Further Results on 3−equitable Labeling

Gaurang V. Ghodasara, Sunil G. Sonchhatra

***Abstract*—A mapping f from the vertex set of a graph G to the set $\{0,1,2\}$ is called 3-equitable labeling, if the edge labels are produced by the absolute difference of labels of end vertices such that the absolute difference of number of vertices of G labeled with 0, 1 and 2 differ by atmost 1 and similarly the absolute difference of number of edges of G labeled with $0,1$ and 2 differ by atmost 1. A graph which admits 3-equitable labeling is called 3-equitable graph. In this paper we prove that the graph obtained by joining two copies of fan graph by a path of arbitrary length is 3-equitable. We also prove similar results for wheel, helm, gear and cycle with one pendant edge.**

***Index Terms*—3-equitable graph, Fan, Wheel, Helm, Gear.**

I. Introduction

We consider simple, finite, undirected graph $G = (V, E)$. In this paper F_n denotes fan graph with $n+1$ vertices, W_n denotes the wheel graph with $n+1$ vertices, H_n denotes helm graph with $2n+1$ vertices and G_n denotes gear graph with $2n+1$ vertices. For all other terminology and notations we follow Gross and Yellen[5].

If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*. A survey on graph labeling is given by Gallian[4].

Definition 1.1 A *fan graph*, denoted by F_n, is the graph with $n+1$ vertices which is the join of the graphs P_n and K_1. i.e. $F_n = P_n + K_1$.

Definition 1.2 A *wheel graph*, denoted by W_n, is the join of the graphs C_n and K_1. i.e. $W_n = C_n + K_1$

Here vertices correspond to C_n are called rim vertices and C_n is called rim of W_n and the vertex corresponding to K_1 is called apex vertex.

Definition 1.3 A *helm graph*, denoted by H_n, is the graph obtained from wheel W_n by adding a pendant edge at each vertex on rim of W_n.

Definition 1.4 A *gear graph*, denoted by G_n, is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the rim of wheel.

Definition 1.5 Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0,1,2\}$ is called *ternary vertex labeling* of G and $f(v)$ is called *label of the vertex* v of G under f.

Let $f^* : E(G) \to \{0,1,2\}$ be the induced edge labeling function defined by $f^*(e) = |f(u) - f(v)|$, for an edge $e = uv$ of G. Let us denote $v_f(i)$ = the number of vertices of G with label i under f and $e_f(i)$ = the number of edges of G with label i under f^*, $0 \le i, j \le 2$.

Definition 1.6 A ternary vertex labeling of a graph G is called *3-equitable labeling* if $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$, $0 \le i, j \le 2$.

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Gaurang V. Ghodasara is an assistant professor of Mathematics in H. & H. B. Kotak Institute of Science, Rajkot, Gujarat. E-mail: gaurang_enjoy@yahoo.co.in.

Sunil G. Sonchhatra is the Ph.D. Scholar in School of Science, R.K. University, Rajkot. He is working as an assistant professor of Mathematics in Lukhdhirji Engineering College, Morbi, Gujarat. E-mail: sonchhabdasunil.20@gmail.com

A graph G is *3-equitable* if it admits 3-equitable labeling.

The concept of 3-equitable labelings was introduced by Cahit[2] in 1990. Seoud and Abdel Maqsoud[6] proved that all fans except $P_2 + K_1$ are 3-equitable. Bapat and Limaye[1] proved that Helm H_n is 3-equitable for $n \ge 4$. Youssef[11] proved that Wheels $W_n = C_n + K_1$ are 3-equitable for $n \ge 4$. In this paper we prove that the graph obtained by joining two copies of fan graphs by a path of arbitrary length is 3-equitable. We also prove similar results for wheel, helm, gear and cycle with one pendant edge.

II. Main Results

Theorem 1 The graph obtained by joining two copies of fan graph F_n by a path of arbitrary length is 3-equitable.

Proof: Let G be the graph obtained by joining two copies of fan graph $F_n = P_n + K_1$ by a path P_k of length $k-1$. Let us denote the successive vertices of first copy of fan graph by $u_1, u_2, \ldots, u_{n+1}$ (where u_1 is the vertex corresponding to K_1) and the successive vertices of second copy of fan graph by $v_1, v_2, \ldots, v_{n+1}$ (where v_1 is the vertex corresponding to K_1). Let $w_1, w_2, \ldots, w_k$ be the vertices of path P_k with $w_1 = u_1$ and $w_k = v_1$.

We define labeling function $f : V(G) \to \{0,1,2\}$ as follows.

Case 1: $n \equiv 0(mod 6)$.

Subcase I: $k \equiv 0(mod 6)$.

$f(u_i) = 0$; if $i \equiv 1,4(mod6)$
$= 1$; if $i \equiv 2,3(mod6)$
$= 2$; if $i \equiv 0,5(mod6)$, $1 \le i \le n+1$.

$f(v_i) = 0$; if $i \equiv 0,3(mod6)$
$= 1$; if $i \equiv 4,5(mod6)$
$= 2$; if $i \equiv 1,2(mod6)$, $1 \le i \le n+1$.

$f(w_j) = 0$; if $j \equiv 1,4(mod6)$
$= 1$; if $j \equiv 2,3(mod6)$
$= 2$; if $j \equiv 0,5(mod6)$, $1 \le j \le k$.

Subcase II: $k \equiv 1(mod6)$.

$f(w_k) = 2$.

$f(w_j) = 0$; if $j \equiv 1,4(mod6)$
$= 1$; if $j \equiv 2,3(mod6)$
$= 2$; if $j \equiv 0,5(mod6)$, $1 \le j \le k-1$.

The remaining vertices are labeled same as in Subcase I.

Subcase III: $k \equiv 2(mod6)$.

