Transreal Calculus

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Abstract—Transreal arithmetic totalises real arithmetic by defining division by zero in terms of three definite, non-finite numbers: positive infinity, negative infinity and nullity. We describe the transreal tangent function and extend continuity and limits from the real domain to the transreal domain. With this preparation, we extend the real derivative to the transreal derivative and extend proper integration from the real domain to the transreal domain. Further, we extend improper integration of absolutely convergent functions from the real domain to the transreal domain. This demonstrates that transreal calculus contains real calculus and operates at singularities where real calculus fails.

Index Terms—transreal arithmetic, transreal tangent, transreal continuity, transreal limit, transreal derivative, transreal integral, transreal calculus.

I. Introduction

TRANSREAL [6] and transcomplex [3][7] arithmetic are developments of Computer Science that are now being normalised in Mathematics [9]. They define division in terms of operations on the lexical reciprocal. This lexical definition contains the usual definition of division, as multiplication by the multiplicative inverse, but also defines division by zero. Consequently transreal and transcomplex arithmetic are supersets of, respectively, real and complex arithmetic. There is a machine proof [6] and a human proof [7] that transreal arithmetic is consistent if real arithmetic is. The hand proof also demonstrates that transreal arithmetic contains real arithmetic and establishes a similar relationship between transcomplex arithmetic and complex arithmetic.

Transreal arithmetic uses a subset of the algorithms of real arithmetic so the general reader will be able to follow any computation in transreal arithmetic but will have little chance of deriving a valid, non-finite, computation until the axioms [6] or algorithms [3] of transreal arithmetic have been properly learned. The reader is further cautioned that the relational operators of transreal arithmetic, less-than ($<$), equal-to ($=$), greater-than ($>$), together with their negations, form a total, irredundant set of independent operations, unlike their real counterparts. These relational operators are described in [4] but there are some typographical errors in that paper. These are corrected in Appendix B below. In order to assist the reader, we explain the special properties of transreal arithmetic where they are first used in a proof.

We begin by describing the transreal tangent [5] so that non-finite tangents are well defined when we use them in the development of the transreal, differential calculus. The trigonometric, tangent function is geometrically defined

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everywhere but it is undefined, in real arithmetic, when it takes on an "infinite" value at multiples of the angle $\pi/2$. In the next section we describe the transreal tangent which is defined everywhere. We then develop transreal limits [5] as a generalisation of real limits. The main results are: wherever infinities occur as symbols in extended-real limits, they occur identically in transreal limits but as definite numbers; wherever the transreal number nullity occurs in transreal limits, the corresponding real limit is undefined. We then develop the transreal derivative [8] so that it contains the real derivative and operates at singularities where the real derivative is undefined. We extend the proper integral of real functions to the transintegral of transreal functions and the improper integral of absolutely convergent, real functions, to the transintegral of absolutely convergent, transreal functions. Thus the transintegral contains all of these real integrals and extends them to operate at singularities. However, this is a rather restricted set of functions. Since the preparation of this integral, a much wider extension of the real integral to the transreal integral has been developed. That material has been submitted for publication elsewhere.

All of the proofs given here are developed in terms of topology, however in Appendix A we appeal to Urysohn's theorem to show that there are metrics on the set of transreal numbers that are compatible with their topology. Furthermore, we construct one such metric.
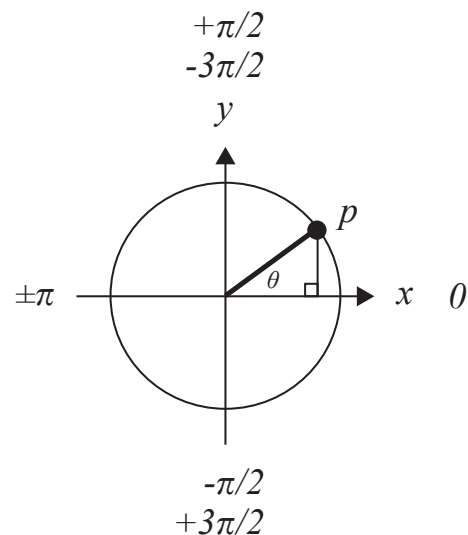
II. Transreal Tangent

