The Multi-level Distance Number for Symmetric Lobster-like Trees about the Weight Center *

Liancui Zuo, Hongfang Guo, and Chunhong Shang†

Abstract

The multi-level distance labeling for a network G is a function $f : V(G) \to \{0, 1, 2, \cdots\}$ so that

$$|f(u) - f(v)| \geq diam(G) + 1 - d(u, v)$$

for any $u, v \in V(G)$, where $diam(G)$ is the diameter of G and $d(u, v)$ is the distance between u and v. The span of f is defined as $\max\{f(u) - f(v) \mid u, v \in V(G)\}$. The multi-level distance number of G is the minimum span of all multi-level distance labelings for G. In the present paper, a class of symmetric lobster-like trees about the weight center is studied, and its multi-level distance number is obtained.

Keywords: multi-level distance number; multi-level distance labeling; symmetric lobster-like tree about the weight center; the minimum span; network

1 Introduction

In recent years, many parameters and classes of graphs are considiered. For example, in [9], different properties of the intrinsic order graph were obtained, namely those dealing with its edges, chains, shadows, neighbors and degrees of its vertices, and some relevant subgraphs, as well as the natural isomorphisms between them. In [17], the n-dimensional cube-connected complete graph is studied. In [22], the linear $(n-1)$-arboricity of $K_{n(m)}$ is obtained. In [23, 24], the hamiltonicity, path t-coloring, and the shortest paths of Sierpińiski-like graphs are researched. In [25], the vertex arboricity of integer distance graph $G(D_{m,k})$ is obtained. In [26], it is obtained that $la_4(K_{n,n}) = \lceil 5n/8 \rceil$ for $n \equiv 0(\mod 5)$.

Multi-level distance labeling (or radio labeling) is motivated by the channel assignment problem introduced by Hale[1]. Given a set of stations (or transmitters) in a communication network, a valid channel assignment is a function that assigns to each station with a channel (nonnegative integer) such that interference is avoided. The level of interference is related to the locations of the stations–the closer the two stations, the stronger the interference that might occur. In order to avoid interference, the separation between the channels assigned a pair of near-by stations must be large enough, and the amount of the required separation depends on the distance between the two stations. The task is to find a valid channel assignment with the minimum span of channels used.

A graph model for this problem is to represent each station by a vertex, and connect any pair of close stations by an edge. A multi-level distance labeling (radio labeling) of a connected graph G is a function $f : V(G) \to \{0, 1, 2, \cdots\}$, such that for any $u, v \in V(G)$,

$$|f(u) - f(v)| \geq diam(G) + 1 - d(u, v),$$

where $diam(G)$ is the diameter (the maximum distance over all pairs of vertices) of G. The span of f is defined as $\max\{f(u) - f(v) \mid u, v \in V(G)\}$. The multi-level distance number for a graph G, denoted by $rn(G)$, is the minimum span of all multi-level distance labeling for G. Multi-level distance labeling is a generalization of the distance-two labeling which has been studied extensively ([2]-[12]), and multi-level distance labeling can be better to reflect the nature of radio channels assignment. In [14, 15, 18, 19], it was studied that the multi-level distance labeling of paths and cycles, square of paths and square of cycles, and in [13], it was determined the radio number of the complete m-ary tree. In [16], it was studied the multi-level distance labeling of trees, and got a lower bound for trees' radio number.

Let T be a tree rooted at a vertex r. For any two vertices u and v, if u is on the (r, v)-path, then u is called an ancestor of v, and v is called a descendent of u. Define the level function on $V(T)$ by $l_r(u) = d(r, u)$ for any $u \in V(T)$. For any $u, v \in V(T)$, define

$$\varphi(u, v) = \max\{l_r(t) : t \text{ is a common ancestor of u and v}\}.$$

For each vertex w in a tree T, the weight of T rooted at w is defined by

$$\omega_T(w) = \sum_{u \in V(T)} l_w(u).$$

The weight of T is the smallest weight among all possible roots of T

$$\omega(T) = \min_{w \in V(T)} \{\omega_T(w)\}.$$

A vertex w' is called a weight center of T if $\omega_T(w') = \omega(T)$. It can be abbreviated as $l(u) = l_{w'}(u)$ if there is no confusion. It is obvious that the weight center cut the tree into a number of branches.

*Manuscript received Dec 23,2014. This work was supported by NSFC for youth with code 61103073. Email:lczuo@163.com. Liancui Zuo and Chunhong Shang are with College of Mathematical Science, Tianjin Normal University, Tianjin, 300387, China; Hongfang Guo is with Department of Mathematics, Institute of Technology, East China Normal University, Shanghai,200241,China.

†The corresponding author:shangchunhong@eyou.com

(Advance online publication: 17 February 2015)

Lemma 1.1. *[16] Suppose that w' is the weight center of T. For any $u, v \in V(T)$, the following two conclusions hold:*

(1) $d(u,v) = l(u) + l(v) - 2\varphi(u,v)$, and

(2) $\varphi(u,v) = 0$ if and only if u and v belong to different branches (unless one of them is w').

Lemma 1.2. *[16] Let T be an n-vertex tree with diameter d. Then*

$$rn(T) \geq (n-1)(d+1) + 1 - 2\omega(T).$$

An arrangement for $V(G)$ can be derived from the radio labeling f, denoted by $V(G) = U(f) = \{u_0, u_1, \cdots, u_{|V|-1}\}$, which satisfies

$$0 = f(u_0) < f(u_1) < f(u_2) < \cdots < f(u_{|V|-1}). \quad (1)$$

If f is a radio labelling, then the span of f is $f(u_{|V|-1})$.

If deleting all suspension vertices and associated edges of T we get a road or an isolated vertex, then we call T a "caterpillar". If deleting all suspension vertices and associated edges of T we get a caterpillar, then we call T a "lobster tree". In 2009, Guo and Zuo gave the exact value of multi-level distance number for a special class of caterpillars in [20]. In 2011, Hou and Zuo [21] obtained the exact value of the multi-level distance number of a class of symmetric lobster trees about weight center. In [13], it was given the following concept.

Definition 1.3. *Let f be a multi-level distance labeling for G, and the vertices of G about f have the sequence as (1). For every $0 \leq i \leq |V|-2$, let*

$$J_f(u_i, u_{i+1}) = f(u_{i+1}) - f(u_i) - [diam(G) + 1 - d(u_i, u_{i+1})].$$

We call $J_f(u_i, u_{i+1})$ a k-jump from u_i to u_{i+1} if $J_f(u_i, u_{i+1}) = k \geq 0$ and say that f has a k-jump from u_i to u_{i+1}. Define the total number of jumps as

$$J(f) = \sum_{i=0}^{|V|-2} J_f(u_i, u_{i+1}).$$

If deleting all suspension vertices and edges associated of T we can get a lobster tree, then we call it a "lobster-like tree". In the present paper, we mainly study the multi-level distance labeling of lobster-like trees, and obtain the multi-level distance number for a special class of lobster-like trees.

2 A lower bound of the radio number of a class of symmetric lobster-like trees about weight center

In the following, we denote a lobster-like tree's all suspension vertices as the C layer points, the corresponding lobster tree's all suspension vertices as the B layer points, and all suspension vertices of its corresponding caterpillar as A layer points.

Let $T = (t_4, t_5, t_6, \cdots, t_i, \cdots, t_{k-4}, t_{k-3})$, and

$$R = (r_{2,0}, r_{3,0}, r_{4,1}, r_{4,2}, \cdots, r_{4,2t_4-4}, \cdots, r_{i,j}, \cdots, r_{k-3,1}, \cdots, r_{k-3,2t_{k-3}-4}, r_{k-2,0}, r_{k-1,0}),$$

where $k \geq 7, t_i \geq 3, r_{2,0}, r_{3,0}, r_{i,j}, r_{k-2,0}, r_{k-1,0} \geq 1$, $1 \leq j \leq 2t_i - 4$, and $4 \leq i \leq k-3$.

Using symbol $P_{k,T,R}$ to represent the lobster-like tree that the diameter is $k-1$, the degree of the ith vertex $v_{i,0}$ on the longest path P_k is t_i, and the others' degree are

$$r_{2,0}+1, r_{3,0}+1, \cdots, r_{i,j}+1, \cdots, r_{k-2,0}+1, r_{k-1,0}+1,$$

respectively, except the suspension vertices, that is, the degree of $v_{3,0}, v_{k-2,0}$ are $r_{3,0}+1, r_{k-2,0}+1$, respectively, the degree of its descendant vertices that belong to B layers are $r_{2,0}+1, r_{k-1,0}+1$, respectively, and among A layer points, the degree of each $v_{i,j}$ is $r_{i,j}+1$, and the degree of its descendant vertices that belong to B layers is $r_{i,j+t_i-2}+1$ for $1 \leq j \leq t_i - 2$ and $4 \leq i \leq k-3$. Thus the number of vertices of $P_{k,T,R}$ is

$$|V(P_{k,T,R})| = \sum_{i=4}^{k-3}[\sum_{j=1}^{t_i-2} r_{i,j}(r_{i,j+t_i-2}+1) + (t_i - 2)] + (r_{2,0}+1)r_{3,0} + (r_{k-2,0}+1)r_{k-1,0} + k - 4. \quad (2)$$

If $r_{2,0} = r_{3,0} = r_{i,j} = r_{k-2,0} = r_{k-1,0} = r(\geq 1)$, we will denote the lobster-like tree that the degree of all vertices are $r+1$ except for the $k-6$ vertices in the middle of P_k and the suspension vertices as $P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3}),r}$. If $r = 1$, then it is denoted by $P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}$, and we have

$$A = \{(v_{3,0}, v_{i,j}, v_{k-2,0}) \mid 1 \leq j \leq t_i - 2, 4 \leq i \leq k-3\},$$

$$B = \{(v_{2,0}, v_{i,j}, v_{k-1,0}) \mid t_i - 1 \leq j \leq 2t_i - 4, 4 \leq i \leq k-3\},$$

and

$$C = \{(v_{1,0}, v_{i,j}, v_{k,0}) \mid 2t_i - 3 \leq j \leq 3t_i - 6, 4 \leq i \leq k-3\}.$$

In this paper, we mainly study the multi-level distance number of $P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}$. Note that $diam(P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}) = k-1$.

Define the vertices of $P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}$ successively $v_{1,0}, v_{2,0}, \cdots, v_{k,0}$, $v_{i,p} (1 \leq p \leq t_i - 2, 4 \leq i \leq k-3)$ is the pth vertex of $v_{i,0}$ that belong to A layer, $v_{i,q} (t_i - 1 \leq q \leq 2t_i - 4, 4 \leq i \leq k-3)$ is the $(q - t_i + 2)$th vertex of $v_{i,0}$ that belong to B layer, and $v_{i,s}$ $(2t_i - 3 \leq s \leq 3t_i - 6, 4 \leq i \leq k-3)$ is the $(s - 2t_i + 4)$th vertex of $v_{i,0}$ that belong to C layer. Please see Fig. 1.