$f(u_i) = 0$; if $i \equiv 1,4(mod6)$
$= 1$; if $i \equiv 2,3(mod6)$
$= 2$; if $i \equiv 0,5(mod6)$, $1 \le i \le n+1$.

$f(v_i) = 0$; if $i \equiv 2,5(mod6)$
$= 1$; if $i \equiv 3,4(mod6)$
$= 2$; if $i \equiv 0,1(mod6)$, $1 \le i \le n+1$.

$f(w_j) = 0$; if $j \equiv 1,4(mod6)$
$= 1$; if $j \equiv 0,5(mod6)$
$= 2$; if $j \equiv 2,3(mod6)$, $1 \le j \le k$.

Subcase IV: $k \equiv 3(mod6)$.

$f(v_2) = 1$, $f(v_4) = 0$, $f(v_{n+1}) = 1$.

$f(v_i) = 0$; if $i \equiv 2,5(mod6)$

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$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$, $i \neq 2$, $i \neq 4$.
The remaining vertices are labeled same as in Subcase III.
Subcase V: $k \equiv 4(mod6)$.
$f(v_{n+1}) = 1$, $f(w_k) = 2$.
The remaining vertices are labeled same as in Subcase III.
Subcase VI: $k \equiv 5(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n + 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n + 1$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k$.
Case 2: $n \equiv 1(mod6)$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n + 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n + 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$.
Subcase II: $k \equiv 1(mod6)$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n + 1$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 2(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n + 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n + 1$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \leq j \leq k - 1$.
Subcase IV: $k \equiv 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n + 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n + 1$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \leq j \leq k$.
Subcase V: $k \equiv 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n + 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n + 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k$.
Subcase VI: $k \equiv 5(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n + 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n + 1$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k$.
Case 3: $n \equiv 2(mod6)$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n + 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n + 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$.
Subcase II: $k \equiv 1(mod6)$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n + 1$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 2, 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n + 1$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n + 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k$.
Subcase IV: $k \equiv 4(mod6)$.
$f(v_2) = 1$, $f(v_4) = 0$, $f(w_k) = 2$.
The remaining vertices are labeled same as defined in Subcase I.
Subcase V: $k \equiv 5(mod6)$.
$f(v_2) = 1$, $f(v_4) = 0$.
The remaining vertices are labeled same as defined in Subcase I.
Case 4: $n \equiv 3(mod6)$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n + 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n + 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$.
Subcase II: $k \equiv 1(mod6)$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$

$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le n + 1$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 2(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n + 1$.
$f(v_{n+1}) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le n$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \le j \le k - 1$.
Subcase IV: $k \equiv 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n + 1$.
$f(v_{n-1}) = 2$, $f(v_{n-3}) = 0$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le n + 1$,
$i \ne n - 3$, $i \ne n - 1$.
$f(w_k) = 2$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \le j \le k - 1$.
Subcase V: $k \equiv 4(mod6)$.
$f(w_k) = 2$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \le j \le k - 1$.
The remaining vertices are labeled same as in Subcase I.
Subcase VI: $k \equiv 5(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le n + 1$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \le i \le n + 1$.
$f(w_k) = 2$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \le j \le k - 1$.
Case 5: $n \equiv 4(mod6)$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \le i \le n + 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n + 1$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \le j \le k$.
Subcase II: $k \equiv 1(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n + 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le n + 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \le j \le k$.
Subcase III: $k \equiv 2(mod6)$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \le j \le k - 1$.
The remaining vertices are labeled same as in Subcase I.
Subcase IV: $k \equiv 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le n + 1$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \le i \le n + 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \le j \le k$.
Subcase V: $k \equiv 4(mod6)$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le n + 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n + 1$.
$f(w_3) = 1$, $f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \le j \le k - 1$, $j \ne 3$.
Subcase VI: $k \equiv 5(mod6)$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \le j \le k$.
The remaining vertices are labeled same as defined in Subcase I.
Case 6: $n \equiv 5(mod6)$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le n + 1$.
$f(v_6) = 0$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n + 1$, $i \ne 6$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \le j \le k$.
Subcase II: $k \equiv 1(mod6)$.
$f(u_{n-1}) = 2$, $f(w_k) = 0$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 2(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n + 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n + 1$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$

$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k-1$.
Subcase IV: $k \equiv 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n+1$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n+1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k$.
Subcase V: $k \equiv 4(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n+1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n+1$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \leq j \leq k-1$.
Subcase VI: $k \equiv 5(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n+1$.
$f(v_{n-2}) = 0$, $f(v_{n-4}) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n+1$, $i \neq n-2$, $i \neq n-4$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k$.

The graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq 2$ in each case. Hence the graph G under consideration is 3-equitable graph.

Theorem 2 The graph obtained by joining two copies of wheel graph by a path of arbitrary length is 3-equitable.
Proof: Let G be the graph obtained by joining two copies of wheel graph W_n by path P_k of length $k-1$. Let us denote the successive vertices of first copy of wheel graph by $u_0, u_1, \dots, u_n$ (where u_0 is apex vertex) and the successive vertices of second copy of wheel graph by $v_0, v_1, \dots, v_n$ (where v_0 is apex vertex). Let $w_1, w_2, \dots, w_k$ be the vertices of path P_k with $w_1 = u_1$ and $w_k = v_1$.
We define labeling function $f : V(G) \to \{0, 1, 2\}$ as follows.
Case 1: $n \equiv 0(mod6)$.
$f(u_0) = 0$, $f(v_0) = 2$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \leq j \leq k$.
Subcase II: $k \equiv 1, 2(mod6)$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(w_{k-1}) = 1$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 0, 1(mod6)$
$= 2$; if $j \equiv 3, 4(mod6)$, $1 \leq j \leq k$, $j \neq k-1$.
Subcase IV: $k \equiv 4(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k$.
Subcase V: $k \equiv 5(mod6)$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.
Case 2: $n \equiv 1(mod6)$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_0) = 2$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 0$, $f(v_1) = 2$, $f(v_{n-2}) = 0$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $2 \leq i \leq n$, $i \neq n-2$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$.
Subcase II: $k \equiv 1(mod6)$.
$f(u_0) = 0$, $f(u_{n-1}) = 1$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$, $i \neq n-1$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k-1$.
Subcase III: $k \equiv 2(mod6)$.
The vertices are labeled same as in Subcase II except for $f(w_k) = 0$.