Fig. 1. Geometrical Construction of the Tangent

Figure 1 shows the well known geometrical construction of the tangent, in which a point, p, lies on a circle, with a unit radius forming the hypotenuse of a right triangle,

whose internal angle is θ. When the sides of the triangle are measured in Cartesian co-ordinates, the tangent is defined as $\tan\theta = y/x$. Part of the graph of this function, for real θ, is shown in Figure 2, where the discs, •, show where the tangent arrives exactly at a signed infinity and the annuli, ○, show where the tangent asymptotes to a signed infinity, that is where it approaches the infinity but does not arrive at it.
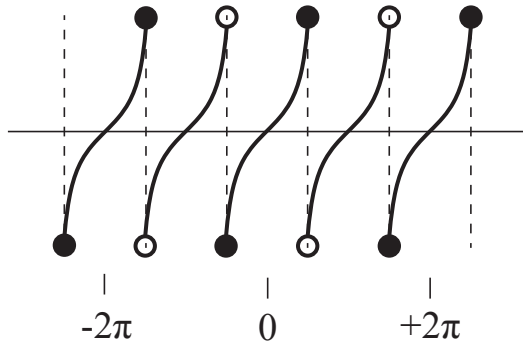

Fig. 2.   Graph of the Transreal Tangent

The reader should examine Figure 2. The abscissa shows the angle in radians; the ordinate shows the value of the tangent function. At zero radians the value of the tangent is $\tan = y/x = 0/1 = 0$. As the angle increases: the value increases until it passes exactly through positive infinity at $\tan(\pi/2) = 1/0 = \infty$; the value then jumps discontinuously so that it passes through all negative, real numbers, each of which is finite, until it arrives at $\tan\pi = 0/1 = 0$; the value continues to increase, asymptoting to positive infinity at $3\pi/(2-\epsilon)$ for small, positive ϵ, then jumps discontinuously to negative infinity at $\tan(3\pi/2) = -1/0 = -\infty$; the value of the tangent then increases to zero at $\tan 2\pi = 0/1 = 0$. Notice that the graph has a least, that is principal, period of 2π, not π as is commonly understood. The results for negative angles are similar. For integral k the value of the tangent is positive infinity at $\theta = 2k\pi + \pi/2$ and negative infinity at $\theta = 2k\pi - \pi/2$. The usual graph for the tangent, computed as the limit of a power series, is similar to Figure 2 but with the difference that the tangent is undefined at $2k\pi \pm \pi/2$ for all integral k. Thus the finite values of the geometrical tangent have period π but the extended-real values have period 2π.

As we lack a geometrical construction for the non-finite, transreal angles, we define that the value of the transreal tangent, at non-finite angles, is the limit of the usual power series, evaluated in transreal arithmetic, so that $\tan(-\infty) = \tan\infty = \tan\Phi = \Phi$. This is justified by Observation 16 in Section III-B *Transreal Sequences* below.

We then take the arctangent as usual, for finite values of the tangent, and augment this with $\arctan(-\infty) = -\pi/2$, $\arctan\infty = \pi/2$, $\arctan\Phi = \Phi$. This defines the principal range of the arctangent for all transreal angles.

III. Transreal Limit and Continuity

In this section we augment the topology of transreal space, derived transarithmetically from ϵ-neighbourhoods [2], with the usual topology of measure theory and integration theory. Amongst other results, we show that transreal space is a compact, separable, Hausdorff space. We then develop transreal sequences and establish the transreal infimum and supremum. We present fundamental results on the limits and continuity of transreal functions. Taken together this implies that transreal calculus contains real calculus.

A. Transreal Topology

Transreal arithmetic implies a topology [2], Figure 3, that gives a definite, numerical value to the result of dividing any real number by zero. Infinity, ∞, is the unique number that results when a positive number is divided by zero; negative infinity, $-\infty$, is the unique number that results when a negative number is divided by zero; nullity, Φ, is the unique number that results when zero is divided by zero. Nullity is not ordered, all other transreal numbers are ordered. Infinity is the largest number and negative infinity is the smallest number. Any particular real number is finite; ∞ and $-\infty$ are infinite; Φ is non-finite. The infinite numbers are also non-finite. The real numbers, $\mathbb{R}$, together with the infinite numbers, $-\infty$ and ∞, make up the extended-real numbers, $\mathbb{R}^E$; the real numbers, together with the non-finite numbers, $-\infty, \infty$ and Φ, make up the transreal numbers, $\mathbb{R}^T$.