If $P_{k,T,R}$ is symmetric about the weight center, then $v_{\frac{k+1}{2},0}$ is the weight center when k is odd, $v_{i,0} (1 \leq i \leq \frac{k-1}{2})$ and its descendent vertices and the associated edges are called "upper branch" of $P_{k,T,R}$, $v_{i,0} (\frac{k+3}{2} \leq i \leq k)$ and its descendent vertices and the associated edges are called "lower branch" of $P_{k,T,R}$, and $v_{\frac{k+1}{2},0}$ and its descendent vertices and the associated edges are called "middle" of $P_{k,T,R}$. "Middle" belong to both the upper branch and lower branch. In this paper, we always assume that they belong to the upper branch. To distinguish them from the upper branch vertices and refer to them as "upper middle" of $P_{k,T,R}$. When k is even, both $v_{\frac{k}{2},0}$ and $v_{\frac{k}{2}+1,0}$ are the weight centers. In this paper, we select $v_{\frac{k}{2},0}$ for the weight center for even k, $v_{i,0}$ $(1 \le i \le \frac{k}{2})$ and its descendent vertices and the associated edges are called "upper branch" of $P_{k,T,R}$, $v_{i,0}$ $(\frac{k}{2}+1 \le i \le k)$ and its descendent vertices and the associated edges are called "lower branch" of $P_{k,T,R}$, $v_{\frac{k}{2},0}$ and its descendent vertices and the associated edges are called "upper middle" of $P_{k,T,R}$, and $v_{\frac{k}{2}+1,0}$ and its descendent vertices and the associated edges are called "lower middle" of $P_{k,T,R}$.

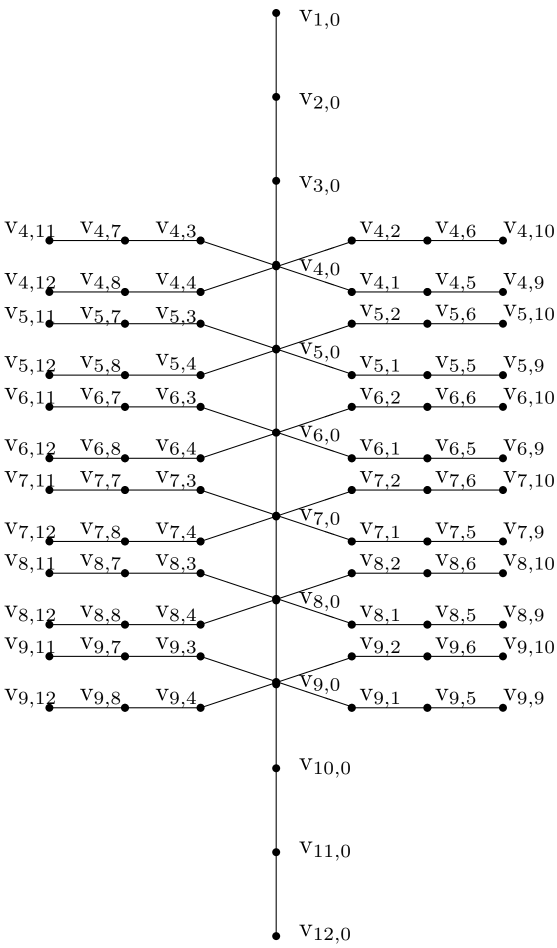

Figure 1. A special symmetric lobster-like tree

Theorem 2.1. *Let $G = P_{k,T,R}$ be a symmetric lobster-like tree about the weight center. By direct calculation, we can obtain that*

$$|V(G)| = \begin{cases} 2\sum_{i=4}^{\frac{k-1}{2}}[\sum_{j=1}^{t_i-2} r_{i,j}(r_{i,j+t_i-2}+1)+(t_i-1)] \\ + \sum_{l=1}^{t_{\frac{k+1}{2}}-2} r_{\frac{k+1}{2},l}(r_{\frac{k+1}{2},l+t_{\frac{k+1}{2}}-2}+1) \\ +t_{\frac{k+1}{2}}+2r_{2,0}[r_{3,0}+1]+1 \quad for\ odd\ k \ge 9, \\ 2\sum_{i=4}^{\frac{k}{2}}[\sum_{j=1}^{t_i-2} r_{i,j}(r_{i,j+t_i-2}+1)+(t_i-1)] \\ +2r_{2,0}(r_{3,0}+1)+2 \quad for\ even\ k, \end{cases}$$

and

$$\omega(G) = \begin{cases} \sum_{i=4}^{\frac{k-1}{2}}\{\sum_{j=1}^{t_i-2} r_{i,j}[(k+7-2i)r_{i,j+t_i-2} \\ +(k+5-2i)]+(k+3-2i)(t_i-1)\} \\ + \sum_{l=1}^{t_{\frac{k+1}{2}}-2} r_{\frac{k+1}{2},l}(3r_{\frac{k+1}{2},l+t_{\frac{k+1}{2}}-2}+2) \\ +t_{\frac{k+1}{2}}+[(k-1)r_{3,0}+(k-3)]r_{2,0} \\ \quad for\ odd\ k \ge 9, \\ \sum_{i=4}^{\frac{k}{2}}\{\sum_{j=1}^{t_i-2} r_{i,j}[(k+7-2i)r_{i,j+t_i-2} \\ +(k+5-2i)]+(k+3-2i)(t_i-1)\} \\ +[(k-1)r_{3,0}+(k-3)]r_{2,0}+1 \\ \quad for\ even\ k. \end{cases}$$

Similarly, the following conclusions can be obtained.

Corollary 2.2. *Let $G = P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3}),r}$. Then*

$$|V(G)| = \begin{cases} [2\sum_{i=4}^{\frac{k-1}{2}}(t_i-2)+t_{\frac{k+1}{2}}](r^2+r+1) \\ +k-6 \quad for\ odd\ k \ge 9, \\ [2\sum_{i=4}^{\frac{k}{2}}(t_i-2)+2](r^2+r+1) \\ +k-6 \quad for\ even\ k, \end{cases}$$

and

$$\omega(G) = \begin{cases} \sum_{i=4}^{\frac{k-1}{2}}\{[(k+7-2i)r^2+(k+5-2i)r](t_i-2) \\ +(k+3-2i)(t_i-1)\} \\ +(t_{\frac{k+1}{2}}-2)(3r^2+2r)+t_{\frac{k+1}{2}} \\ +(k-1)r^2+(k-3)r \quad for\ odd\ k \ge 9, \\ \sum_{i=4}^{\frac{k}{2}}\{[(k+7-2i)r^2+(k+5-2i)r](t_i-2) \\ +(k+3-2i)(t_i-1)\}+(k-1)r^2 \\ +(k-3)r+1 \quad for\ even\ k. \end{cases}$$

Corollary 2.3. *Let $G = P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}$. Then*

$$|V(G)| = \begin{cases} 6\sum_{i=4}^{\frac{k-1}{2}}(t_i-2)+3t_{\frac{k+1}{2}}+k-6 \\ \quad for\ odd\ k \ge 9, \\ 6\sum_{i=4}^{\frac{k}{2}}(t_i-2)+k \quad for\ even\ k, \end{cases}$$

and

$$\omega(G) = \begin{cases} \sum_{i=4}^{\frac{k-1}{2}}[(3k+15-6i)(t_i-2)]+6t_{\frac{k+1}{2}} \\ +\frac{k^2}{4}-\frac{49}{4} \quad for\ odd\ k \ge 9, \\ \sum_{i=4}^{\frac{k}{2}}[(3k+15-6i)(t_i-2)]+\frac{k^2}{4} \quad for\ even\ k. \end{cases}$$

By Lemma 1.2 and equation (2), it is easy to verify that

$$rn(G) \ge (3k-12)t_{\frac{k+1}{2}}+k^2-7k+1 \ for\ k=7. \quad (3)$$

By Theorem 2.1, Corollaries 2.2-2.3, and Lemma 1.2, we have the following conclusions.

Theorem 2.4. *Let $G = P_{k,T,R}$ be a symmetric lobster-like tree about the weight center. Then*

$$rn(G) \geq \begin{cases} \sum_{i=4}^{\frac{k-1}{2}} \{ \sum_{j=1}^{t_i-2} r_{i,j}[(4i-14)r_{i,j+t_i-2} + (4i-10)] \\ +(4i-6)(t_i-1)\} \\ + \sum_{l=1}^{t_{\frac{k+1}{2}}-2} r_{\frac{k+1}{2},l}[(k-6)r_{\frac{k+1}{2},l+t_{\frac{k+1}{2}}-2} \\ +(k-4)] + (k-2)t_{\frac{k+1}{2}} \\ +2r_{2,0}(r_{3,0}+3)+1 \quad \text{for odd } k \geq 9, \\ \sum_{i=4}^{\frac{k}{2}} \{ \sum_{j=1}^{t_i-2} r_{i,j}[(4i-14)r_{i,j+t_i-2} + (4i-10)] \\ +(4i-6)(t_i-1)\} + 2r_{2,0}(r_{3,0}+3) \\ +k-1 \quad \text{for even } k. \end{cases}$$

Theorem 2.5. *Let $G = P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3}),r}$. Then*

$$rn(G) \geq \begin{cases} \sum_{i=4}^{\frac{k-1}{2}} \{[(4i-14)r^2 + (4i-10)r](t_i-2) \\ +(4i-6)(t_i-1)\} \\ +[(k-6)r^2 + (k-4)r](t_{\frac{k+1}{2}}-2) \\ +(k-2)t_{\frac{k+1}{2}} + 2r^2 + 6r + 1 \\ \quad \text{for odd } k \geq 9, \\ \sum_{i=4}^{\frac{k}{2}} \{[(4i-14)r^2 + (4i-10)r](t_i-2) \\ +(4i-6)(t_i-1)\} \\ +2r^2+6r+k-1 \quad \text{for even } k. \end{cases}$$

Theorem 2.6. *Let $G = P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}$. Then*

$$rn(G) \geq \begin{cases} 6\sum_{i=4}^{\frac{k-1}{2}} [(2i-5)(t_i-2)] + (3k-12)t_{\frac{k+1}{2}} \\ +\frac{1}{2}(k-7)^2+1 \quad \text{for odd } k \geq 9, \\ 6\sum_{i=4}^{\frac{k}{2}} [(2i-5)(t_i-2)] \\ +\frac{k^2}{2}-k+1 \quad \text{for even } k. \end{cases}$$

In order to improve the lower bound of the multi-level distance number of G for odd $k \geq 9$, we give the following lemma.