Subcase IV: $k \equiv 3(mod6)$.
$f(w_{k-1}) = 1$.
The remaining vertices are labeled same as in Subcase II except for $f(u_{n-1}) = 1$.
Subcase V: $k \equiv 4(mod6)$.
$f(v_n) = 1$.
The remaining vertices are labeled same as in Subcase II except for $f(u_{n-1}) = 1$.
Subcase VI: $k \equiv 5(mod6)$.
The vertices are labeled same as in Subcase II except for $f(w_k) = 0$ and $f(u_{n-1}) = 1$.
Case 3: $n \equiv 2(mod6)$.
$f(u_0) = 0$, $f(v_0) = 2$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(v_n) = 1$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n-1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$,
$= 2$; if $j \equiv 0, 5(mod6)$ $1 \leq j \leq k$.
Subcase II: $k \equiv 1(mod6)$.
$f(u_n) = 1$, $f(w_k) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 2(mod6)$.
$f(w_{k-4}) = 1$.
The remaining vertices are labeled same as in Subcase II.
Subcase IV: $k \equiv 3(mod6)$.
$f(w_{k-5}) = 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.
Subcase V: $k \equiv 4(mod6)$.
$f(w_k) = 1$.
The remaining vertices are labeled same as in Subcase IV except for $f(w_{k-5}) = 1$.
Subcase VI: $k \equiv 5(mod6)$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(w_3) = 1$, $f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 1, 2(mod6)$
$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k-1$, $j \neq 3$.
Case 4: $n \equiv 3(mod6)$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_n) = 2$, $f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n-1$
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \leq j \leq k$.
Subcase II: $k \equiv 1(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$.
Subcase III: $k \equiv 2(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq j \leq n$.
$f(w_{k-3}) = 1$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k$, $j \neq k-3$.
Subcase IV: $k \equiv 3(mod6)$.
$f(w_{k-4}) = 1$.
The remaining vertices are labeled same as in Subcase III except for $f(w_{k-3}) = 1$.
Subcase V: $k \equiv 4(mod6)$.
$f(u_0) = 2$, $f(v_0) = 0$ and $f(w_k) = 2$.
The remaining vertices are labeled same as in Subcase II except for $f(u_0) = 0$ and $f(v_0) = 2$.
Subcase VI: $k \equiv 5(mod6)$.
The vertices are labeled same as defined in Subcase V except for $f(w_k) = 2$.
Case 5: $n \equiv 4(mod6)$.
$f(u_0) = 0$, $f(v_0) = 2$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(v_n) = 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n-1$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k-1$.
Subcase II: $k \equiv 1(mod6)$.
The vertices are labeled same as in Subcase I except for $f(w_k) = 0$.
Subcase III: $k \equiv 2(mod6)$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$

$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(w_k) = 2$, $f(w_{k-1}) = 1$.
The remaining vertices are labeled same as in Subcase I except for $f(w_k) = 0$.

Subcase IV: $k \equiv 3, 4(mod6)$.
The vertices are labeled same as in Subcase III except for $f(w_{k-1}) = 1$.

Subcase V: $k \equiv 5(mod6)$.
The vertices are labeled same as in Subcase IV except for $f(w_k) = 2$.

Case 6: $n \equiv 5(mod6)$.

Subcase I: $k \equiv 0(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(w_{k-2}) = 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$, $j \neq k-2$.

Subcase II: $k \equiv 1(mod6)$.
$f(w_{k-3}) = 1$, $f(w_k) = 2$.
The remaining vertices are labeled same as in Subcase I except for $f(w_{k-2}) = 1$.

Subcase III: $k \equiv 2(mod6)$.
$f(u_0) = 2$, $f(v_0) = 0$, $f(w_k) = 2$, $f(w_{k-4}) = 1$.
The remaining vertices are labeled same as in Subcase I except for $f(w_{k-2}) = 1$.

Subcase IV: $k \equiv 3(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(v_n) = 2$, $f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n-1$.
$f(w_k) = 2$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k-1$.

Subcase V: $k \equiv 4(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k-1$.

Subcase VI: $k \equiv 5(mod6)$.
$f(u_0) = 2$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.
$f(w_k) = 0$, $f(w_{k-2}) = 1$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 1, 2(mod6)$,
$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k-1$, $j \neq k-2$.

The graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq 2$ in each case. Hence the graph G under consideration is 3-equitable graph.

Theorem 3 The graph obtained by joining two copies of helm graph H_n by a path of arbitrary length is 3-equitable.
Proof: Let G be the graph obtained by joining two copies of helm graph H_n by a path P_k of length $k-1$. Let u_0 be the apex vertex, $u_1, u_2, \ldots, u_n$ be the rim vertices and $u'_1, u'_2, \ldots, u'_n$ be the pendant vertices of first copy of helm H_n. Similarly let v_0 be the apex vertex, $v_1, v_2, \ldots, v_n$ be the rim vertices and $v'_1, v'_2, \ldots, v'_n$ be the pendant vertices of second copy of helm H_n. Let $w_1, w_2, \ldots, w_k$ be the vertices of path P_k with $w_1 = u_1$ and $w_k = v_1$. We define labeling function $f : V(G) \rightarrow \{0, 1, 2\}$ as follows.