In summary the arithmetic and order relation defined in $\mathbb{R}^T$ is such that, for each $x, y \in \mathbb{R}^T$, it follows that [6]:

i) $-\Phi = \Phi$, $-(\infty) = -\infty$ and $-(-\infty) = \infty$,

ii) $0^{-1} = \infty$, $\Phi^{-1} = \Phi$, $\infty^{-1} = 0$ and $(-\infty)^{-1} = 0$,

iii) $\Phi + x = \Phi$, $\infty + x = \begin{cases} \Phi & , \text{ if } x \in \{-\infty, \Phi\} \\ \infty & , \text{ otherwise} \end{cases}$ and $-\infty + x = \begin{cases} \Phi & , \text{ if } x \in \{\infty, \Phi\} \\ -\infty & , \text{ otherwise} \end{cases}$,

iv) $\Phi \times x = \Phi$, $\infty \times x = \begin{cases} \Phi & , \text{ if } x \in \{0, \Phi\} \\ -\infty & , \text{ if } x < 0 \\ \infty & , \text{ if } x > 0 \end{cases}$ and $-\infty \times x = -(\infty \times x)$,

v) $x - y = x + (-y)$ and

vi) $x \div y = x \times y^{-1}$.

vii) If $x \in \mathbb{R}$ then $-\infty < x < \infty$.

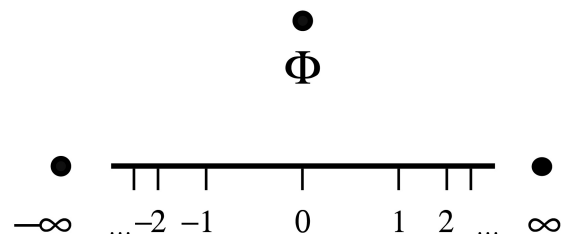viii) The following does not hold $x < \Phi$ or $\Phi < x$.

Fig. 3.   Transreal Number-Line

We now define a topology for the whole of $\mathbb{R}^T = \mathbb{R} \cup \{-\infty, \infty, \Phi\}$ which contains the usual topology on $\mathbb{R}^E = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ that is used in measure theory and integration theory. Specifically we note that $\{-\infty\}$, $\{\infty\}$ and $\{\Phi\}$ are singleton sets that are not finitely path connected to any other numbers. This retains compatibility with an older view of the topology of the transreal numbers, based

on computing ϵ-neighbourhoods using transreal arithmetic [2]. In our new topology we have that $\{-\infty\}$ and $\{\infty\}$ are closed and not open, while $\{\Phi\}$ is both closed and open. We impose neighbourhoods on $\{-\infty\}$ and $\{\infty\}$ so that the usual topology of measure theory and integration holds, with the possibility that real and extended-real functions have limits of $-\infty$ and ∞. The number Φ is then left as the unique, isolated point, reflecting its status as the unique, unordered number, Φ, in transreal arithmetic. We then rehearse various theorems of sequences, limits and continuity, all of which show that wherever $-\infty$ and ∞ occur as limits in transreal calculus they occur identically in (extended) real-calculus, with the difference that $-\infty$ and ∞ are abstract symbols in (extended) real calculus and are numbers in transreal arithmetic and transreal calculus. Furthermore $0/0$ is undefined in (extended) real calculus but in transreal arithmetic $\Phi = 0/0$ is a definite number and, in transreal calculus, it is the limit, for example, of constant, transreal functions of the form $f(x) = \Phi$. Thus real calculus is extended to transreal calculus.

Definition 1: Let $A \subset \mathbb{R}^T$. We say that $x \in \mathbb{R}^T$ is a *transinterior* point related to A if and only if one of the following conditions holds:

1) $x \in \mathbb{R}$ and there is a positive $\varepsilon \in \mathbb{R}$ such that $(x - \varepsilon, x + \varepsilon) \subset A$,
2) $x = -\infty$ and there is $b \in \mathbb{R}$ such that $[-\infty, b) \subset A$,
3) $x = \infty$ and there is $a \in \mathbb{R}$ such that $(a, \infty] \subset A$ or
4) $x = \Phi$ and $\{\Phi\} \subset A$.

We denote the set of all transinterior points related to A as transint$A$. We say that a set $A \subset \mathbb{R}^T$ is *transopen* if and only if $A = $ transint$A$.

Notice that for every $A \subset \mathbb{R}^T$ it is the case that transint$A \subset A$.

Theorem 2: The class of all transopen sets in $\mathbb{R}^T$ is a topology on $\mathbb{R}^T$. That is to say:

1) $\emptyset, \mathbb{R}^T$ are transopen,
2) Any union of transopen sets is a transopen set and
3) A finite intersection of transopen sets is a transopen set.

