widetilde{X}$ such that $\widetilde{Y}N+\widetilde{X}D=I_m$. Then by Theorem 3, there exist matrices X and Y over $\mathcal{A}$ such that

$$\begin{bmatrix}\widetilde{X} & \widetilde{Y}\\ -\widetilde{N} & \widetilde{D}\end{bmatrix}\begin{bmatrix}D & -Y\\ N & X\end{bmatrix}=I_{m+n}. \tag{12}$$

Let U_1 and U_2 as in (9). Then by (12), they are unimodular. Applying $G=[-Y^t \quad X^t]^t$, we have (10). Analogously, applying $F=[\,\widetilde{N} \quad \widetilde{Y}\,]$ we have (11). ■

Lemma 2: (cf. Lemma 3.1 of [4]) Suppose that $P\in\mathcal{F}^{n\times m}$ and $C\in\mathcal{F}^{m\times n}$ and let (N,D) be a right-coprime factorization over $\mathcal{A}$ of P and $(\widetilde{Y},\widetilde{X})$ a left-coprime factorization over $\mathcal{A}$ of C. Under these conditions, C is a stabilizing controller of P if and on ly if

$$\Delta_1=\widetilde{Y}N+\widetilde{X}D$$

is a unimodular of $\mathcal{A}$.

Proof: "If": Suppose that Δ_1 is a unimodular. Then Δ_1^{-1} is over $\mathcal{A}$. First, since $I_m+CP=\widetilde{X}^{-1}\Delta_1 D^{-1}$, we see that $\det(I_m+CP)=\det(I_n+PC)$ is a nonzerodivisor. Next, we show $H(P,C)$ is over $\mathcal{A}$. By direct substitution in $H(P,C)$, we have

$$\begin{aligned}
&H(P,C)\\
=&\begin{bmatrix}I_n-P(I_m+CP)^{-1}C & -P(I_m+CP)^{-1}\\ (I_m+CP)^{-1}C & (I_m+CP)^{-1}\end{bmatrix}\\
=&\begin{bmatrix}I_n-PD\Delta_1^{-1}\widetilde{X}C & -PD\Delta_1^{-1}\widetilde{X}\\ D\Delta_1^{-1}\widetilde{X}C & D\Delta_1^{-1}\widetilde{X}\end{bmatrix}\\
=&\begin{bmatrix}I_n-N\Delta_1^{-1}\widetilde{Y} & -N\Delta_1^{-1}\widetilde{X}\\ D\Delta_1^{-1}\widetilde{Y} & D\Delta_1^{-1}\widetilde{X}\end{bmatrix}.
\end{aligned} \tag{13}$$

Because Δ_1^{-1} is unimodular, this $H(P,C)$ is over $\mathcal{A}$.

"Only If": Suppose that C is a stabilizing controller of P. Then $\det(I_m+CP)\neq 0$ and $H(P,C)$ is over $\mathcal{A}$.

Because $H(P,C)$ is over $\mathcal{A}$. the following matrix, which is modified from (13), is also over $\mathcal{A}$.

$$\begin{bmatrix}N\Delta_1^{-1}\widetilde{Y} & N\Delta_1^{-1}\widetilde{X}\\ D\Delta_1^{-1}\widetilde{Y} & D\Delta_1^{-1}\widetilde{X}\end{bmatrix}. \tag{14}$$

Recall that (N,D) is a right-coprime factorization over $\mathcal{A}$ of P and $(\widetilde{Y},\widetilde{X})$ a left-coprime factorization over $\mathcal{A}$ of C. Then there exist matrices A, B, R, S over $\mathcal{A}$ such that

$$\widetilde{A}N+\widetilde{B}D=I_m,\quad \widetilde{Y}R+\widetilde{X}S=I_n.$$

By using these relation and (14), we have

$$\Delta_1^{-1}=[\,\widetilde{A} \quad \widetilde{B}\,]\begin{bmatrix}N\Delta_1^{-1}\widetilde{Y} & N\Delta_1^{-1}\widetilde{X}\\ D\Delta_1^{-1}\widetilde{Y} & D\Delta_1^{-1}\widetilde{X}\end{bmatrix}\begin{bmatrix}R\\ S\end{bmatrix}. \tag{15}$$

All matrices of the right hand side of (15) is over $\mathcal{A}$. Hence Δ_1^{-1} is also over $\mathcal{A}$ and so Δ_1 is unimodular. ■

Theorem 5: (cf. Corollary 5.1.30 of [3]) Suppose that $P\in\mathcal{F}^{n\times m}$ and let (N,D) and $(\widetilde{N},\widetilde{D})$ be any right- and left-coprime factorization, respectively, over $\mathcal{A}$. Suppose also that C admits right- and left-coprime factorizations over $\mathcal{A}$. Then the following are equivalent:

(i) C stabilizes P.