Case 1: $n \equiv 0(mod6)$.
$f(u_0) = 0$, $f(v_0) = 2$.

Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \leq j \leq k$.

Subcase II: $k \equiv 1, 2(mod6)$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$
$f(v'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.

Subcase III: $k \equiv 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$

$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(w_{k-1}) = 1$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 0, 1(mod6)$
$= 2$; if $j \equiv 3, 4(mod6)$, $1 \leq j \leq k$, $j \neq k-1$.

Subcase IV: $k \equiv 4(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.
$f(v'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k$.

Subcase V: $k \equiv 5(mod6)$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.

Case 2: $n \equiv 1(mod6)$.

Subcase I: $k \equiv 0(mod6)$.
$f(u_0) = 2$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 0$, $f(v_1) = 2$, $f(v_{n-2}) = 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $2 \leq i \leq n$, $i \neq n-2$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$.

Subcase II: $k \equiv 1(mod6)$.
$f(u_0) = 0$, $f(u_{n-1}) = 1$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$, $i \neq n-1$.
$f(u'_n) = 0$.
$f(u'_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n-1$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(v'_2) = 2$, $f(v'_n) = 0$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n-1$, $i \neq 2$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k-1$.

Subcase III: $k \equiv 2(mod6)$.
The vertices are labeled same as in Subcase II except for $f(v'_2) = 2$, $f(w_k) = 0$.

Subcase IV: $k \equiv 3(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(v'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
$f(w_k) = 0$,$f(w_{k-1}) = 1$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k-2$.

Subcase V: $k \equiv 4(mod6)$.
$f(v_n) = 1$, $f(v'_3) = 0$.
The remaining vertices are labeled same as in Subcase IV except for $f(w_{k-1}) = 1$.

Subcase VI: $k \equiv 5(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(u'_n) = 0$, $f(u'_{n-2}) = 2$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n-1$, $i \neq n-2$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(v'_n) = 0$, $f(v'_{n-1}) = 1$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n-2$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k$.

Case 3: $n \equiv 2(mod6)$.
$f(u_0) = 0$, $f(v_0) = 2$.

Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \le i \le n$.
$f(v_n) = 1$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \le i \le n-1$.
$f(v'_3) = 0$, $f(v'_4) = 1$.
$f(v'_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le n$, $i \ne 3$, $i \ne 4$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \le j \le k$.

Subcase II: $k \equiv 1(mod6)$.
$f(u_n) = 1$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n-1$.
$f(u'_3) = 2$, $f(u'_{n-2}) = 1$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le n$,$i \ne 3$, $i \ne n-2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le n$.
$f(v'_4) = 1$.
$f(v'_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le n$, $i \ne 4$.
$f(w_k) = 2$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \le j \le k-1$.

Subcase III: $k \equiv 2(mod6)$.
$f(w_{k-4}) = 1$.
The remaining vertices are labeled same as in Subcase II.

Subcase IV: $k \equiv 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n$.
$f(u'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \le i \le n$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \le i \le n$
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \le i \le n$.
$f(w_{k-5}) = 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \le j \le k$, $j \ne k-5$.

Subcase V: $k \equiv 4(mod6)$.
$f(v'_1) = 0$, $f(v'_{n-1}) = 1$.
$f(v'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $2 \le i \le n$, $i \ne n-1$.
The remaining vertices are labeled same as in Subcase IV except for $f(w_{k-5}) = 1$.

Subcase VI: $k \equiv 5(mod6)$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \le i \le n$.
$f(u'_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n$.
$f(v'_n) = 1$.
$f(v'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \le i \le n-1$.
$f(w_3) = 1$, $f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 1, 2(mod6)$
$= 2$; if $j \equiv 4, 5(mod6)$, $1 \le j \le k-1$, $j \ne 3$.

Case 4: $n \equiv 3(mod6)$.

Subcase I: $k \equiv 0(mod6)$.
$f(u_n) = 2$, $f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le n-1$.
$f(u'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \le i \le n$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le n$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \le i \le n$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \le j \le k$.

Subcase II: $k \equiv 1(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n$.
$f(u'_n) = 2$.
$f(u'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \le i \le n-1$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le n$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \le i \le n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \le j \le k$.

Subcase III: $k \equiv 2(mod6)$.
$f(u_0) = 2$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$

$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.
$f(v'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(w_{k-3}) = 1$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k$, $j \neq k-3$.

Subcase IV: $k \equiv 3(mod6)$.
$f(u_0) = 0$, $f(v_0) = 2$, $f(w_5) = 1$, $f(w_k) = 0$.
The remaining vertices are labeled same as in Subcase III except for $f(w_{k-3}) = 1$.

Subcase V: $k \equiv 4(mod6)$.
$f(u_0) = 2$, $f(v_0) = 0$, $f(w_k) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(v'_5) = 1$.
$f(v'_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$, $i \neq 5$.
The remaining vertices are labeled same as in Subcase II except for $f(u_0) = 0$, $f(u'_n) = 2$.

Subcase VI: $k \equiv 5(mod6)$
The vertices are labeled same as in Subcase V except for $f(w_k) = 2$.

Case 5: $n \equiv 4(mod6)$.
$f(u_0) = 0$, $f(v_0) = 2$.

Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(v_n) = 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n-1$.
$f(v'_1) = 0$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $2 \leq i \leq n$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k-1$.

Subcase II: $k \equiv 1(mod6)$.
$f(u'_1) = 1$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $2 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I except for $f(v'_1) = 0$, $f(w_k) = 0$.

Subcase III: $k \equiv 2(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(u'_1) = 1$.
$f(u'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $2 \leq i \leq n$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(v'_1) = 0$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $2 \leq i \leq n$.
$f(w_k) = 2$, $f(w_{k-1}) = 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k-2$.