Proof:

1) Notice that transint$\emptyset = \emptyset$ and $\mathbb{R}^T \subset$ transint$\mathbb{R}^T$ follow directly from the definition of a transopen set.
2) Let I be any set and $A = \bigcup_{\alpha \in I} A_\alpha$, where A_α is transopen for all $\alpha \in I$. If $x \in A$ then $x \in A_\alpha$ for some $\alpha \in I$, whence $x \in$ transintA_α. We have several cases: $x \in \mathbb{R}$, whence there is a positive $\varepsilon \in \mathbb{R}$ such that $(x - \varepsilon, x + \varepsilon) \subset A_\alpha \subset A$; or $x = -\infty$, whence there is $b \in \mathbb{R}$ such that $[-\infty, b) \subset A_\alpha \subset A$; or $x = \infty$, whence there is $a \in \mathbb{R}$ such that $(a, \infty] \subset A_\alpha \subset A$; or $x = \Phi$, whence $\{\Phi\} \subset A_\alpha \subset A$. In every case, $x \in$ transint$A$, whence $A \subset$ transint$A$.
3) Let $A_1, A_2 \subset \mathbb{R}^T$ be transopen sets. If $x \in A_1 \cap A_2$ then $x \in A_1$ and $x \in A_2$, whence $x \in$ transintA_1 and $x \in$ transintA_2. If $x \in \mathbb{R}$ then there are positive $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ such that $(x - \varepsilon_1, x + \varepsilon_1) \subset A_1$ and $(x - \varepsilon_2, x + \varepsilon_2) \subset A_2$. Taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have $(x - \varepsilon, x + \varepsilon) \subset A_1 \cap A_2$. If $x = -\infty$ then there are $b_1, b_2 \in \mathbb{R}$ such that $[\infty, b_1) \subset A_1$ and $[-\infty, b_1) \subset A_1$. Taking $b = \min\{b_1, b_2\}$, we have

$[-\infty, b) \subset A_1 \cap A_2$. If $x = \infty$ then there are $a_1, a_2 \in \mathbb{R}$ such that $(a_1, \infty] \subset A_1$ and $(a_2, \infty] \subset A_1$. Taking $a = \max\{a_1, a_2\}$, we have $(a\infty] \subset A_1 \cap A_2$. Finally if $x = \Phi$ then $\{\Phi\} \subset A_1$ and $\{\Phi\} \subset A_2$, whence $\{\Phi\} \subset A_1 \cap A_2$. In any case $x \in$ transit$(A_1 \cap A_2)$, whence $A_1 \cap A_2 \subset$ transit$(A_1 \cap A_2)$. $\blacksquare$

Reverting now to ordinary terminology, we call a transopen set an open set, we call a transinterior point an interior point, and we denote transint$A$ by int-A.

Observation 3: Notice that the class of sets of the form $(a, b), [-\infty, b), (a, \infty]$ and $\{\Phi\}$ make a basis[1] for the topology of $\mathbb{R}^T$.

We recall that a subset of topological space is closed if and only if its complement is open.

Example 4: The sets $\{\Phi\}$, $(-\infty, x)$, (x, ∞), $[-\infty, x)$, $(x, \infty]$, $(-\infty, \infty) = \mathbb{R}$, $[-\infty, \infty]$, $[-\infty, \infty)$, $(-\infty, \infty]$ and (x, y) are open on $\mathbb{R}^T$ where $x, y \in \mathbb{R}$ and $x < y$.

Example 5: The sets $\{-\infty\}$, $\{\infty\}$, $\{x\}$, $[-\infty, x]$, $[x, \infty]$, $(-\infty, x]$, $[x, \infty)$, $(x, y]$, $[x, y)$ and $[x, y]$ are not open on $\mathbb{R}^T$ where $x, y \in \mathbb{R}$ and $x < y$.

Example 6: The sets $\{\Phi\}$, $\{-\infty\}$, $\{\infty\}$, $\{x\}$, $[-\infty, \infty]$, $[-\infty, x]$, $[x, \infty]$ and $[x, y]$ are closed on $\mathbb{R}^T$ where $x, y \in \mathbb{R}$ and $x < y$. In fact, $\mathbb{R}^T \setminus \{\Phi\} = [-\infty, \infty]$, $\mathbb{R}^T \setminus \{-\infty\} = \mathbb{R} \cup (1, \infty] \cup \{\Phi\}$, $\mathbb{R}^T \setminus \{\infty\} = \mathbb{R} \cup [-\infty, 1) \cup \{\Phi\}$, $\mathbb{R}^T \setminus \{x\} = [-\infty, x) \cup (x, \infty] \cup \{\Phi\}$, $\mathbb{R}^T \setminus [-\infty, \infty] = \{\Phi\}$, $\mathbb{R}^T \setminus [-\infty, x] = (x, \infty] \cup \{\Phi\}$, $\mathbb{R}^T \setminus [x, \infty] = [-\infty, x) \cup \{\Phi\}$ and $\mathbb{R}^T \setminus [x, y] = [-\infty, x) \cup (y, \infty] \cup \{\Phi\}$ are open.