(ii) C has a left-coprime factorization $(\widetilde{Y},\widetilde{X})$ over $\mathcal{A}$ with $\widetilde{Y}N+\widetilde{X}D=I_m$.

(iii) C has a right-coprime factorization (Y,X) over $\mathcal{A}$ with $\widetilde{N}Y+\widetilde{N}X=I_n$.

Proof: (i)→(ii): Suppose that C stabilizes P. Suppose that $(\widetilde{Y}_1,\widetilde{X}_1)$ is a left-coprime factorization over $\mathcal{A}$ of C, which may be different from $(\widetilde{Y},\widetilde{X})$. Then by Lemma 2,

$$\Delta_1=\widetilde{Y}_1N+\widetilde{X}_1D$$

is a unimodular of $\mathcal{A}$. By letting $\widetilde{Y}=\Delta_1^{-1}\widetilde{Y}_1$ and $\widetilde{X}=\Delta_1^{-1}\widetilde{X}_1$, we have (ii).

(ii)→(i): Suppose that C has a left-coprime factorization $(\widetilde{Y},\widetilde{X})$ over $\mathcal{A}$ with $\widetilde{Y}N+\widetilde{X}D=I_m$. From this identity, we have $CP+I_m=\widetilde{X}^{-1}D^{-1}$, which is nonsingular. Now $H(P,C)$ is

$$\begin{aligned}
&H(P,C)\\
=&\begin{bmatrix}I-P(I_m+CP)^{-1}C & -P(I_m+CP)^{-1}\\ (I_m+CP)^{-1}C & (I_m+CP)^{-1}\end{bmatrix}\\
=&\begin{bmatrix}I-PD\widetilde{X}C & -PD\widetilde{X}\\ D\widetilde{X}C & D\widetilde{X}\end{bmatrix}\\
=&\begin{bmatrix}I_n-N\widetilde{Y} & -N\widetilde{X}\\ D\widetilde{Y} & D\widetilde{X}\end{bmatrix},
\end{aligned} \tag{16}$$

which is over $\mathcal{A}$. Thus C stabilizies P.

The equivalence of (i) and (iii) is proved analogously. ■

Before presenting the generalization of Lemma 4.1.32 of [3], we should give a lemma which is a generalization of Corollary 4.1.26 of [3].

Lemma 3: (cf. Corollary 4.1.26 of [3]) Suppose $P\in\mathcal{F}^{n\times m}$ and that $P=ND^{-1}=\widetilde{D}^{-1}\widetilde{N}$ with the matrices $N,D,\widetilde{N},\widetilde{D}$ over $\mathcal{A}$. Let $F_1=[-\widetilde{N} \quad \widetilde{D}]$ and $F_2=[D^t \quad N^t]^t$. Then the following are equivalent:

(i) N and D are right-coprime over $\mathcal{A}$, and $\widetilde{N}$ and $\widetilde{D}$ left-coprime over $\mathcal{A}$.

(ii) There exist unimodular matrices U_1 and U_2 of the forms $U_1=[\,G_1^t \quad F_1^t\,]^t$ and $U_2=[\,F_2 \quad G_2\,]$ for some matrices G_1 and G_2 over $\mathcal{A}$.

Proof: "(i)→(ii)". By Theorem 3, there exist matrices $\widetilde{Y}$, $\widetilde{X}$, Y, and X over $\mathcal{A}$ such that (7) holds. Then U_1 is the first matrix of the left hand side of (7). Also U_2 is the second matrix of the left hand side of (7). That is, $G_1=[\,\widetilde{X} \quad \widetilde{Y}\,]$ and $G_2=[\,-Y^t \quad X^t\,]^t$.

"(ii)→(i)". Let V be U_1^{-1} and decompose it into 4 blocks as follows:

$$U_1^{-1}=V=\begin{matrix} & \begin{matrix}m & n\end{matrix}\\ \begin{matrix}m\\ n\end{matrix} & \begin{bmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{bmatrix}\end{matrix}.$$

Because $U_1 V = I_{m+n}$, we have $\widetilde{N}(-V_{12}) + \widetilde{D}V_{22} = I_n$. Hence $\widetilde{N}$ and $\widetilde{D}$ are left-coprime factorization over $\mathcal{A}$.