Subcase IV: $k \equiv 3(mod6)$.
$f(u'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase III except for $f(v'_1) = 0$, $f(w_{k-1}) = 1$.

Subcase V: $k \equiv 4(mod6)$.
$f(v'_1) = 1$.
The remaining vertices are labeled same as in Subcase IV.

Subcase VI: $k \equiv 5(mod6)$.
The vertices are labeled same as in subcase V except for $f(w'_k) = 2$.

Case 6: $n \equiv 5(mod6)$.

Subcase I: $k \equiv 0(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(v'_2) = 0$.
$f(v'_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$, $i \neq 2$.
$f(w_{k-2}) = 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$,$j \neq k-2$.

Subcase II: $k \equiv 1(mod6)$.
$f(u_0) = 2$, $f(v_0) = 0$, $f(w_{k-3}) = 1$, $f(w_k) = 2$.
The remaining vertices are labeled same as in Subcase I except for $f(v'_2) = 0$ and $f(w_{k-2}) = 1$.

Subcase III: $k \equiv 2(mod6)$.
$f(u_0) = 2$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$

$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(w_4) = 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \leq j \leq k$, $j \neq 4$.

Subcase IV: $k \equiv 3(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$
$f(v_0) = 2$, $f(v_n) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n-1$,
$f(v'_n) = 0$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n-1$.
$f(w_k) = 2$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 0, 5(mod6)$
$= 2$; if $j \equiv 2, 3(mod6)$, $1 \leq j \leq k-1$.

Subcase V: $k \equiv 4(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \leq j \leq k$.

Subcase VI: $k \equiv 5(mod6)$.
$f(u_0) = 2$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \leq i \leq n$.
$f(u'_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq n$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \leq i \leq n$.
$f(v'_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq n$.
$f(w_{k-2}) = 1$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 1, 2(mod6)$
$= 2$; if $j \equiv 4, 5(mod6)$, $1 \leq j \leq k$, $j \neq k-2$.

The graph G under consideration satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq 2$ in each case. Hence the graph G under consideration is 3-equitable graph.

Theorem 4 The graph obtained by joining two copies of gear graph G_n by a path of arbitrary length is 3-equitable.

Proof: Let G be the graph obtained by joining two copies of gear graph G_n by path P_k of length $k-1$. Let us denote the successive vertices of first copy of gear graph by $u_0, u_1, \ldots, u_{2n}$, where u_0 is apex vertex, $u_1, u_3, \ldots, u_{2n-1}$ are rim vertices of wheel and $u_2, u_4, \ldots, u_{2n}$ are the vertices inserted between two consecutive rim vertices corresponding to $u_1, u_3, \ldots, u_{2n-1}$ respectively. Similarly let $v_0, v_1, \ldots, v_{2n}$ be the successive vertices of second copy of gear graph, where v_0 is apex vertex, $v_1, v_3, \ldots, v_{2n-1}$ are rim vertices of wheel and $v_2, v_4, \ldots, v_{2n}$ are the vertices inserted between two consecutive rim vertices corresponding to $v_1, v_3, \ldots, v_{2n-1}$ respectively. Let $w_1, w_2, \ldots, w_k$ be the vertices of path P_k with $w_1 = u_1$ and $w_k = v_1$.

We define labeling function $f : V(G) \rightarrow \{0, 1, 2\}$ as follows.

Case 1: $n \equiv 0, 3(mod6)$.
$f(u_0) = 0$, $f(v_0) = 2$.

Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq 2n$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \leq i \leq 2n$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \leq j \leq k$.

Subcase II: $k \equiv 1, 2(mod6)$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \leq i \leq 2n$.
The remaining vertices are labeled same as in Subcase I.

Subcase III: $k \equiv 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \leq i \leq 2n$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \leq i \leq 2n$.
$f(w_{k-1}) = 1$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$

$= 1$; if $j \equiv 0, 1(mod6)$
$= 2$; if $j \equiv 3, 4(mod6)$, $1 \le j \le k$, $j \ne k-1$.
Subcase IV: $k \equiv 4(mod6)$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \le i \le 2n$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \le i \le 2n$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \le j \le k$.
Subcase V: $k \equiv 5(mod6)$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \le i \le 2n$.
The remaining vertices are labeled same as in Subcase I.
Case 2: $n \equiv 1, 4(mod6)$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le 2n$.
$f(v_0) = 2$, $f(v_{2n}) = 1$,
$f(v_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \le i \le 2n-1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \le j \le k$.
Subcase II: $k \equiv 1(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le 2n$.
$f(v_0) = 2$, $f(v_2) = 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le 2n$, $n \ne 2$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \le j \le k$.
Subcase III: $k \equiv 2(mod6)$.
$f(u_0) = 0$, $f(u_{2n}) = 1$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le 2n-1$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le 2n$.
$f(w_4) = 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \le j \le k$.
Subcase IV: $k \equiv 3(mod6)$.
$f(u_0) = 2$,$f(v_0) = 0$, $f(v_{2n}) = 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \le i \le 2n-1$.
The remaining vertices are labeled same as in Subcase I.
Subcase V: $k \equiv 4(mod6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le 2n$.
$f(v_0) = 2$, $f(v_{2n-1}) = 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le 2n$, $i \ne 2n-1$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 4, 5(mod6)$
$= 2$; if $j \equiv 1, 2(mod6)$, $1 \le j \le k-1$.
Subcase VI: $k \equiv 5(mod6)$.
$f(u_0) = 2$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \le i \le 2n$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le 2n$.
$f(w_3) = 1$, $f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod6)$
$= 1$; if $j \equiv 1, 2(mod6)$
$= 2$; if $j \equiv 4, 5(mod6)$, $1 \le j \le k-1$, $j \ne 3$.
Case 3: $n \equiv 2, 5(mod6)$.
$f(u_0) = 0$, $f(v_0) = 2$.
Subcase I: $k \equiv 0(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le 2n$.
$f(v_{2n}) = 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le 2n-1$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod6)$
$= 1$; if $j \equiv 2, 3(mod6)$
$= 2$; if $j \equiv 0, 5(mod6)$, $1 \le j \le k-1$.
Subcase II: $k \equiv 1(mod6)$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le 2n$.
$f(v_{2n-3}) = 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le 2n$, $i \ne 2n-3$.
$f(w_k) = 0$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \le j \le k-1$.
Subcase III: $k \equiv 2(mod6)$.
All the vertices are labeled same as in Subcase II except for $f(w_k) = 0$.
Subcase IV: $k \equiv 3(mod6)$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 2, 3(mod6)$
$= 2$; if $i \equiv 0, 5(mod6)$, $1 \le i \le 2n$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$