Example 7: The sets $(-\infty, x)$, (x, ∞), $[-\infty, x)$, $(x, \infty]$, $(-\infty, \infty) = \mathbb{R}$, $[-\infty, \infty)$, $(-\infty, \infty]$, $(-\infty, x]$, $[x, \infty)$, (x, y), $(x, y]$ and $[x, y)$ are not closed on $\mathbb{R}^T$ where $x, y \in \mathbb{R}$ and $x < y$.

Proposition 8: $\mathbb{R}^T$ is a Hausdorff[2] space.

Proof: Let there be distinct $x, y \in \mathbb{R}^T$. If x or y is Φ, say $x = \Phi$, then it is enough to take $A = \{\Phi\}$, with B a neighbourhood[3] of y, such that $\Phi \notin B$. If one of them is equal to $-\infty$ and the other is equal ∞, say $x = -\infty$ and $y = \infty$, it is enough to take $a, b \in \mathbb{R}$ such that $a < b$, $A = [-\infty, a)$ and $B = (b, \infty]$. If one of them is equal to $-\infty$ and the other is a real number, say $x = -\infty$ and $y \in \mathbb{R}$, it is enough to take a positive $\varepsilon \in \mathbb{R}$, $b \in \mathbb{R}$ such that $b < y - \varepsilon$, $A = [-\infty, b)$ and $B = (y - \varepsilon, y + \varepsilon)$. If one of them is equal to ∞ and the other is a real number, say $x = \infty$ and $y \in \mathbb{R}$, it is enough to take a positive $\varepsilon \in \mathbb{R}$, $a \in \mathbb{R}$ such that $y + \varepsilon < a$, $A = (a, \infty]$ and $B = (y - \varepsilon, y + \varepsilon)$. If $x, y \in \mathbb{R}$, it is enough to take a positive $\varepsilon \in \mathbb{R}$ such that $2\varepsilon < |x - y|$, $A = (x - \varepsilon, x + \varepsilon)$ and $B = (y - \varepsilon, y + \varepsilon)$. In every case, A is a neighbourhood of x, B is a neighbourhood of y and $A \cap B = \emptyset$. $\blacksquare$

Proposition 9: The topology on $\mathbb{R}$, induced by the topology of $\mathbb{R}^T$, is the usual topology of $\mathbb{R}$. That is if $A \subset \mathbb{R}^T$ is open on $\mathbb{R}^T$ then $A \cap \mathbb{R}$ is open (in the usual sense) on $\mathbb{R}$ and if $A \subset \mathbb{R}$ is open (in the usual sense) on $\mathbb{R}$ then A is open on $\mathbb{R}^T$.

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[1]A basis for a topological space X is a class $\mathcal{B}$ of open subsets of X such that for every open set A of X and $x \in A$ there is a element B from $\mathcal{B}$ such that $x \in B$ and $B \subset A$.

[2]A topological space, X, is a Haussdorf space if and only if for any distinct $x, y \in X$, there are open sets $U, V \subset X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. See [10].

[3]A subset U, of a topological space, is a neighbourhood of x if and only if $x \in U$ and U is open.

Proof: Let $A \subset \mathbb{R}^T$ be an open set on $\mathbb{R}^T$. If $x \in A \cap \mathbb{R}$ then $x \in \text{int} A$ because $x \in A$. This fact, together with $x \in \mathbb{R}$, implies that there is a positive $\varepsilon \in \mathbb{R}$ such that $(x - \varepsilon, x + \varepsilon) \subset A$, whence $(x - \varepsilon, x + \varepsilon) \subset A \cap \mathbb{R}$. Thus $x \in \text{int}(A \cap \mathbb{R})$, where $\text{int}(A \cap \mathbb{R})$ denotes the interior of $A \cap \mathbb{R}$ in the usual topology on $\mathbb{R}$.

Now let $A \subset \mathbb{R}$ be open (in usual sense) on $\mathbb{R}$. If $x \in A$ then there is a positive $\varepsilon \in \mathbb{R}$ such that $(x - \varepsilon, x + \varepsilon) \subset A$. Thus $x \in \text{int} A$. ∎

Corollary 10: If $A \subset \mathbb{R}^T$ is closed on $\mathbb{R}^T$ then $A \cap \mathbb{R}$ is closed (in the usual sense) on $\mathbb{R}$.

Proposition 11: $\mathbb{R}^T$ is disconnected[4].