The right-coprimeness can be proved analogously. Let W be U_2^{-1} and decompose it into 4 blocks as follows:

$$U_2^{-1} = W = \begin{matrix} & m & n \\ m & W_{11} & W_{12} \\ n & W_{21} & W_{22} \end{matrix}.$$

Because $WU_2 = I_{m+n}$, we have $W_{12}N + W_{11}D = I_m$. Hence N and D are right-coprime factorization over $\mathcal{A}$. ■

Lemma 4: (cf. Lemma 4.1.32 of [3]) Suppose $P \in \mathcal{F}^{n\times m}$. Let (N, D) and $(\widetilde{N}, \widetilde{D})$ be right-/left-coprime factorizations, respectively, over $\mathcal{A}$ of P. Let U_1 and U_2 be the unimodular matrices of the form in (ii) of Lemma 3. Then the set of all matrices $\widetilde{Y} \in \mathcal{A}^{m\times n}$ and $\widetilde{X} \in \mathcal{A}^{m\times m}$ with $\widetilde{Y}N + \widetilde{X}D = I_m$ is given by

$$[\,\widetilde{X} \quad \widetilde{Y}\,] = [\,I_m \quad R\,]\,U_2^{-1}, \tag{17}$$

where $R \in \mathcal{A}^{m\times n}$. Similarly the set of all matrices $Y \in \mathcal{A}^{m\times n}$ and $X \in \mathcal{A}^{n\times n}$ with $\widetilde{N}Y + \widetilde{D}X = I_n$ is given by

$$\begin{bmatrix} -Y \\ X \end{bmatrix} = U_1^{-1} \begin{bmatrix} S \\ I_n \end{bmatrix}, \tag{18}$$

where $S \in \mathcal{A}^{m\times n}$.

The proof of Lemma 4 is analogous to that of Lemma 4.1.32 of [3], in which Lemma 3 above is used instead of Corollary 4.1.26 of [3].

Proof of Lemma 4: It is necessary to show that (i) every $\widetilde{Y}$ and $\widetilde{X}$ of the form (17) satisfies $\widetilde{Y}N + \widetilde{N}D = I_m$, and (ii) every $\widetilde{Y}$ and $\widetilde{X}$ satisfy $\widetilde{Y}N + \widetilde{X}D = I_m$ are of the form (17) for some R.

To prove (i), observe that $U_2^{-1}U_2 = I_{m+n}$. Hence

$$\begin{aligned} \widetilde{Y}N + \widetilde{X}D &= [\,\widetilde{X} \quad \widetilde{Y}\,] \begin{bmatrix} D \\ N \end{bmatrix} \\ &= [\,I_m \quad R\,]\,U_2^{-1} \begin{bmatrix} D \\ N \end{bmatrix} \\ &= [\,I_m \quad R\,] \begin{bmatrix} I_m \\ O \end{bmatrix} \\ &= I_m. \end{aligned}$$

To prove (ii), suppose that $\widetilde{Y}'$ and $\widetilde{X}'$ satisfies $\widetilde{Y}'N + \widetilde{X}'D = I_m$. Decompose U_2 as follows:

$$\begin{bmatrix} D & G_{21} \\ N & G_{22} \end{bmatrix} := U_2.$$

Define $R = \widetilde{Y}'G_{22} + \widetilde{X}'G_{21}$. Then

$$[\,\widetilde{X}' \quad \widetilde{Y}'\,]\,U_2 = [\,\widetilde{X}' \quad \widetilde{Y}'\,] \begin{bmatrix} D & G_{21} \\ N & G_{22} \end{bmatrix} = [\,I_m \quad R\,].$$

The proof concerning Y and X can be given analogously. It is necessary to show that (i) every Y and X of the form (18) satisfies $\widetilde{N}Y + \widetilde{D}X = I_n$, and (ii) every Y and X satisfy $\widetilde{N}Y + \widetilde{D}X = I_n$ are of the form (18) for some S.