$= 2$; if $i \equiv 0, 1(mod 6)$, $1 \le i \le 2n$.
$f(w_{k-1}) = 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod 6)$
$= 1$; if $j \equiv 0, 5(mod 6)$
$= 2$; if $j \equiv 2, 3(mod 6)$, $1 \le j \le k$, $j \ne k-1$.
Subcase V: $k \equiv 4(mod 6)$.
$f(v_{2n}) = 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 2, 3(mod 6)$
$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \le i \le 2n-1$.
The remaining vertices are labeled same as in Subcase IV.
Subcase VI: $k \equiv 5(mod 6)$.
$f(w_j) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 2, 3(mod 6)$
$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \le j \le k$.
The remaining vertices are labeled same as in Subcase IV.

The graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$, $0 \le i, j \le 2$ in each case. Hence the graph G under consideration is 3-equitable graph.

Theorem 5 The graph obtained by joining two copies of cycle with one pendant edge by a path of arbitrary length is 3-equitable.

Proof: Let G be the graph obtained by joining two copies of cycle with one pendant edge by path P_k of length $k-1$. Let us denote the successive vertices of first copy of cycle by $u_1, u_2, \dots, u_n$, $e = u_0u_1$ be the pendant edge and u_0 be the pendant vertex. Similarly let $v_1, v_2, \dots, v_n$ be the successive vertices of second copy of cycle, $e' = v_0v_1$ be the pendant edge and v_0 be the pendant vertex. Let $w_1, w_2, \dots, w_k$ be the successive vertices of path P_k with $w_1 = u_1$ and $w_k = v_1$. We define labeling function $f : V(G) \to \{0, 1, 2\}$ as follows.

Case 1: $n \equiv 0(mod 6)$.
Subcase I: $k \equiv 0(mod 6)$.
$f(u_0) = 2$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 0, 5(mod 6)$
$= 2$; if $i \equiv 2, 3(mod 6)$, $1 \le i \le n$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 3, 4(mod 6)$
$= 2$; if $i \equiv 0, 1(mod 6)$, $1 \le i \le n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod 6)$
$= 1$; if $j \equiv 2, 3(mod 6)$
$= 2$; if $j \equiv 0, 5(mod 6)$, $1 \le j \le k$.
Subcase II: $k \equiv 1, 4(mod 6)$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 2, 3(mod 6)$
$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \le i \le n$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 2(mod 6)$.
$f(v_0) = 1$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod 6)$
$= 1$; if $i \equiv 1, 2(mod 6)$
$= 2$; if $i \equiv 4, 5(mod 6)$, $1 \le i \le n$.
The remaining vertices are labeled same as in Subcase I.
Subcase IV: $k \equiv 3(mod 6)$.
$f(v_0) = 2$.
The remaining vertices are labeled same as in Subcase III.
Subcase V: $k \equiv 5(mod 6)$.
$f(v_0) = 2$.
The remaining vertices are labeled same as in Subcase I.

Case 2: $n \equiv 1(mod 6)$.
Subcase I: $k \equiv 0(mod 6)$.
$f(u_0) = 2$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 0, 5(mod 6)$
$= 2$; if $i \equiv 2, 3(mod 6)$, $1 \le i \le n$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod 6)$
$= 1$; if $i \equiv 4, 5(mod 6)$
$= 2$; if $i \equiv 1, 2(mod 6)$, $1 \le i \le n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod 6)$
$= 1$; if $j \equiv 2, 3(mod 6)$
$= 2$; if $j \equiv 0, 5(mod 6)$, $1 \le j \le k$.
Subcase II: $k \equiv 1(mod 6)$.
$f(u_0) = 1$, $f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 2, 3(mod 6)$
$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \le i \le n$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 2(mod 6)$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 0, 1(mod 6)$
$= 2$; if $i \equiv 3, 4(mod 6)$, $1 \le i \le n$.
The remaining vertices are labeled same as in Subcase I.
Subcase IV: $k \equiv 3(mod 6)$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 3, 4(mod 6)$
$= 2$; if $i \equiv 0, 1(mod 6)$, $1 \le i \le n$.
$f(w_{k-1}) = 1$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod 6)$
$= 1$; if $j \equiv 0, 5(mod 6)$
$= 2$; if $j \equiv 2, 3(mod 6)$, $1 \le j \le k$, $j \ne k-1$.
The remaining vertices are labeled same as in Subcase I.
Subcase V: $k \equiv 4(mod 6)$.
$f(u_0) = 1$, $f(v_0) = 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 2, 3(mod 6)$
$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \le i \le n$.
The remaining vertices are labeled same as in Subcase IV.
Subcase VI: $k \equiv 5(mod 6)$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 0, 1(mod 6)$
$= 2$; if $i \equiv 3, 4(mod 6)$, $1 \le i \le n$.
The remaining vertices are labeled same as in Subcase V.