Lemma 2.7. *Suppose that $G = P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}$, f is a one-to-one non-negative integer function on $V(G)$, and the vertices in G about f have the sequence as (1). Then f is a radio labeling of G if and only if for any consecutive subset of vertices $\{u_i, u_{i+1}, \cdots, u_j\}$, $0 \leq i < j \leq |V|-1$, the following results hold:*

(1) if u_i, u_j belong to different branches of G, then

$$\sum_{t=i}^{j-1} [J_f(u_t,u_{t+1}) + 2\varphi(u_t,u_{t+1})] \geq 2\sum_{t=i+1}^{j-1} l(u_t) - k(j-i-1),$$

(2) If u_i, u_j belong to the same branches of G, then

(i)when any one isn't the ancestor of another one, we have

$$\sum_{t=i}^{j-1} [J_f(u_t,u_{t+1}) + 2\varphi(u_t,u_{t+1})] \geq \begin{cases} 2\sum_{t=i+1}^{j-1} l(u_t) - k(j-i-1) \\ +2\min\{l(u_i)-1, l(u_j)-1\} \\ \quad if\, u_i \in A \text{ or } deg(u_i) = t_i, u_j \in A, \\ 2\sum_{t=i+1}^{j-1} l(u_t) - k(j-i-1) \\ +2\min\{l(u_i)-1, l(u_j)-2\} \\ \quad if\, u_i \in A \text{ or } deg(u_i) = t_i, u_j \in B, \\ 2\sum_{t=i+1}^{j-1} l(u_t) - k(j-i-1) \\ +2\min\{l(u_i)-1, l(u_j)-3\} \\ \quad if\, u_i \in A \text{ or } deg(u_i) = t_i, u_j \in C, \\ 2\sum_{t=i+1}^{j-1} l(u_t) - k(j-i-1) \\ +2\min\{l(u_i)-2, l(u_j)-2\} \quad if\, u_i, u_j \in B, \\ 2\sum_{t=i+1}^{j-1} l(u_t) - k(j-i-1) \\ +2\min\{l(u_i)-2, l(u_j)-3\}\, if\, u_i \in B, u_j \in C, \\ 2\sum_{t=i+1}^{j-1} l(u_t) - k(j-i-1) \\ +2\min\{l(u_i)-3, l(u_j)-3\} \quad if\, u_i, u_j \in C, \end{cases}$$

where u_i and u_j may exchange their positions.

(ii) when one vertex is the ancestor of another one, we have

$$\sum_{t=i}^{j-1} [J_f(u_t,u_{t+1}) + 2\varphi(u_t,u_{t+1})] \geq 2\sum_{t=i+1}^{j-1} l(u_t) - k(j-i-1) + 2\min\{l(u_i), l(u_j)\},$$

Proof. Suppose that f is a multi-level distance labeling of G with $diam(G) = k-1$. For any $0 \leq i < j \leq |V|-1$, add up the following equations

$$J_f(u_t,u_{t+1}) + 2\varphi(u_t,u_{t+1}) = f(u_{t+1}) - f(u_t) + l(u_{t+1}) + l(u_t) - diam(G) - 1, \quad i \leq t \leq j-1,$$

we obtain that

$$\sum_{t=i}^{j-1} [J_f(u_t,u_{t+1}) + 2\varphi(u_t,u_{t+1})] = f(u_j) - f(u_i) - k(j-i) + 2[\sum_{t=i+1}^{j-1} l(u_t)] + l(u_i) + l(u_j).$$

By the definition of f, we have

$$f(u_j) - f(u_i) \geq k - l(u_i) - l(u_j),$$

and then the condition (1) holds.

Since

$$f(u_j) - f(u_i) = k(j-i) - 2[\sum_{t=i+1}^{j-1} l(u_t)] - l(u_i) - l(u_j) + \sum_{t=i}^{j-1} [J_f(u_t,u_{t+1}) + 2\varphi(u_t,u_{t+1})]$$

$$\geq \begin{cases} k-l(u_i)-l(u_j)+2\min\{l(u_i),l(u_j)\} \\ \qquad if\ u_i\ (u_j)\ is\,the\,ancestor\,of\ u_j\ (u_i), \\ k-l(u_i)-l(u_j)+2\min\{l(u_i)-1,l(u_j)-1\} \\ \qquad if\ u_i\in A\,or\,deg(u_i)=t_i, u_j\in A, \\ k-l(u_i)-l(u_j)+2\min\{l(u_i)-1,l(u_j)-2\} \\ \qquad if\ u_i\in A\,or\,deg(u_i)=t_i, u_j\in B, \\ k-l(u_i)-l(u_j)+2\min\{l(u_i)-1,l(u_j)-3\} \\ \qquad if\ u_i\in A\,or\,deg(u_i)=t_i, u_j\in C, \\ k-l(u_i)-l(u_j)+2\min\{l(u_i)-2,l(u_j)-2\} \\ \qquad if\ u_i,u_j\in B, \\ k-l(u_i)-l(u_j)+2\min\{l(u_i)-2,l(u_j)-3\} \\ \qquad if u_i\in B, u_j\in C, \\ k-l(u_i)-l(u_j)+2\min\{l(u_i)-3,l(u_j)-3\} \\ \qquad if u_i,u_j\in C, \end{cases}$$

the condition (2) holds.

Assume that f satisfies conditions (1) and (2). If u_i, u_j are in different branches, then

$$d(u_i,u_j)=l(u_i)+l(u_j),$$

and

$$f(u_j)-f(u_i)\geq k-l(u_i)-l(u_j)=k-d(u_i,u_j).$$

If u_i, u_j are in a same branch, then

$d(u_i,u_j)=$

$$\begin{cases} l(u_i)+l(u_j)-2\min\{l(u_i),l(u_j)\} \\ \qquad if\ u_i\ (u_j)\ is\ the\ ancestor\ of\ u_j\ (u_i), \\ l(u_i)+l(u_j)-2\min\{l(u_i)-1,l(u_j)-1\} \\ \qquad if\ u_i\in A\ or\ deg(u_i)=t_i, u_j\in A, \\ l(u_i)+l(u_j)-2\min\{l(u_i)-1,l(u_j)-2\} \\ \qquad if\ u_i\in A\ or\ deg(u_i)=t_i, u_j\in B, \\ l(u_i)+l(u_j)-2\min\{l(u_i)-1,l(u_j)-3\} \\ \qquad if\ u_i\in A\ or\ deg(u_i)=t_i, u_j\in C, \\ l(u_i)+l(u_j)-2\min\{l(u_i)-2,l(u_j)-2\} \\ \qquad if u_i,u_j\in B, \\ l(u_i)+l(u_j)-2\min\{l(u_i)-2,l(u_j)-3\} \\ \qquad if u_i\in B, u_j\in C, \\ l(u_i)+l(u_j)-2\min\{l(u_i)-3,l(u_j)-3\} \\ \qquad if u_i,u_j\in C, \end{cases}$$

and thus $f(u_j)-f(u_i)\geq k-d(u_i,u_j)$. Hence f is a multi-level distance labeling of G. □

Now we revise the lower bound of the multi-level distance number for the symmetric lobster-like tree about weight center in Theorem 2.6 for odd $k\geq 9$.

Theorem 2.8. *Let $G=P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}$ be a symmetric lobster-like tree about weight center. For odd $k\geq 9$ and $t_4=t_{\frac{k+1}{2}}=t_{k-3}$, we have*

$$\begin{aligned} rn(G)\geq\ & 6\sum_{i=4}^{\frac{k-1}{2}}[(2i-5)(t_i-2)] \\ & +(3k-12)t_{\frac{k+1}{2}}+\tfrac{1}{2}(k-7)^2+2. \end{aligned}$$

Proof. Let f be a radio labeling of G, and the vertices in G about f have the sequence as (1). By Definition 1.3 and Lemma 2.7, we have

$$\begin{aligned} & f(u_{|V|-1})=\sum_{i=0}^{|V|-2}[f(u_{i+1})-f(u_i)] \\ & =k(|V|-1)-\sum_{i=0}^{|V|-2}d(u_i,u_{i+1})+\sum_{i=0}^{|V|-2}J_f(u_i,u_{i+1}) \\ & =k(|V|-1)-2\omega(G)+l(u_0)+l(u_{|V|-1})+\sigma(f) \\ & \geq k[6\sum_{i=4}^{\frac{k-1}{2}}(t_i-2)+3t_{\frac{k+1}{2}}+k-7]+1+\sigma(f) \\ & -2\{\sum_{i=4}^{\frac{k-1}{2}}[(3k+15-6i)(t_i-2)]+6t_{\frac{k+1}{2}}+\tfrac{k^2}{4}-\tfrac{49}{4}\} \\ & =6\sum_{i=4}^{\frac{k-1}{2}}[(2i-5)(t_i-2)]+(3k-12)t_{\frac{k+1}{2}} \\ & \quad+\tfrac{1}{2}(k-7)^2+1+\sigma(f), \end{aligned}$$

where $\sigma(f)=\sum_{i=0}^{|V|-2}[J_f(u_i,u_{i+1})+2\varphi(u_i,u_{i+1})]$. The weights of all vertices appear twice except for $l(u_0)$ and $l(u_{|V|-1})$. Note that $l(u_i)\geq 0$ for $0\leq i\leq |V|-1$. Therefore, only if $u_0=v_{\frac{k+1}{2},0}$ and the distance from $u_{|V|-1}$ to u_0 is one, that is, $l(u_0)=0$ and $l(u_{|V|-1})=1$, the right-hand side of above formulae gets its minimum value.

Claim 1. Under the condition $t_4=t_{\frac{k+1}{2}}=t_{k-3}$, there must exist a vertex

$$u_i\in\{v_{1,0},v_{4,j},v_{k-3,j},v_{k,0}|2t_4-3\leq j\leq 3t_4-6\}$$

such that $u_{i-1},u_{i+1}\neq v_{\frac{k+1}{2},q}$, where $2t_{\frac{k+1}{2}}-3\leq q\leq 3t_{\frac{k+1}{2}}-6$.

Assume that every vertex

$$u_i\in\{v_{1,0},v_{4,j},v_{k-3,j},v_{k,0}|2t_4-3\leq j\leq 3t_4-6\}$$

for $G=P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}(k\geq 9)$ is adjacent to one vertex $v_{\frac{k+1}{2},q}$ for some $2t_{\frac{k+1}{2}}-3\leq q\leq 3t_{\frac{k+1}{2}}-6$, then

$$t_{\frac{k+1}{2}}-2\geq\frac{1}{2}(2t_{\frac{k+1}{2}}-4+2)=t_{\frac{k+1}{2}}-1,$$

a contradiction. Hence the claim holds.

Claim 2. Suppose that u_i satisfies Claim 1, then

$$\sigma(f)=\sum_{t=0}^{|V|-2}[J_f(u_t,u_{t+1})+2\varphi(u_t,u_{t+1})]\geq 1.$$

Consider a consecutive subset of vertices $\{u_{i-1},u_i,u_{i+1}\}$ with $2\leq i\leq |V|-2$. Then it is clear that there are two vertices belong to the same branch.