Proof: In fact $\mathbb{R}^T = [-\infty, \infty] \cup \{\Phi\}$ and the sets $[-\infty, \infty]$ and $\{\Phi\}$ are open. ∎

Notice that Φ is the unique isolated point[5] of $\mathbb{R}^T$.

Proposition 12: $\mathbb{R}^T$ is a separable[6] space.

Proof: $\mathbb{Q} \cup \{\Phi\}$ is dense in $\mathbb{R}^T$. ∎

Proposition 13: $\mathbb{R}^T$ is compact[7].

Proof: Let I be any set and $\{A_\alpha; \ \alpha \in I\}$ be an open covering of $\mathbb{R}^T$. We have that $\Phi, -\infty, \infty \in \bigcup_{\alpha \in I} A_\alpha$. Thus there are $\alpha_1, \alpha_2, \alpha_3 \in I$ such that $\Phi \in A_{\alpha_1}$, $-\infty \in A_{\alpha_2}$ and $\infty \in A_{\alpha_3}$. So $\{\Phi\} \subset A_{\alpha_1}$ and there are $a, b \in \mathbb{R}$ with $a < b$ such that $[-\infty, a) \subset A_{\alpha_2}$ and $(b, \infty] \subset A_{\alpha_3}$. Furthermore $[a, b] \subset \bigcup_{\alpha \in I} A_\alpha$, whence $[a, b] \subset \left(\bigcup_{\alpha \in I} A_\alpha \right) \cap \mathbb{R} = \bigcup_{\alpha \in I} (A_\alpha \cap \mathbb{R})$. So $\{A_\alpha \cap \mathbb{R}; \ \alpha \in I\}$ is an open covering of $[a, b]$ on $\mathbb{R}$. As $[a, b]$ is compact on $\mathbb{R}$, there are $n \in \mathbb{N}$ and $\alpha_4, \ldots, \alpha_n$ such that $[a, b] \subset \bigcup_{i=4}^{n} (A_{\alpha_i} \cap \mathbb{R}) = \left(\bigcup_{i=4}^{n} A_{\alpha_i} \right) \cap \mathbb{R} \subset \bigcup_{i=4}^{n} A_{\alpha_i}$. Thus $\mathbb{R}^T = ([-\infty, a) \cup [a, b] \cup (b, \infty] \cup \{\Phi\}) \subset \bigcup_{i=1}^{n} A_{\alpha_i}$. ∎

Corollary 14: Let $A \subset \mathbb{R}^T$. It follows that A is compact if and only if A is closed.

Proof: Let $A \subset \mathbb{R}^T$. If A is compact, since $\mathbb{R}^T$ is Hausdorff space, A is closed. See [10], Theorem 26.3. If A is closed, since $\mathbb{R}^T$ is compact, A is compact. See [10], Theorem 26.2. ∎

B. Transreal Sequences

We use the usual definition for the convergence of a sequence in a topological space. That is a sequence, $(x_n)_{n \in \mathbb{N}} \subset$

$\mathbb{R}^T$, converges to $x \in \mathbb{R}^T$ if and only if for each neighbourhood, $V \subset \mathbb{R}^T$ of x, there is $n_V \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq n_V$.

Notice that since $\mathbb{R}^T$ is a Hausdorff space, the limit of a sequence, when it exists, is unique.

Observation 15: Let $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and let $L \in \mathbb{R}$. Notice that $\lim_{n \to \infty} x_n = L$ in $\mathbb{R}^T$ if and only if $\lim_{n \to \infty} x_n = L$ in the usual sense in $\mathbb{R}$. Furthermore, $(x_n)_{n \in \mathbb{N}}$ diverges, in the usual sense, to negative infinity if and only if $\lim_{n \to \infty} x_n = -\infty$ in $\mathbb{R}^T$. Similarly $(x_n)_{n \in \mathbb{N}}$ diverges, in the usual sense, to infinity if and only if $\lim_{n \to \infty} x_n = \infty$ in $\mathbb{R}^T$.

Observation 16: Let $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^T$. Notice that $\lim_{n \to \infty} x_n = \Phi$ if and only if there is $k \in \mathbb{N}$ such that $x_n = \Phi$ for all $n \geq k$.

Proposition 17: Every monotone sequence of transreal numbers is convergent.