To prove (i), observe that $U_1^{-1}U_1 = I_{m+n}$. Hence

$$\begin{aligned} \widetilde{N}Y + \widetilde{D}X &= [\,-\widetilde{N} \quad \widetilde{D}\,] \begin{bmatrix} -Y \\ X \end{bmatrix} \\ &= [\,-\widetilde{N} \quad \widetilde{D}\,]\,U_1^{-1} \begin{bmatrix} S \\ I_n \end{bmatrix} \\ &= [\,O \quad I_n\,] \begin{bmatrix} S \\ I_n \end{bmatrix} \\ &= I_n. \end{aligned}$$

To prove (ii), suppose that Y' and X' satisfies $\widetilde{N}Y' + \widetilde{D}X' = I_n$. Decompose U_1 as follows:

$$\begin{bmatrix} G_{11} & G_{12} \\ -\widetilde{N} & \widetilde{D} \end{bmatrix} := U_1.$$

Then, define $S = -G_{11}Y' + G_{12}X'$. Now we have

$$\begin{aligned} & U_1 \begin{bmatrix} -Y' \\ X' \end{bmatrix} \\ =& \begin{bmatrix} G_{11} & G_{12} \\ -\widetilde{N} & \widetilde{D} \end{bmatrix} \begin{bmatrix} -Y' \\ X' \end{bmatrix} \\ =& \begin{bmatrix} S \\ I_n \end{bmatrix}. \end{aligned}$$

■

Now we can prove Theorem 2.

Proof of Theorem 2:

We prove the first representation only. The second one is proved analogously.

$\subset$. By Lamma 1, any stabilizing controller has both right-/right-coprime factorizations over $\mathcal{A}$.

It is shown in Theorem 5 that C is stabilized by P if and only if C has a left-coprime factorization $(\widetilde{Y}, \widetilde{X})$ over $\mathcal{A}$ such that $\widetilde{Y}N + \widetilde{X}D = I_m$. Now consider another left-coprime factorization $(\widetilde{Y}', \widetilde{X}')$ such that

$$\widetilde{Y}'N + \widetilde{X}'N = I_m \tag{19}$$

in the unknown matrices $\widetilde{Y}'$ and $\widetilde{X}'$. Then, again by Theorem 5, C stabilizes P if and only if C is of the form $\widetilde{X}'^{-1}\widetilde{Y}'$ for $\widetilde{X}' \in \mathcal{A}^{m\times m}$ and $\widetilde{Y}' \in \mathcal{A}^{m\times n}$ such that (19) holds and $\widetilde{X}'$ is nonsingular. From Theorem 4, the matrix

$$U_1 = \begin{bmatrix} \widetilde{X} & \widetilde{Y} \\ -\widetilde{N} & \widetilde{D} \end{bmatrix} \tag{20}$$

is unimodular, and moreover U_1^{-1} is of the form

$$U_1^{-1} = \begin{bmatrix} D & \\ & G \\ N & \end{bmatrix},$$

where G is a matrix over over $\mathcal{A}$. By Lemma 4, we have all solutions for $(\widetilde{Y}', \widetilde{X}')$ of (19) are of the form

$$[\,\widetilde{X}' \quad \widetilde{Y}'\,] = [\,I \quad R\,]\,U_1 = [\,\widetilde{X} - R\widetilde{N} \quad \widetilde{Y} + R\widetilde{D}\,]$$

for some $R \in \mathcal{A}^{m\times n}$. ■

V. Example

Let us consider Anantharam's example. Anantharam [7] considered the case $\mathcal{A} = \mathbb{Z}[\sqrt{-5}] = \{u + v\sqrt{-5}\,|\,u, v \in \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the set of integers (This ring [14, pp.134–135] is isomorphic to $\mathbb{Z}[x]/(x^2+5)$ and is an integral domain but not a unique factorization domain. In fact, $6 \in \mathbb{Z}[\sqrt{-5}]$ has two factorizations, $2 \cdot 3$ and $(1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$). He showed that a single-input single-output plant $p = (1 + \sqrt{-5})/2$ does not admit a coprime factorization but is stabilizable and $c = (1 - \sqrt{-5})/(-2)$ is a stabilizing controller.

Let us consider $p = 5/2$, then we have coprime factorization

$$y \cdot 5 + x \cdot 2 = 1,$$

where $y = 1$ and $x = -2$. Thus the set of all stbilizing controllers is given as

$$\{\ (y + 2r)/(x - 5r) \mid r \in \mathcal{A}\ \}.$$

VI. Conclusions

In this paper, we consider the factorization approach to control systems with plants admitting coprime factorizations and with the set of stable causal transfer functions being a general commutative ring. We have shown that the parametrization of stabilizing controllers is still achieved by the Youla-parametrization.

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