(1) (i) If u_{i-1},u_{i+1} belong to the same branch of G and any one vertex isn't the ancestor of the other one, then

$$\begin{aligned} & \sigma(f)=\sum_{t=0}^{|V|-2}[J_f(u_t,u_{t+1})+2\varphi(u_t,u_{t+1})] \\ & \geq\sum_{t=i-1}^{i}[J_f(u_t,u_{t+1})+2\varphi(u_t,u_{t+1})] \end{aligned}$$

$$\geq \begin{cases} 2l(u_t)-k+2\min\{l(u_i)-1,l(u_j)-1\}, \\ \quad if\ u_i\in A\ or\ \deg(u_i)=t_i, u_j\in A \\ 2l(u_t)-k+2\min\{l(u_i)-1,l(u_j)-2\}, \\ \quad if\ u_i\in A\ or\ \deg(u_i)=t_i, u_j\in B, \\ 2l(u_t)-k+2\min\{l(u_i)-1,l(u_j)-3\}, \\ \quad if\ u_i\in A\ or\ \deg(u_i)=t_i, u_j\in C. \\ 2l(u_t)-k+2\min\{l(u_i)-2,l(u_j)-2\}, \\ \quad if\ u_i,u_j\in B, \\ 2l(u_t)-k+2\min\{l(u_i)-2,l(u_j)-3\}, \\ \quad if\ u_i\in B, u_j\in C, \\ 2l(u_t)-k+2\min\{l(u_i)-3,l(u_j)-3\}, \\ \quad if\ u_i,u_j\in C, \end{cases}$$
$$\geq 2\cdot\tfrac{k-1}{2}-k+2=1.$$

Note that the upper result also holds when we exchange the positions of u_i and u_j.

(ii) If u_{i-1}, u_{i+1} belong to the same branch of G and one vertex is the ancestor of another one, then

$$\begin{aligned}\sigma(f) &= \sum_{t=0}^{|V|-2}[J_f(u_t,u_{t+1})+2\varphi(u_t,u_{t+1})] \\ &\geq \sum_{t=i-1}^{i}[J_f(u_t,u_{t+1})+2\varphi(u_t,u_{t+1})] \\ &\geq 2l(u_i)-k+2\min\{l(u_{i-1}),l(u_{i+1})\} \\ &\geq 2\cdot\tfrac{k-1}{2}-k+2=1.\end{aligned}$$

(2) (i) If u_{i-1}, u_i belong to the same branch of G and one vertex is the ancestor of the other one, then

$$\begin{aligned}\sigma(f) &= \sum_{t=0}^{|V|-2}[J_f(u_t,u_{t+1})+2\varphi(u_t,u_{t+1})] \\ &\geq J_f(u_{i-1},u_i)+2\varphi(u_{i-1},u_i) \\ &\geq 2\min\{l(u_{i-1}),l(u_i)\}\geq 2.\end{aligned}$$

(ii) If u_{i-1}, u_i belong to the same branch of G and any one vertex isn't the ancestor of the another one, then

$$\sigma(f) = \sum_{t=0}^{|V|-2}[J_f(u_t,u_{t+1})+2\varphi(u_t,u_{t+1})]$$
$$\geq J_f(u_{i-1},u_i)+2\varphi(u_{i-1},u_i)$$
$$\geq \begin{cases} 2\min\{l(u_i)-1,l(u_j)-1\}, \\ \quad if\ u_i\in A\ or\ \deg(u_i)=t_i, u_j\in A \\ 2\min\{l(u_i)-1,l(u_j)-2\}, \\ \quad if\ u_i\in A\ or\ \deg(u_i)=t_i, u_j\in B, \\ 2\min\{l(u_i)-1,l(u_j)-3\}, \\ \quad if\ u_i\in A\ or\ \deg(u_i)=t_i, u_j\in C, \\ 2\min\{l(u_i)-2,l(u_j)-2\}, \\ \quad if\ u_i,u_j\in B, \\ 2\min\{l(u_i)-2,l(u_j)-3\}, \\ \quad if\ u_i\in B, u_j\in C, \\ 2\min\{l(u_i)-3,l(u_j)-3\}, \\ \quad if\ u_i,u_j\in C. \end{cases}$$
$$\geq 2,$$

where the positions of u_i and u_j may exchange.

(3) For the same reason as (2), if u_i, u_{i+1} belong to the same branch, then we have

$$\begin{aligned}\sigma(f) &= \sum_{t=0}^{|V|-2}[J_f(u_t,u_{t+1})+2\varphi(u_t,u_{t+1})] \\ &\geq J_f(u_i,u_{i+1})+2\varphi(u_i,u_{i+1})\geq 2.\end{aligned}$$

Thus Claim 2 holds.

By arbitrary of f, we have

$$\begin{aligned}rn(G)\geq\ & 6\sum_{i=4}^{\frac{k-1}{2}}[(2i-5)(t_i-2)] \\ &+(3k-12)t_{\frac{k+1}{2}}+\tfrac{1}{2}(k-7)^2+2.\end{aligned}$$

□

3 The radio number of a class of symmetric lobster-like trees about weight center

The radio number of $G=P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}$ is given below.

Theorem 3.1. *Let $G=P_{k,(t_4,\cdots,t_i,\cdots,t_{k-3})}$ which satisfies*

(i) all t_i have the same parity,

(ii) $t_4=t_{\frac{k+1}{2}}=t_{k-3}$ and $t_{\frac{k+1}{2}-j}\geq t_{4+j}$ for $1\leq j\leq\lfloor\frac{k-7}{4}\rfloor$, when k is odd,

(iii) $t_{\frac{k}{2}}\geq\frac{1}{2}(t_4-2)+2$ and $t_{\frac{k}{2}-j}\geq t_{4+j}$ for $1\leq j\leq\lfloor\frac{k-8}{4}\rfloor$ if both k and t_i are even, and $t_{\frac{k}{2}}\geq\frac{1}{2}(t_4-1)+2$ and $t_{\frac{k}{2}-j}\geq t_{4+j}$ for $1\leq j\leq\lfloor\frac{k-8}{4}\rfloor$ if k is even and t_i is odd.

Then

$$rn(G)=\begin{cases}(3k-12)t_{\frac{k+1}{2}}+k^2-7k+1 & for\ k=7, \\ 6\sum_{i=4}^{\frac{k-1}{2}}[(2i-5)(t_i-2)]+(3k-12)t_{\frac{k+1}{2}} \\ +\frac{1}{2}(k-7)^2+2 & for\ odd\ k\geq 9, \\ 6\sum_{i=4}^{\frac{k}{2}}[(2i-5)(t_i-2)]+\frac{k^2}{2}-k+1 \\ & for\ even\ k.\end{cases}$$

Proof. By Theorems 2.6, 2.8, and equation (3), it is only need to prove the opposite inequality. Rearrange the sequence of vertices of G as $V(G)=U(f)=\{u_0,u_1,\cdots,u_{|V|-1}\}$. In the following, we will use an algorithm to construct a multi-level distance labeling f. The symbol $\rightarrow(l)$ indicates that the jump between the two successive vertices u_i and u_{i+1} is l, that is, $J_f(u_i,u_{i+1})=l$, and the symbol $\rightarrow$ indicates that the jump between the two successive vertices u_i and u_{i+1} is 0, that is, $J_f(u_i,u_{i+1})=0$. Let $f(u_0)=0$, and

$f(u_{i+1})=f(u_i)+diam(G)+1-d(u_i,u_{i+1})+J_f(u_i,u_{i+1})$ for $0\leq i\leq|V|-2$.

Case 1. If k is even, t_i is odd, $t_{\frac{k}{2}}\geq\frac{1}{2}(t_4-1)+2$ and $t_{\frac{k}{2}-j}\geq t_{4+j}$ for $1\leq j\leq\lfloor\frac{k-8}{4}\rfloor$, then the algorithm is defined as following:

$$\begin{aligned}&u_0=v_{\frac{k}{2},0}\rightarrow v_{k,0}\rightarrow v_{1,0}\rightarrow v_{\frac{k}{2}+1,2t_{\frac{k}{2}+1}-3} \\ &\rightarrow v_{4,2t_4-3}\rightarrow v_{k-3,2t_{k-3}-3}\rightarrow v_{\frac{k}{2},2t_{\frac{k}{2}}-3}\rightarrow \\ &v_{k-3,2t_{k-3}-2}\rightarrow v_{4,2t_4-2}\rightarrow v_{\frac{k}{2}+1,2t_{\frac{k}{2}+1}-2} \\ &\rightarrow\cdots\rightarrow v_{\frac{k}{2}+1,\frac{t_4}{2}+2t_{\frac{k}{2}+1}-\frac{9}{2}}\rightarrow v_{4,3t_4-6}\rightarrow\end{aligned}$$