Case 3: $n \equiv 2(mod 6)$.
Subcase I: $k \equiv 0(mod 6)$.
$f(u_0) = 1$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 0, 5(mod 6)$
$= 2$; if $i \equiv 2, 3(mod 6)$, $1 \le i \le n$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod 6)$
$= 1$; if $i \equiv 4, 5(mod 6)$
$= 2$; if $i \equiv 1, 2(mod 6)$, $1 \le i \le n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod 6)$

$= 1$; if $j \equiv 2, 3(mod 6)$
$= 2$; if $j \equiv 0, 5(mod 6)$, $1 \leq j \leq k$.
Subcase II: $k \equiv 1(mod 6)$.
$f(u_0) = f(v_0) = 0$.
$f(v_1) = f(w_k) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 2, 3(mod 6)$
$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 2(mod 6)$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod 6)$
$= 1$; if $i \equiv 1, 2(mod 6)$
$= 2$; if $i \equiv 4, 5(mod 6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.
Subcase IV: $k \equiv 3(mod 6)$.
$f(u_0) = 2$.
$f(u_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 3, 4(mod 6)$
$= 2$; if $i \equiv 0, 1(mod 6)$, $1 \leq i \leq n$.
$f(v_0) = 1$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 2, 3(mod 6)$
$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 0, 3(mod 6)$
$= 1$; if $j \equiv 4, 5(mod 6)$
$= 2$; if $j \equiv 1, 2(mod 6)$, $1 \leq j \leq k$.
Subcase V: $k \equiv 4(mod 6)$.
$f(u_0) = 1$, $f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod 6)$
$= 1$; if $i \equiv 1, 2(mod 6)$
$= 2$; if $i \equiv 4, 5(mod 6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase IV.
Subcase VI: $k \equiv 5(mod 6)$.
$f(v_0) = 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 0, 1(mod 6)$
$= 2$; if $i \equiv 3, 4(mod 6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase IV.
Case 4: $n \equiv 3(mod 6)$.
Subcase I: $k \equiv 0(mod 6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 0, 5(mod 6)$
$= 2$; if $i \equiv 2, 3(mod 6)$, $1 \leq i \leq n$.
$f(v_0) = 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 3, 4(mod 6)$
$= 2$; if $i \equiv 0, 1(mod 6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod 6)$
$= 1$; if $j \equiv 2, 3(mod 6)$
$= 2$; if $j \equiv 0, 5(mod 6)$, $1 \leq j \leq k$.
Subcase II: $k \equiv 1(mod 6)$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 2, 3(mod 6)$
$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 2, 3(mod 6)$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 0, 1(mod 6)$
$= 2$; if $i \equiv 3, 4(mod 6)$, $1 \leq j \leq n$.
The remaining vertices are labeled same as in Subcase I.
Subcase IV: $k \equiv 4(mod 6)$.
$f(v_0) = 2$.
The remaining vertices are labeled same as in Subcase II.
Subcase V: $k \equiv 5(mod 6)$.
$f(v_0) = 2$.
The remaining vertices are labeled same as in Subcase I.
Case 5: $n \equiv 4(mod 6)$.
Subcase I: $k \equiv 0(mod 6)$.
$f(u_0) = 1$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 0, 5(mod 6)$
$= 2$; if $i \equiv 2, 3(mod 6)$, $1 \leq i \leq n$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 3, 4(mod 6)$
$= 2$; if $i \equiv 0, 1(mod 6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod 6)$
$= 1$; if $j \equiv 2, 3(mod 6)$
$= 2$; if $j \equiv 0, 5(mod 6)$, $1 \leq j \leq k$.
Subcase II: $k \equiv 1(mod 6)$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 2, 3(mod 6)$
$= 2$; if $i \equiv 0, 5(mod 6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.
Subcase III: $k \equiv 2(mod 6)$.
$f(v_0) = 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 0, 1(mod 6)$
$= 2$; if $i \equiv 3, 4(mod 6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.
Subcase IV: $k \equiv 3(mod 6)$.
$f(u_0) = 2$, $f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 0, 3(mod 6)$
$= 1$; if $i \equiv 1, 2(mod 6)$
$= 2$; if $i \equiv 4, 5(mod 6)$, $1 \leq i \leq n$.
The remaining vertices are labeled same as in Subcase I.
Subcase V: $k \equiv 4(mod 6)$.
$f(u_0) = 2$.
The remaining vertices are labeled same as in Subcase II.
Subcase VI: $k \equiv 5(mod 6)$.
$f(v_0) = 2$.
The remaining vertices are labeled same as in Subcase I.
Case 6: $n \equiv 5(mod 6)$.
Subcase I: $k \equiv 0(mod 6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 1, 4(mod 6)$
$= 1$; if $i \equiv 0, 5(mod 6)$
$= 2$; if $i \equiv 2, 3(mod 6)$, $1 \leq i \leq n$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod 6)$
$= 1$; if $i \equiv 3, 4(mod 6)$
$= 2$; if $i \equiv 0, 1(mod 6)$, $1 \leq i \leq n$.
$f(w_j) = 0$; if $j \equiv 1, 4(mod 6)$
$= 1$; if $j \equiv 2, 3(mod 6)$
$= 2$; if $j \equiv 0, 5(mod 6)$, $1 \leq j \leq k$.
Subcase II: $k \equiv 1(mod 6)$.
$f(u_0) = 0$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod 6)$

$= 1$; if $i \equiv 4, 5(mod6)$
$= 2$; if $i \equiv 1, 2(mod6)$, $1 \le i \le n$.
$f(v_0) = 2$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le n$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 3, 4(mod6)$
$= 2$; if $j \equiv 0, 1(mod6)$, $1 \le j \le k$.