Proof: Suppose $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^T$ is increasing. The case for decreasing, transreal $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^T$ is similar. If $x_k = \Phi$, for some $k \in \mathbb{N}$, then $x_n = \Phi$, for all $n \in \mathbb{N}$, because $x_i \leq \Phi \leq x_j$ for all $i \leq k$ and $j \geq k$, whence $\lim_{n \to \infty} x_n = \Phi$. If $x_n = -\infty$, for all $n \in \mathbb{N}$, then $\lim_{n \to \infty} x_n = -\infty$. If $x_n \neq \Phi$, for all $n \in \mathbb{N}$, and $x_k \neq -\infty$, for some $k \in \mathbb{N}$, then $x_n > -\infty$ for all $n \geq k$, whence there is $s = \sup\{x_n; \ n \in \mathbb{N}\}$ and $s \in \mathbb{R} \cup \{\infty\}$. If $s = \infty$ then, for each $a \in \mathbb{R}$, there is $n_a \in \mathbb{N}$ such that $x_{n_a} > a$. Since $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, $x_n \in (a, \infty]$ for all $n \geq n_a$, whence $\lim_{n \to \infty} x_n = \infty$. If $s \in \mathbb{R}$ then $(x_{k+n})_{n \in \mathbb{N}}$ is a monotone, bounded sequence of real numbers, thus it is convergent. Hence $(x_n)_{n \in \mathbb{N}}$ is convergent. ∎

Theorem 18: Every sequence of transreal numbers has a convergent subsequence.

Proof: Let $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^T$. If $\{n; \ x_n \neq \Phi\}$ is a finite set then clearly $\lim_{n \to \infty} x_n = \Phi$. If $\{n; \ x_n \neq \Phi\}$ is an infinite set then denote, by $(y_k)_{k \in \mathbb{N}}$, the subsequence of $(x_n)_{n \in \mathbb{N}}$ of all elements of $(x_n)_{n \in \mathbb{N}}$ that are distinct from Φ. Let $J = \{k; \ y_k > y_m \text{ for all } m > k\}$. If J is a infinite set, we write $J = \{k_1, k_2, \ldots\}$, with $k_1 < k_2 < \cdots$. Since for each $i \in \mathbb{N}$, $k_i \in J$, we have that $y_{k_i} > y_{k_j}$ for all $i < j$. Thus $(y_{k_i})_{i \in \mathbb{N}}$ is a decreasing subsequence of $(x_n)_{n \in \mathbb{N}}$. If J is finite, let k_1 be greater than all of the elements of J. Since $k_1 \notin J$, there is $k_2 > k_1$ such that $y_{k_2} \geq y_{k_1}$. Since $k_2 > k_1$, it follows that $k_2 \notin J$. So there is $k_3 > k_2$ such that $y_{k_3} \geq y_{k_2}$. By induction we build an increasing subsequence $(y_{k_i})_{i \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$. In both cases, in agreement with Proposition 17, $(y_{k_i})_{i \in \mathbb{N}}$ is convergent. ∎

Proposition 19: Let $x, y \in \mathbb{R}^T$ and let $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subset \mathbb{R}^T$ such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$. It follows that:

1) If $x, y \in \{-\infty, \infty\}$ and $x + y = \Phi$ do not occur simultaneously then $\lim_{n \to \infty} (x_n + y_n) = x + y$;

2) If $x, y \in \{0, \infty, -\infty\}$ and $xy = \Phi$ do not occur simultaneously then $\lim_{n \to \infty} (x_n y_n) = xy$;

3) If $y \neq 0$ then $\lim_{n \to \infty} (y_n^{-1}) = y^{-1}$ and

4) If $y = 0$ and there is $k \in \mathbb{N}$ such that $y_n < 0$ for all $n \geq k$ then $\lim_{n \to \infty} (y_n^{-1}) = -(y^{-1})$. If $y = 0$ and there is $k \in \mathbb{N}$, such that $y_n > 0$ for all $n \geq k$, then $\lim_{n \to \infty} (y_n^{-1}) = y^{-1}$.

Theorem 20 (Sandwiches): Let $L \in \mathbb{R}^T$ and let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$ such that $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} z_n = L$. If there is $N \in \mathbb{N}$, such that $x_n \leq y_n \leq z_n$ for all $n \geq N$, then $\lim_{n\to\infty} y_n = L$.

Proof: Let $L \in \mathbb{R}^T$, let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$ and let $N \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = L$, $\lim_{n\to\infty} z_n = L$ and $x_n \leq y_n \leq z_n$ for all $n \geq N$.

If $L = \Phi$, the result follows immediately from Observation 16.

If $L \in \mathbb{R}$, let there be an arbitrary, positive $\varepsilon \in \mathbb{R}$. Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, there are $N_1, N_2 \in \mathbb{N}$ such that $L - \varepsilon < x_n$ for all $n \geq N_1$ and $z_n < L + \varepsilon$ for all $n \geq N_2$. Taking $N_3 = \max\{N, N_1, N_2\}$, we have that $L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$ for all $n \geq N_3$.