$v_{k-3,3t_{k-3}-6} \to v_{\frac{k}{2},\frac{t_4}{2}+2t_{\frac{k}{2}}-\frac{9}{2}} \to v_{k-1,0} \to$
$v_{2,0} \to v_{\frac{k}{2}+1,\frac{t_4}{2}+2t_{\frac{k}{2}+1}-\frac{7}{2}} \to v_{\frac{k}{2},\frac{t_4}{2}+2t_{\frac{k}{2}}-\frac{7}{2}}$
$\to \cdots \to v_{\frac{k}{2}+1,3t_{\frac{k}{2}+1}-6} \to v_{\frac{k}{2},3t_{\frac{k}{2}}-6} \to v_{k-2,0}$
$\to v_{\frac{k}{2}-2,0} \to v_{\frac{k}{2}+2,2t_{\frac{k}{2}+2}-3} \to v_{5,2t_5-3} \to \cdots$
$\to v_{\frac{k}{2}+2,3t_5-6} \to v_{5,3t_5-6} \to v_{k-3,0} \to v_{\frac{k}{2}-3,0}$
$\to \cdots \to v_{\lfloor \frac{3k-4}{4} \rfloor,2t_{\lfloor \frac{3k-4}{4} \rfloor}-3}$
$\to v_{\lfloor \frac{k+8}{4} \rfloor,2t_{\lfloor \frac{k+8}{4} \rfloor}-3} \to \cdots \to v_{\lfloor \frac{3k-4}{4} \rfloor,3t_{\lfloor \frac{k+8}{4} \rfloor}-6}$
$\to v_{\lfloor \frac{k+8}{4} \rfloor,3t_{\lfloor \frac{k+8}{4} \rfloor}-6} \to v_{\lfloor \frac{3k+2}{4} \rfloor,0} \to v_{\lfloor \frac{k+2}{4} \rfloor,0}$
$\to v_{\lfloor \frac{3k}{4} \rfloor,2t_{\lfloor \frac{3k}{4} \rfloor}-3} \to v_{\lfloor \frac{k+12}{4} \rfloor,2t_{\lfloor \frac{k+12}{4} \rfloor}-3} \to \cdots$
$\to v_{\lfloor \frac{3k}{4} \rfloor,3t_{\lfloor \frac{3k}{4} \rfloor}-6} \to v_{\lfloor \frac{k+12}{4} \rfloor,3t_{\lfloor \frac{3k}{4} \rfloor}-6}$
$\to v_{\lfloor \frac{3k-2}{4} \rfloor,0} \to v_{\lfloor \frac{k-2}{4} \rfloor,0} \to \cdots \to$
$v_{k-4,2t_{k-4}-3} \to v_{\frac{k}{2}-1,2t_{\frac{k}{2}-1}-3} \to \cdots$
$\to v_{k-4,3t_{k-4}-6} \to v_{\frac{k}{2}-1,3t_{k-4}-6} \to v_{\frac{k}{2}+3,0} \to$
$v_{3,0} \to v_{\frac{k}{2}+2,3t_5-5} \to v_{\lfloor \frac{k+12}{4} \rfloor,3t_{\lfloor \frac{3k}{4} \rfloor}-5} \to \cdots$
$\to v_{\lfloor \frac{3k-4}{4} \rfloor,3t_{\lfloor \frac{3k-4}{4} \rfloor}-6} \to v_{\frac{k}{2}-1,3t_{\frac{k}{2}-1}-6} \to$
$v_{\frac{k}{2}+1,t_{\frac{k}{2}+1}-1} \to v_{4,t_4-1} \to v_{k-3,t_{k-3}-1} \to v_{\frac{k}{2},t_{\frac{k}{2}}-1}$
$\to v_{k-3,t_{k-3}} \to v_{4,t_4} \to v_{\frac{k}{2}+1,t_{\frac{k}{2}+1}} \to \cdots \to$
$v_{\frac{k}{2}+1,\frac{t_4}{2}+t_{\frac{k}{2}+1}-\frac{5}{2}} \to v_{4,2t_4-4} \to v_{k-3,2t_{k-3}-4} \to$
$v_{\frac{k}{2},\frac{t_4}{2}+t_{\frac{k}{2}}-\frac{5}{2}} \to v_{\frac{k}{2}+1,\frac{t_4}{2}+t_{\frac{k}{2}+1}-\frac{3}{2}} \to$
$v_{\frac{k}{2},\frac{t_4}{2}+t_{\frac{k}{2}}-\frac{3}{2}} \to \cdots \to v_{\frac{k}{2}+1,2t_{\frac{k}{2}+1}-4} \to v_{\frac{k}{2},2t_{\frac{k}{2}}-4}$
$\to v_{\frac{k}{2}+2,t_{\frac{k}{2}+2}-1} \to v_{5,t_5-1} \to \cdots \to v_{\frac{k}{2}+2,2t_5-4}$
$\to v_{5,2t_5-4} \to \cdots \to v_{\lfloor \frac{3k-4}{4} \rfloor,t_{\lfloor \frac{3k-4}{4} \rfloor}-1}$
$\to v_{\lfloor \frac{k+8}{4} \rfloor,t_{\lfloor \frac{k+8}{4} \rfloor}-1} \to \cdots \to v_{\lfloor \frac{3k-4}{4} \rfloor,2t_{\lfloor \frac{k+8}{4} \rfloor}-4}$
$\to v_{\lfloor \frac{k+8}{4} \rfloor,2t_{\lfloor \frac{k+8}{4} \rfloor}-4} \to v_{\lfloor \frac{3k}{4} \rfloor,t_{\lfloor \frac{3k}{4} \rfloor}-1}$
$\to v_{\lfloor \frac{k+12}{4} \rfloor,t_{\lfloor \frac{3k}{4} \rfloor}-1} \to \cdots \to v_{\lfloor \frac{3k}{4} \rfloor,2t_{\lfloor \frac{3k}{4} \rfloor}-4}$
$\to v_{\lfloor \frac{k+12}{4} \rfloor,2t_{\lfloor \frac{3k}{4} \rfloor}-4} \to \cdots \to v_{k-4,t_{k-4}-1} \to$
$v_{\frac{k}{2}-1,t_{\frac{k}{2}-1}-1} \to \cdots \to v_{k-4,2t_{k-4}-4} \to v_{\frac{k}{2}-1,2t_{k-4}-4}$
$\to v_{\frac{k}{2}+2,2t_5-3} \to v_{\lfloor \frac{k+12}{4} \rfloor,2t_{\lfloor \frac{3k}{4} \rfloor}-3} \to \cdots$
$\to v_{\lfloor \frac{3k-4}{4} \rfloor,2t_{\lfloor \frac{3k-4}{4} \rfloor}-4} \to v_{\frac{k}{2}-1,2t_{\frac{k}{2}-1}-4} \to v_{\frac{k}{2}+1,1}$
$\to v_{4,1} \to v_{k-3,1} \to v_{\frac{k}{2},1} \to v_{k-3,2}$
$\to v_{4,2} \to v_{\frac{k}{2}+1,2} \to v_{4,3} \to v_{k-3,3} \to$
$\cdots \to v_{\frac{k}{2}+1,\frac{t_4-1}{2}} \to v_{4,t_4-2} \to v_{k-3,t_{k-3}-2} \to$
$v_{\frac{k}{2},\frac{t_4-1}{2}} \to v_{\frac{k}{2}+1,\frac{t_4+1}{2}} \to v_{\frac{k}{2},\frac{t_4+1}{2}} \to v_{\frac{k}{2}+1,\frac{t_4+3}{2}} \to$
$v_{\frac{k}{2},\frac{t_4+3}{2}} \to \cdots \to v_{\frac{k}{2}+1,t_{\frac{k}{2}+1}-2} \to v_{\frac{k}{2},t_{\frac{k}{2}}-2}$
$\to v_{\frac{k}{2}+2,1} \to v_{5,1} \to v_{\frac{k}{2}+2,2} \to v_{5,2} \to \cdots$
$\to v_{\frac{k}{2}+2,t_5-2} \to v_{5,t_5-2} \to \cdots \to v_{\lfloor \frac{3k-4}{4} \rfloor,1}$
$\to v_{\lfloor \frac{k+8}{4} \rfloor,1} \to \cdots \to v_{\lfloor \frac{3k-4}{4} \rfloor,t_{\lfloor \frac{k+8}{4} \rfloor}-2} \to$
$v_{\lfloor \frac{k+8}{4} \rfloor,t_{\lfloor \frac{k+8}{4} \rfloor}-2} \to v_{\lfloor \frac{3k}{4} \rfloor,1} \to v_{\lfloor \frac{k+12}{4} \rfloor,1} \to \cdots \to$
$v_{\lfloor \frac{3k}{4} \rfloor,t_{\lfloor \frac{3k}{4} \rfloor}-2} \to v_{\lfloor \frac{k+12}{4} \rfloor,t_{\lfloor \frac{3k}{4} \rfloor}-2} \to \cdots \to v_{k-4,1}$
$\to v_{\frac{k}{2}-1,1} \to \cdots \to v_{k-4,t_{k-4}-2}$
$\to v_{\frac{k}{2}-1,t_{k-4}-2} \to v_{\frac{k}{2}+2,t_5-1} \to v_{\lfloor \frac{k+12}{4} \rfloor,t_{\lfloor \frac{3k}{4} \rfloor}-1}$
$\to \cdots \to v_{\lfloor \frac{3k-4}{4} \rfloor,t_{\lfloor \frac{3k-4}{4} \rfloor}-2} \to v_{\frac{k}{2}-1,t_{\frac{k}{2}-1}-2}$
$\to v_{\frac{k}{2}+2,0} \to v_{\frac{k}{2}-1,0} \to v_{\frac{k}{2}+1,0}$.

By the definition of f, we know that $\sigma(f) = 0$. Hence

$$f(u_{|V|-1}) = 6\sum_{i=4}^{\frac{k}{2}}(2i-5)(t_i-2) + \frac{k^2}{2} - k + 1.$$

In the following we show that f is a multi-level distance labelling.

Because $2l(u_t) \le k$ for $0 \le t \le |V|-1$, we have

$$2\sum_{t=1}^{|V|-2} l(u_t) - k(|V|-1) \le 0,$$

but

$$\sum_{t=0}^{|V|-2}[J_f(u_t,u_{t+1}) + 2\varphi(u_t,u_{t+1})] \ge 0,$$

so (1) of Lemma 2.1 holds.

In the algorithm above, by definition of f, there must be a consecutive subset of vertices $\{u_{t-1}, u_t, u_{t+1}\}$ for $2 \le t \le |V|-2$ so that u_{t-1}, u_{t+1} belong to the same branch. If one of u_{t-1}, u_{t+1} is the others' ancestor, by the construction of the algorithm, we always have

$$2l(u_t) - k + 2\min\{l(u_{t-1}), l(u_{t+1})\} \le 0$$

holds. Otherwise, we have

$$2l(u_t) - k + 2\min\{l(u_{t-1})-1, l(u_{t+1})-1\} \le 0$$

or

$$2l(u_t) - k + 2\min\{l(u_{t-1})-1, l(u_{t+1})-2\} \le 0$$

or

$$2l(u_t) - k + 2\min\{l(u_{t-1})-1, l(u_{t+1})-3\} \le 0$$

or

$$2l(u_t) - k + 2\min\{l(u_{t-1})-2, l(u_{t+1})-2\} \le 0$$

or

$$2l(u_t) - k + 2\min\{l(u_{t-1})-2, l(u_{t+1})-3\} \le 0$$

or

$$2l(u_t) - k + 2\min\{l(u_{t-1})-3, l(u_{t+1})-3\} \le 0$$

holds, and

$$\sum_{t=0}^{|V|-2}[J_f(u_t,u_{t+1}) + 2\varphi(u_t,u_{t+1})] \ge 0.$$

Therefore, (2) of Lemma 3 holds.