Subcase III: $k \equiv 2(mod6)$.
$f(u_0) = 1$, $f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 1, 4(mod6)$
$= 1$; if $i \equiv 0, 5(mod6)$
$= 2$; if $i \equiv 2, 3(mod6)$, $1 \le i \le n$.
The remaining vertices are labeled same as in Subcase II.

subcase IV: $k \equiv 3, 4(mod6)$.
$f(v_0) = 1$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 0, 1(mod6)$
$= 2$; if $i \equiv 3, 4(mod6)$, $1 \le i \le n$.
The remaining vertices are labeled same as in Subcase II.

Subcase V: $k \equiv 5(mod6)$.
$f(u_0) = 1$.
$f(u_i) = 0$; if $i \equiv 0, 3(mod6)$
$= 1$; if $i \equiv 1, 2(mod6)$
$= 2$; if $i \equiv 4, 5(mod6)$, $1 \le i \le n$.
$f(v_0) = 0$.
$f(v_i) = 0$; if $i \equiv 2, 5(mod6)$
$= 1$; if $i \equiv 3, 4(mod6)$
$= 2$; if $i \equiv 0, 1(mod6)$, $1 \le i \le n$.
$f(w_k) = 2$.
$f(w_j) = 0$; if $j \equiv 2, 5(mod6)$
$= 1$; if $j \equiv 0, 1(mod6)$
$= 2$; if $j \equiv 3, 4(mod6)$, $1 \le j \le k - 1$.

The graph G under consideration satisfies the conditions $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$, $0 \le i, j \le 2$ in each case. Hence the graph G under consideration is 3-equitable graph.

III. ILLUSTRATIONS

Illustration 1 As an illustration of *Theorem 1*, 3-equitable labeling of the graph G obtained by joining two copies of fan graph F_7 by path P_9 is shown in *Fig. 1*. It is the case related to $n \equiv 1(mod6)$ and $k \equiv 3(mod6)$.

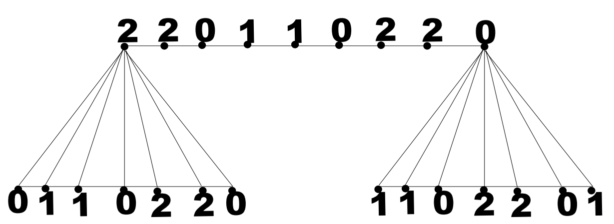

Fig. 1. : 3-equitable labeling of the graph G obtained by joining two copies of F_7 by P_9.

Illustration 2 As an illustration of labeling pattern defined in *Theorem 2*, 3-equitable labeling of the graph G obtained by joining two copies of wheel graph W_8 by path P_6 is shown in *Fig. 2*. It is the case related to $n \equiv 2(mod6)$ and $k \equiv 0(mod6)$.

Illustration 3 As an illustration of labeling pattern defined in

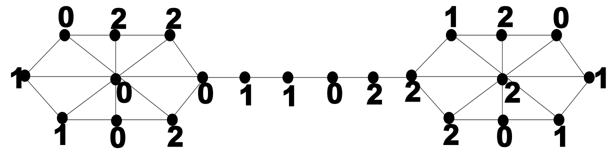

Fig. 2. : 3-equitable labeling of the graph G obtained by joining two copies of W_8 by P_6.

Theorem 3, 3-equitable labeling of the graph G obtained by joining two copies of helm graph H_6 by path P_6 is shown in *Fig. 3*. It is the case related to $n \equiv 0(mod6)$ and $k \equiv 0(mod6)$.

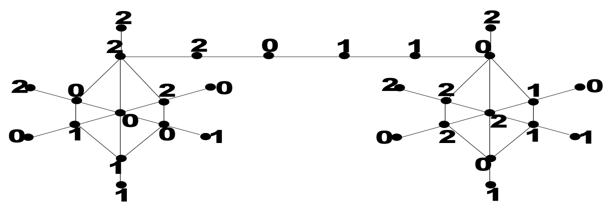

Fig. 3. : 3-equitable labeling of the graph G obtained by joining two copies of H_6 by P_6.

Illustration 4 As an illustration of labeling pattern defined in *Theorem 4*, 3-equitable labeling of the graph G obtained by joining two copies of gear graph G_6 by path P_6 is shown in *Fig. 4*. It is the case related to $n \equiv 0(mod6)$ and $k \equiv 0(mod6)$.

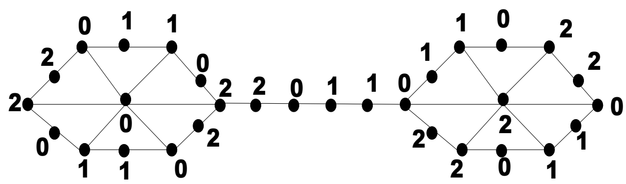

Fig. 4. : 3-equitable labeling of the graph G obtained by joining two copies of G_6 by P_6.

Illustration 5 As an illustration of labeling pattern defined in *Theorem 5*, 3-equitable labeling of the graph G obtained by joining two copies of cycle C_6 with one pendant edge by path P_6 is shown in *Fig. 5*. It is the case related to $n \equiv 0(mod6)$ and $k \equiv 0(mod6)$.

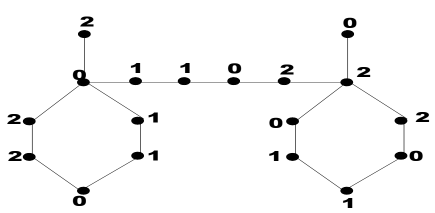

Fig. 5. : 3-equitable labeling of the graph G obtained by joining two copies of cycle C_6 with one pendant edge by P_6.

IV. CONCLUSION

The research work presented here provide five new results in the theory of 3-equitable labeling of graphs. The entire work is focused on joining two copies of some graph by a path of arbitrary length. In this work two copies of fans, wheels helms, gears and cycle with one pendant edge are considered.

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