So f is a multi-level distance labeling of G, then

$$rn(G) \le f(u_{|V|-1}) = 6\sum_{i=4}^{\frac{k}{2}}[(2i-5)(t_i-2)] + \frac{k^2}{2} - k + 1.$$

Case 2. If both k and t_i are even, $t_{\frac{k}{2}} \geq \frac{1}{2}(t_4-2)+2$ and $t_{\frac{k}{2}-j} \geq t_{4+j}$ for $1 \leq j \leq \lfloor \frac{k-8}{4} \rfloor$, then the algorithm is defined as following:

$$u_0 = v_{\frac{k}{2},0} \rightarrow v_{k,0} \rightarrow v_{1,0} \rightarrow v_{\frac{k}{2}+1,2t_{\frac{k}{2}+1}-3}$$
$$\rightarrow v_{4,2t_4-3} \rightarrow v_{k-3,2t_{k-3}-3} \rightarrow v_{\frac{k}{2},2t_{\frac{k}{2}}-3}$$
$$\rightarrow v_{k-3,2t_{k-3}-2} \rightarrow v_{4,2t_4-2} \rightarrow v_{\frac{k}{2}+1,2t_{\frac{k}{2}+1}-2}$$
$$\rightarrow \cdots \rightarrow v_{k-3,3t_{k-3}-6} \rightarrow v_{4,3t_4-6} \rightarrow v_{k-1,0}$$
$$\rightarrow v_{2,0} \rightarrow v_{\frac{k}{2}+1,\frac{t_4}{2}+2t_{\frac{k}{2}+1}-4} \rightarrow v_{\frac{k}{2},\frac{t_4}{2}+2t_{\frac{k}{2}}-4}$$
$$\rightarrow v_{\frac{k}{2}+1,\frac{t_4}{2}+2t_{\frac{k}{2}+1}-3} \rightarrow v_{\frac{k}{2},\frac{t_4}{2}+2t_{\frac{k}{2}}-3} \rightarrow \cdots \rightarrow$$
$$v_{\frac{k}{2}+1,3t_{\frac{k}{2}+1}-6} \rightarrow v_{\frac{k}{2},3t_{\frac{k}{2}}-6} \rightarrow v_{k-2,0}$$
$$\rightarrow v_{\frac{k}{2}-2,0} \rightarrow v_{\frac{k}{2}+2,2t_{\frac{k}{2}+2}-3} \rightarrow v_{5,2t_5-3} \rightarrow \cdots \rightarrow$$
$$v_{\frac{k}{2}+2,3t_5-6} \rightarrow v_{5,3t_5-6} \rightarrow v_{k-3,0}$$
$$\rightarrow v_{\frac{k}{2}-3,0} \rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-4}{4} \rfloor,2t_{\lfloor \frac{3k-4}{4} \rfloor}-3} \rightarrow$$
$$v_{\lfloor \frac{k+8}{4} \rfloor,2t_{\lfloor \frac{k+8}{4} \rfloor}-3} \rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-4}{4} \rfloor,3t_{\lfloor \frac{k+8}{4} \rfloor}-6} \rightarrow$$
$$v_{\lfloor \frac{k+8}{4} \rfloor,3t_{\lfloor \frac{k+8}{4} \rfloor}-6} \rightarrow v_{\lfloor \frac{3k+2}{4} \rfloor,0}$$
$$\rightarrow v_{\lfloor \frac{k+2}{4} \rfloor,0} \rightarrow v_{\lfloor \frac{3k}{4} \rfloor,2t_{\lfloor \frac{3k}{4} \rfloor}-3} \rightarrow v_{\lfloor \frac{k+12}{4} \rfloor,2t_{\lfloor \frac{k+12}{4} \rfloor}-3}$$
$$\rightarrow \cdots \rightarrow v_{\lfloor \frac{3k}{4} \rfloor,3t_{\lfloor \frac{3k}{4} \rfloor}-6} \rightarrow v_{\lfloor \frac{k+12}{4} \rfloor,3t_{\lfloor \frac{3k}{4} \rfloor}-6}$$
$$\rightarrow v_{\lfloor \frac{3k-2}{4} \rfloor,0} \rightarrow v_{\lfloor \frac{k-2}{4} \rfloor,0} \rightarrow \cdots \rightarrow v_{k-4,2t_{k-4}-3} \rightarrow$$
$$v_{\frac{k}{2}-1,2t_{\frac{k}{2}-1}-3} \rightarrow \cdots \rightarrow v_{k-4,3t_{k-4}-6} \rightarrow v_{\frac{k}{2}-1,3t_{k-4}-6}$$
$$\rightarrow v_{\frac{k}{2}+3,0} \rightarrow v_{3,0} \rightarrow v_{\frac{k}{2}+2,3t_5-6} \rightarrow v_{\lfloor \frac{k+12}{4} \rfloor,3t_{\lfloor \frac{3k}{4} \rfloor}-6}$$
$$\rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-4}{4} \rfloor,3t_{\lfloor \frac{3k-4}{4} \rfloor}-6} \rightarrow v_{\frac{k}{2}-1,3t_{\frac{k}{2}-1}-6} \rightarrow$$
$$v_{\frac{k}{2}+1,t_{\frac{k}{2}+1}-1} \rightarrow v_{4,t_4-1} \rightarrow v_{k-3,t_{k-3}-1} \rightarrow v_{\frac{k}{2},t_{\frac{k}{2}}-1} \rightarrow$$
$$v_{k-3,t_{k-3}} \rightarrow v_{4,t_4} \rightarrow v_{\frac{k}{2}+1,t_{\frac{k}{2}+1}} \rightarrow \cdots \rightarrow v_{k-3,2t_{k-3}-4}$$
$$\rightarrow v_{4,2t_4-4} \rightarrow v_{\frac{k}{2}+1,\frac{t_4}{2}+t_{\frac{k}{2}+1}-2} \rightarrow v_{\frac{k}{2},\frac{t_4}{2}+t_{\frac{k}{2}}-2}$$
$$\rightarrow \cdots \rightarrow v_{\frac{k}{2}+1,2t_{\frac{k}{2}+1}-4} \rightarrow v_{\frac{k}{2},2t_{\frac{k}{2}}-4} \rightarrow v_{\frac{k}{2}+2,t_{\frac{k}{2}+2}-1}$$
$$\rightarrow v_{5,t_5-1} \rightarrow \cdots \rightarrow v_{\frac{k}{2}+2,2t_5-4} \rightarrow v_{5,2t_5-4}$$
$$\rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-4}{4} \rfloor,t_{\lfloor \frac{3k-4}{4} \rfloor}-1} \rightarrow v_{\lfloor \frac{k+8}{4} \rfloor,t_{\lfloor \frac{k+8}{4} \rfloor}-1}$$
$$\rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-4}{4} \rfloor,2t_{\lfloor \frac{k+8}{4} \rfloor}-4} \rightarrow v_{\lfloor \frac{k+8}{4} \rfloor,2t_{\lfloor \frac{k+8}{4} \rfloor}-4}$$
$$\rightarrow v_{\lfloor \frac{3k}{4} \rfloor,t_{\lfloor \frac{3k}{4} \rfloor}-1} \rightarrow v_{\lfloor \frac{k+12}{4} \rfloor,t_{\lfloor \frac{k+12}{4} \rfloor}-1} \rightarrow \cdots \rightarrow$$
$$v_{\lfloor \frac{3k}{4} \rfloor,2t_{\lfloor \frac{3k}{4} \rfloor}-4} \rightarrow v_{\lfloor \frac{k+12}{4} \rfloor,2t_{\lfloor \frac{3k}{4} \rfloor}-4} \rightarrow \cdots \rightarrow$$
$$v_{k-4,t_{k-4}-1} \rightarrow v_{\frac{k}{2}-1,t_{\frac{k}{2}-1}-1} \rightarrow \cdots \rightarrow v_{k-4,2t_{k-4}-4}$$
$$\rightarrow v_{\frac{k}{2}-1,2t_{k-4}-4} \rightarrow v_{\frac{k}{2}+2,2t_5-3} \rightarrow v_{\lfloor \frac{k+12}{4} \rfloor,2t_{\lfloor \frac{3k}{4} \rfloor}-3}$$
$$\rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-4}{4} \rfloor,2t_{\lfloor \frac{3k-4}{4} \rfloor}-4} \rightarrow v_{\frac{k}{2}-1,2t_{\frac{k}{2}-1}-4} \rightarrow$$
$$v_{\frac{k}{2}+1,1} \rightarrow v_{4,1} \rightarrow v_{k-3,1} \rightarrow v_{\frac{k}{2},1} \rightarrow v_{k-3,2} \rightarrow v_{4,2}$$
$$\rightarrow v_{\frac{k}{2}+1,2} \rightarrow v_{4,3} \rightarrow v_{k-3,3} \rightarrow \cdots \rightarrow v_{\frac{k}{2},\frac{t_4}{2}-1}$$
$$\rightarrow v_{k-3,t_{k-3}-2} \rightarrow v_{4,t_4-2} \rightarrow v_{\frac{k}{2}+1,\frac{t_4}{2}} \rightarrow v_{\frac{k}{2},\frac{t_4}{2}} \rightarrow$$
$$v_{\frac{k}{2}+1,\frac{t_4}{2}+1} \rightarrow \cdots \rightarrow v_{\frac{k}{2},t_{\frac{k}{2}}-2} \rightarrow v_{\frac{k}{2}+2,1}$$
$$\rightarrow v_{5,1} \rightarrow \cdots \rightarrow v_{\frac{k}{2}+2,t_5-2} \rightarrow v_{5,t_5-2} \rightarrow \cdots \rightarrow$$
$$v_{\lfloor \frac{3k-4}{4} \rfloor,1} \rightarrow v_{\lfloor \frac{k+8}{4} \rfloor,1} \rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-4}{4} \rfloor,t_{\lfloor \frac{k+8}{4} \rfloor}-2}$$
$$\rightarrow v_{\lfloor \frac{k+8}{4} \rfloor,t_{\lfloor \frac{k+8}{4} \rfloor}-2} \rightarrow v_{\lfloor \frac{3k}{4} \rfloor,1} \rightarrow v_{\lfloor \frac{k+12}{4} \rfloor,1} \rightarrow \cdots$$
$$\rightarrow v_{\lfloor \frac{3k}{4} \rfloor,t_{\lfloor \frac{3k}{4} \rfloor}-2} \rightarrow v_{\lfloor \frac{k+12}{4} \rfloor,t_{\lfloor \frac{3k}{4} \rfloor}-2} \rightarrow \cdots$$
$$\rightarrow v_{k-4,1} \rightarrow v_{\frac{k}{2}-1,1} \rightarrow \cdots \rightarrow v_{k-4,t_{k-4}-2} \rightarrow$$
$$v_{\frac{k}{2}-1,t_{k-4}-2} \rightarrow v_{\frac{k}{2}+2,t_5-1} \rightarrow v_{\lfloor \frac{k+12}{4} \rfloor,t_{\lfloor \frac{3k}{4} \rfloor}-1}$$
$$\rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-4}{4} \rfloor,t_{\lfloor \frac{3k-4}{4} \rfloor}-2} \rightarrow$$
$$v_{\frac{k}{2}-1,t_{\frac{k}{2}-1}-2} \rightarrow v_{\frac{k}{2}+2,0} \rightarrow v_{\frac{k}{2}-1,0} \rightarrow v_{\frac{k}{2}+1,0}.$$

Similar as Case 1, we can obtain that f is a multi-level distance labelling of G, and then

$$rn(G) \leq f(u_{|V|-1}) = 6\sum_{i=4}^{\frac{k}{2}}[(2i-5)(t_i-2)] + \frac{k^2}{2} - k + 1.$$

Case 3. If both k and t_i are odd, then $t_4 = t_{\frac{k+1}{2}} = t_{k-3}$ and $t_{\frac{k+1}{2}-j} \geq t_{4+j}$ for $1 \leq j \leq \lfloor \frac{k-7}{4} \rfloor$. The algorithm is defined as following:

$$u_0 = v_{\frac{k+1}{2},0} \rightarrow v_{k,0} \rightarrow v_{1,0} \rightarrow v_{\frac{k+1}{2},2t_{\frac{k+1}{2}}-3} \rightarrow v_{4,2t_4-3}$$
$$\rightarrow v_{k-3,2t_{k-3}-3} \rightarrow v_{\frac{k+1}{2},2t_{\frac{k+1}{2}}-2} \rightarrow v_{k-3,2t_{k-3}-2}$$
$$\rightarrow v_{4,2t_4-2} \rightarrow v_{\frac{k+1}{2},2t_{\frac{k+1}{2}}-1} \rightarrow \cdots \rightarrow v_{\frac{k+1}{2},3t_{\frac{k+1}{2}}-6}$$
$$\rightarrow v_{4,3t_4-6} \rightarrow v_{k-3,3t_{k-3}-6} \rightarrow (1)v_{\frac{k-1}{2},2t_{\frac{k-1}{2}}-3}$$
$$\rightarrow v_{k-1,0} \rightarrow v_{2,0} \rightarrow v_{\frac{k+3}{2},2t_{\frac{k+3}{2}}-3} \rightarrow v_{5,2t_5-3} \rightarrow$$
$$v_{\frac{k+3}{2},2t_{\frac{k+3}{2}}-2} \rightarrow v_{5,2t_5-2} \rightarrow \cdots \rightarrow v_{\frac{k+3}{2},3t_5-6} \rightarrow$$
$$v_{5,3t_5-6} \rightarrow v_{\frac{k+7}{2},0} \rightarrow v_{3,0} \rightarrow v_{\frac{k+5}{2},2t_{\frac{k+5}{2}}-3} \rightarrow v_{6,2t_6-3}$$
$$\rightarrow \cdots \rightarrow v_{\frac{k+5}{2},3t_6-6} \rightarrow v_{6,3t_6-6} \rightarrow v_{\frac{k+9}{2},0} \rightarrow v_{4,0}$$
$$\rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor,2t_{\lfloor \frac{3k-5}{4} \rfloor}-3} \rightarrow v_{\lfloor \frac{k+9}{4} \rfloor,2t_{\lfloor \frac{k+9}{4} \rfloor}-3} \rightarrow$$
$$\cdots \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor,3t_{\lfloor \frac{k+9}{4} \rfloor}-6} \rightarrow v_{\lfloor \frac{k+9}{4} \rfloor,3t_{\lfloor \frac{k+9}{4} \rfloor}-6} \rightarrow$$
$$v_{\lfloor \frac{3k+3}{4} \rfloor,0} \rightarrow v_{\lfloor \frac{k+5}{4} \rfloor,0} \rightarrow v_{\lfloor \frac{3k-1}{4} \rfloor,2t_{\lfloor \frac{3k-1}{4} \rfloor}-3} \rightarrow$$
$$v_{\lfloor \frac{k+13}{4} \rfloor,2t_{\lfloor \frac{k+13}{4} \rfloor}-3} \rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-1}{4} \rfloor,3t_{\lfloor \frac{3k-1}{4} \rfloor}-6} \rightarrow$$
$$v_{\lfloor \frac{k+13}{4} \rfloor,3t_{\lfloor \frac{3k-1}{4} \rfloor}-6} \rightarrow v_{\lfloor \frac{3k+7}{4} \rfloor,0} \rightarrow v_{\lfloor \frac{k+9}{4} \rfloor,0} \rightarrow \cdots \rightarrow$$
$$v_{k-4,2t_{k-4}-3} \rightarrow v_{\frac{k-1}{2},2t_{\frac{k-1}{2}}-2} \rightarrow \cdots \rightarrow v_{\frac{k-1}{2},3t_{k-4}-6}$$
$$\rightarrow v_{k-4,3t_{k-4}-6} \rightarrow v_{\frac{k-5}{2},0} \rightarrow v_{k-2,0} \rightarrow v_{\frac{k-1}{2},3t_{k-4}-5}$$
$$\rightarrow v_{\frac{k+3}{2},3t_5-5} \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor,3t_{\lfloor \frac{3k-1}{4} \rfloor}-5} \rightarrow \cdots \rightarrow$$
$$v_{\frac{k-1}{2},3t_{\frac{k-1}{2}}-6} \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor,3t_{\lfloor \frac{3k-5}{4} \rfloor}-6} \rightarrow v_{\frac{k+1}{2},t_{\frac{k+1}{2}}-1}$$
$$\rightarrow v_{k-3,t_{k-3}-1} \rightarrow v_{4,t_4-1} \rightarrow v_{\frac{k+1}{2},t_{\frac{k+1}{2}}} \rightarrow v_{4,t_4} \rightarrow$$
$$v_{k-3,t_{k-3}} \rightarrow \cdots \rightarrow v_{\frac{k+1}{2},2t_{\frac{k+1}{2}}-4} \rightarrow v_{k-3,2t_{k-3}-4}$$
$$\rightarrow v_{4,2t_4-4} \rightarrow v_{\frac{k+3}{2},t_{\frac{k+3}{2}}-1} \rightarrow v_{5,t_5-1} \rightarrow \cdots \rightarrow$$
$$v_{\frac{k+3}{2},2t_5-4} \rightarrow v_{5,2t_5-4} \rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor,t_{\lfloor \frac{3k-5}{4} \rfloor}-1}$$
$$\rightarrow v_{\lfloor \frac{k+9}{4} \rfloor,t_{\lfloor \frac{k+9}{4} \rfloor}-1} \rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor,2t_{\lfloor \frac{k+9}{4} \rfloor}-4}$$
$$\rightarrow v_{\lfloor \frac{k+9}{4} \rfloor,2t_{\lfloor \frac{k+9}{4} \rfloor}-4} \rightarrow v_{\lfloor \frac{3k-1}{4} \rfloor,t_{\lfloor \frac{3k-1}{4} \rfloor}-1}$$
$$\rightarrow v_{\lfloor \frac{k+13}{4} \rfloor,t_{\lfloor \frac{k+13}{4} \rfloor}-1} \rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-1}{4} \rfloor,2t_{\lfloor \frac{3k-1}{4} \rfloor}-4}$$
$$\rightarrow v_{\lfloor \frac{k+13}{4} \rfloor,2t_{\lfloor \frac{3k-1}{4} \rfloor}-4} \rightarrow \cdots \rightarrow v_{k-4,t_{k-4}-1}$$
$$\rightarrow v_{\frac{k-1}{2},t_{\frac{k-1}{2}}-1} \rightarrow \cdots \rightarrow v_{k-4,2t_{k-4}-4}$$
$$\rightarrow v_{\frac{k-1}{2},2t_{k-4}-4} \rightarrow v_{\frac{k+3}{2},2t_5-3} \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor,2t_{\lfloor \frac{3k-1}{4} \rfloor}-3}$$
$$\rightarrow \cdots \rightarrow v_{\frac{k-1}{2},2t_{\frac{k-1}{2}}-4} \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor,2t_{\lfloor \frac{3k-5}{4} \rfloor}-4} \rightarrow$$
$$v_{\frac{k+1}{2},1} \rightarrow v_{k-3,1} \rightarrow v_{4,1} \rightarrow v_{\frac{k+1}{2},2} \rightarrow v_{4,2} \rightarrow v_{k-3,2}$$
$$\rightarrow \cdots \rightarrow v_{\frac{k+1}{2},t_{\frac{k+1}{2}}-2} \rightarrow v_{4,t_4-2} \rightarrow v_{k-3,t_{k-3}-2}$$
$$\rightarrow v_{5,1} \rightarrow v_{\frac{k+3}{2},1} \rightarrow \cdots \rightarrow v_{5,t_5-2} \rightarrow v_{\frac{k+3}{2},t_5-2}$$

$$
\begin{array}{l}
\rightarrow \cdots \rightarrow v_{\lfloor \frac{k+9}{4} \rfloor,1} \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor,1} \rightarrow \cdots \rightarrow \\
v_{\lfloor \frac{k+9}{4} \rfloor, t_{\lfloor \frac{k+9}{4} \rfloor}-2} \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor, t_{\lfloor \frac{k+9}{4} \rfloor}-2} \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor,1} \\
\rightarrow v_{\lfloor \frac{3k-1}{4} \rfloor,1} \rightarrow \cdots \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor, t_{\lfloor \frac{3k-1}{4} \rfloor}-2} \rightarrow \\
v_{\lfloor \frac{3k-1}{4} \rfloor, t_{\lfloor \frac{3k-1}{4} \rfloor}-2} \rightarrow \cdots \rightarrow v_{\frac{k-1}{2},1} \rightarrow v_{k-4,1} \\
\rightarrow \cdots \rightarrow v_{\frac{k-1}{2}, t_{k-4}-2} \rightarrow v_{k-4, t_{k-4}-2} \rightarrow \\
v_{\lfloor \frac{k+13}{4} \rfloor, t_{\lfloor \frac{3k-1}{4} \rfloor}-1} \rightarrow v_{\frac{k+3}{2}, t_5-1} \rightarrow \cdots \rightarrow \\
v_{\frac{k-1}{2}, t_{\frac{k-1}{2}}-4} \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor, t_{\lfloor \frac{3k-5}{4} \rfloor}-2} \rightarrow v_{\frac{k-3}{2},0} \\
\rightarrow v_{\frac{k+5}{2},0} \rightarrow v_{\frac{k-1}{2},0} \rightarrow v_{\frac{k+3}{2},0}.
\end{array}
$$

Similar as Case 1, we can show that f is a multi-level distance labelling of G.

By definition of f, if $k=7$, then there is no vertex satisfying Claim 1, so $\sigma(f)=0$, thus $rn(G) \leq f(u_{|V|-1}) = (3k-12)t_{\frac{k+1}{2}} + k^2 - 7k + 1$.

If $k \geq 9$, then there is only one vertex

$$u_t \in \{v_{1,0}, v_{4,j}, v_{k-3,j}, v_{k,0} | 2t_4 - 3 \leq j \leq 3t_4 - 6\}$$

such that u_{t-1}, u_{t+1} belong to the same branch, $l(u_{t-1}) = l(v_{4,3t_4-6}) = \frac{k-1}{2}, l(u_{t+1}) = l(v_{\frac{k-1}{2}, 2t_{\frac{k-1}{2}}-3}) = 4$ and u_{t+1} is not the ancestor of u_{t-1}, thus

$$\varphi(u_{t-1}, u_{t+1}) = \min\{l(u_{t-1}) - 3, l(u_{t+1}) - 3\} = 1.$$

By Theorem 2.8 and the above algorithm, we have $\sigma(f) = 1$, hence

$$
\begin{array}{rl}
rn(G) \leq & 6 \sum\limits_{i=4}^{\frac{k-1}{2}} [(2i-5)(t_i-2)] + (3k-12)t_{\frac{k+1}{2}} \\
& + \frac{1}{2}(k-7)^2 + 2.
\end{array}
$$

Case 4. If k is odd and t_i is even, then $t_4 = t_{\frac{k+1}{2}} = t_{k-3}$ and $t_{\frac{k+1}{2}-j} \geq t_{4+j}$ for $1 \leq j \leq \lfloor \frac{k-7}{4} \rfloor$. The algorithm is defined as following:

$$
\begin{array}{l}
u_0 = v_{\frac{k+1}{2},0} \rightarrow v_{k,0} \rightarrow v_{1,0} \rightarrow v_{\frac{k+1}{2}, 2t_{\frac{k+1}{2}}-3} \rightarrow v_{4,2t_4-3} \\
\rightarrow v_{k-3,2t_{k-3}-3} \rightarrow v_{\frac{k+1}{2}, 2t_{\frac{k+1}{2}}-2} \rightarrow v_{k-3,2t_{k-3}-2} \\
\rightarrow v_{4,2t_4-2} \rightarrow v_{\frac{k+1}{2}, 2t_{\frac{k+1}{2}}-1} \rightarrow \cdots \rightarrow v_{\frac{k+1}{2}, 3t_{\frac{k+1}{2}}-6} \\
\rightarrow v_{k-3,3t_{k-3}-6} \rightarrow v_{4,3t_4-6} \rightarrow (1) v_{\frac{k+3}{2}, 2t_{\frac{k+3}{2}}-3} \rightarrow \\
v_{2,0} \rightarrow v_{k-1,0} \rightarrow v_{\frac{k-1}{2}, 2t_{\frac{k-1}{2}}-3} \rightarrow v_{k-4,2t_{k-4}-3} \\
\rightarrow v_{\frac{k-1}{2}, 2t_{\frac{k-1}{2}}-2} \rightarrow v_{k-4,2t_{k-4}-2} \rightarrow \cdots \rightarrow v_{\frac{k-1}{2}, 3t_{k-4}-6} \\
\rightarrow v_{k-4,3t_{k-4}-6} \rightarrow v_{\frac{k-5}{2},0} \rightarrow v_{k-2,0} \rightarrow v_{\frac{k-3}{2}, 2t_{\frac{k-3}{2}}-3} \rightarrow \\
v_{k-5,2t_{k-5}-3} \rightarrow \cdots \rightarrow v_{\frac{k-3}{2}, 3t_{k-5}-6} \rightarrow v_{k-5,3t_{k-5}-6} \rightarrow \\
v_{\frac{k-7}{2},0} \rightarrow v_{k-3,0} \rightarrow \cdots \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor, 2t_{\lfloor \frac{k+13}{4} \rfloor}-3} \\
\rightarrow v_{\lfloor \frac{3k-1}{4} \rfloor, 2t_{\lfloor \frac{3k-1}{4} \rfloor}-3} \rightarrow \cdots \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor, 3t_{\lfloor \frac{3k-1}{4} \rfloor}-6} \\
\rightarrow v_{\lfloor \frac{3k-1}{4} \rfloor, 3t_{\lfloor \frac{3k-1}{4} \rfloor}-6} \rightarrow v_{\lfloor \frac{k+5}{4} \rfloor,0} \rightarrow v_{\lfloor \frac{3k+7}{4} \rfloor,0} \rightarrow \\
v_{\lfloor \frac{k+9}{4} \rfloor, 2t_{\lfloor \frac{k+9}{4} \rfloor}-3} \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor, 2t_{\lfloor \frac{3k-5}{4} \rfloor}-3} \rightarrow \cdots \rightarrow \\
v_{\lfloor \frac{k+9}{4} \rfloor, 3t_{\lfloor \frac{k+9}{4} \rfloor}-6} \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor, 3t_{\lfloor \frac{k+9}{4} \rfloor}-6} \rightarrow v_{\lfloor \frac{k+1}{4} \rfloor,0} \rightarrow
\end{array}
$$

$$
\begin{array}{l}
v_{\lfloor \frac{3k+3}{4} \rfloor,0} \rightarrow \cdots \rightarrow v_{6,2t_6-3} \rightarrow v_{\frac{k+5}{2}, 2t_{\frac{k+5}{2}}-3} \rightarrow \cdots \rightarrow \\
v_{6,3t_6-6} \rightarrow v_{\frac{k+5}{2}, 3t_6-6} \rightarrow v_{4,0} \rightarrow v_{\frac{k+9}{2},0} \rightarrow v_{5,2t_5-3} \rightarrow \\
v_{\frac{k+3}{2}, 2t_{\frac{k+3}{2}}-2} \rightarrow v_{5,2t_5-2} \rightarrow v_{\frac{k+3}{2}, 2t_{\frac{k+3}{2}}-1} \rightarrow \cdots \\
\rightarrow v_{5,3t_5-6} \rightarrow v_{\frac{k+3}{2}, 3t_5-5} \rightarrow v_{4,0} \rightarrow v_{\frac{k+9}{2},0} \rightarrow \\
v_{\lfloor \frac{k+13}{4} \rfloor, 3t_{\lfloor \frac{3k-1}{4} \rfloor}-6} \rightarrow v_{\frac{k+3}{2}, 3t_5-4} \rightarrow \cdots \rightarrow \\
v_{\lfloor \frac{3k-5}{4} \rfloor, 3t_{\lfloor \frac{3k-5}{4} \rfloor}-6} \rightarrow v_{\frac{k-1}{2}, 3t_{\frac{k-1}{2}}-6} \rightarrow v_{\frac{k+1}{2}, t_{\frac{k+1}{2}}-1} \\
\rightarrow v_{4,t_4-1} \rightarrow v_{k-3,t_{k-3}-1} \rightarrow v_{\frac{k+1}{2}, t_{\frac{k+1}{2}}} \rightarrow v_{k-3,t_{k-3}} \\
\rightarrow v_{4,t_4} \rightarrow \cdots \rightarrow v_{\frac{k+1}{2}, 2t_{\frac{k+1}{2}}-4} \rightarrow v_{k-3,2t_{k-3}-4} \\
\rightarrow v_{4,2t_4-4} \rightarrow v_{\frac{k+3}{2}, t_{\frac{k+3}{2}}-1} \rightarrow v_{5,t_5-1} \rightarrow \cdots \rightarrow \\
v_{\frac{k+3}{2}, 2t_5-4} \rightarrow v_{5,2t_5-4} \rightarrow \cdots \rightarrow \\
v_{\lfloor \frac{3k-5}{4} \rfloor, t_{\lfloor \frac{3k-5}{4} \rfloor}-1} \rightarrow v_{\lfloor \frac{k+9}{4} \rfloor, t_{\lfloor \frac{k+9}{4} \rfloor}-1} \rightarrow \cdots \rightarrow \\
v_{\lfloor \frac{3k-5}{4} \rfloor, 2t_{\lfloor \frac{k+9}{4} \rfloor}-4} \rightarrow v_{\lfloor \frac{k+9}{4} \rfloor, 2t_{\lfloor \frac{k+9}{4} \rfloor}-4} \rightarrow \\
v_{\lfloor \frac{3k-1}{4} \rfloor, t_{\lfloor \frac{3k-1}{4} \rfloor}-1} \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor, t_{\lfloor \frac{k+13}{4} \rfloor}-1} \rightarrow \cdots \rightarrow \\
v_{\lfloor \frac{3k-1}{4} \rfloor, 2t_{\lfloor \frac{3k-1}{4} \rfloor}-4} \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor, 2t_{\lfloor \frac{3k-1}{4} \rfloor}-4} \rightarrow \cdots \rightarrow \\
v_{k-4,t_{k-4}-1} \rightarrow v_{\frac{k-1}{2}, t_{\frac{k-1}{2}}-1} \rightarrow \cdots \rightarrow v_{k-4,2t_{k-4}-4} \rightarrow \\
v_{\frac{k-1}{2}, 2t_{k-4}-4} \rightarrow v_{\frac{k+3}{2}, 2t_5-3} \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor, 2t_{\lfloor \frac{3k-1}{4} \rfloor}-3} \rightarrow \\
\cdots \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor, 2t_{\lfloor \frac{3k-5}{4} \rfloor}-4} \rightarrow v_{\frac{k-1}{2}, 2t_{\frac{k-1}{2}}-4} \rightarrow v_{\frac{k+1}{2},1} \\
\rightarrow v_{4,1} \rightarrow v_{k-3,1} \rightarrow v_{\frac{k+1}{2},2} \rightarrow v_{k-3,2} \rightarrow v_{4,2} \rightarrow \cdots \\
\rightarrow v_{\frac{k+1}{2}, t_{\frac{k+1}{2}}-2} \rightarrow v_{k-3,t_{k-3}-2} \rightarrow v_{4,t_4-2} \rightarrow v_{\frac{k+3}{2},1} \rightarrow \\
v_{5,1} \rightarrow \cdots \rightarrow v_{\frac{k+3}{2}, t_5-2} \rightarrow v_{5,t_5-2} \rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor,1} \\
\rightarrow v_{\lfloor \frac{k+9}{4} \rfloor,1} \rightarrow \cdots \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor, t_{\lfloor \frac{k+9}{4} \rfloor}-2} \rightarrow \\
v_{\lfloor \frac{k+9}{4} \rfloor, t_{\lfloor \frac{k+9}{4} \rfloor}-2} \rightarrow v_{\lfloor \frac{3k-1}{4} \rfloor,1} \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor,1} \rightarrow \cdots \rightarrow \\
v_{\lfloor \frac{3k-1}{4} \rfloor, t_{\lfloor \frac{3k-1}{4} \rfloor}-2} \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor, t_{\lfloor \frac{3k-1}{4} \rfloor}-2} \rightarrow \cdots \rightarrow \\
v_{k-4,1} \rightarrow v_{\frac{k-1}{2},1} \rightarrow \cdots \rightarrow v_{k-4,t_{k-4}-2} \rightarrow v_{\frac{k-1}{2}, t_{k-4}-2} \\
\rightarrow v_{\frac{k+3}{2}, t_5-1} \rightarrow v_{\lfloor \frac{k+13}{4} \rfloor, t_{\lfloor \frac{3k-1}{4} \rfloor}-1} \rightarrow \\
\cdots \rightarrow v_{\lfloor \frac{3k-5}{4} \rfloor, t_{\lfloor \frac{3k-5}{4} \rfloor}-2} \rightarrow v_{\frac{k-1}{2}, t_{\frac{k-1}{2}}-2} \rightarrow \cdots \rightarrow \\
v_{\frac{k+5}{2},0} \rightarrow v_{\frac{k-3}{2},0} \rightarrow v_{\frac{k+3}{2},0} \rightarrow v_{\frac{k-1}{2},0}.
\end{array}
$$

Similar as Case 3, we can show that f is a multi-level distance labelling of G.

If $k = 7$, then

$$rn(G) \leq (3k-12)t_{\frac{k+1}{2}} + k^2 - 7k + 1.$$

If $k \geq 9$, then

$$
\begin{array}{rl}
rn(G) \leq & 6 \sum\limits_{i=4}^{\frac{k-1}{2}} [(2i-5)(t_i-2)] + (3k-12)t_{\frac{k+1}{2}} \\
& + \frac{1}{2}(k-7)^2 + 2.
\end{array}
$$

Applying Theorems 2.4-2.6, and 2.8, we have shown that the theorem holds. □

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(Advance online publication: 17 February 2015)

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