If $L = -\infty$, let there be an arbitrary $b \in \mathbb{R}$. Since $\lim_{n\to\infty} z_n = L$, there is $N_1 \in \mathbb{N}$ such that $z_n \in [-\infty, b)$ for all $n \geq N_1$. Taking $N_2 = \max\{N, N_1\}$, we have that $y_n \leq z_n < b$ for all $n \geq N_2$, whence $y_n \in [-\infty, b)$ for all $n \geq N_2$.

If $L = \infty$, the result follows similarly to the previous case. ∎

If $x, y \in \mathbb{R}^T$, we write $x \not< y$, if and only if $x < y$ does not hold and we write $x \not> y$, if and only if $x > y$ does not hold. Notice that $\not<$ is not equivalent to $\geq$. For example $\Phi \not< 0$ but $\Phi \geq 0$ does not hold. See [4] and Appendix B.

Definition 21: Let there be a set $A \subset \mathbb{R}^T$. We say that $u \in \mathbb{R}^T$ is the *supremum of A* and we write $u = \sup A$ if and only if one of the following conditions occurs:

i) $A = \{u\}$ or

ii) $u \not< x$ for all $x \in A$ and for each positive $\varepsilon \in \mathbb{R}$ there is $x \in A$ such that $u - \varepsilon < x$.

And we say that $v \in \mathbb{R}^T$ is the *infimum of A* and we write $v = \inf A$ if and only if one of the following conditions occurs:

iii) $A = \{v\}$ or

iv) $x \not< v$ for all $x \in A$ and for each positive $\varepsilon \in \mathbb{R}$ there is $x \in A$ such that $x < v + \varepsilon$.

Definition 22: Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$. Let $v_n = \inf\{x_k, \ k \geq n\}$ and let $u_n = \sup\{x_k, \ k \geq n\}$. We define and denote the *lower limit* and the *upper limit* of $(x_n)_{n\in\mathbb{N}}$, respectively, by

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} v_n \quad \text{and} \quad \limsup_{n\to\infty} x_n := \lim_{n\to\infty} u_n.$$

Notice that $v_n \not> v_{n+1}$ and $u_n \not< u_{n+1}$ for all $n \in \mathbb{N}$, whence $\lim_{n\to\infty} v_n = \sup_{n\in\mathbb{N}}\{v_n\}$ and $\lim_{n\to\infty} u_n = \inf_{n\in\mathbb{N}}\{u_n\}$. Therefore the notations $\sup_{n\in\mathbb{N}} \inf_{k\geq n}\{x_k\}$ and $\inf_{n\in\mathbb{N}} \sup_{k\geq n}\{x_k\}$ denote, respectively, the lower limit and the upper limit.

Proposition 23: Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$. It follows that there is a limit $\lim_{n\to\infty} x_n$ if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$. In this case, $\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

C. Limit and Continuity of Transreal Functions

We remember that if X is a topological space then $x_0 \in A \subset X$ is a limit point of A if and only if for every neighbourhood V of x_0 it follows that $V \cap (A \setminus \{x_0\}) = \emptyset$. The set of all limit points of A is denoted as A'.

We use the usual definition of the limit of functions in a topological space. That is, if A is a subset of $\mathbb{R}^T$, $f : A \to \mathbb{R}^T$ is a function, x_0 is a limit point of A and L is a transreal number, we say that $\lim_{x\to x_0} f(x) = L$ if and only if, for each neighbourhood V of L, there is a neighbourhood U of x_0 such that $f(A \cap U \setminus \{x_0\}) \subset V$.

Observation 24: Notice that given $x_0, L \in \mathbb{R}$, the transreal limit $\lim_{x\to x_0} f(x) = L$ in $\mathbb{R}^T$ exists if and only if the real limit $\lim_{x\to x_0} f(x) = L$ exists in the usual sense in $\mathbb{R}$. The same can be said about $\lim_{x\to x_0} f(x) = -\infty$, $\lim_{x\to x_0} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = L$, $\lim_{x\to -\infty} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = \infty$, $\lim_{x\to \infty} f(x) = L$, $\lim_{x\to \infty} f(x) = -\infty$ and $\lim_{x\to \infty} f(x) = \infty$.

Observation 25: Let $x_0 \in \mathbb{R}^T$, notice that $\lim_{x\to x_0} f(x) = \Phi$ if and only if there is a neighbourhood U of x_0 such that $f(x) = \Phi$ for all $x \in U \setminus \{x_0\}$.

Proposition 26: Let $A \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$, $x_0 \in A'$ and $L \in \mathbb{R}^T$. The following two statements are equivalent:

1) $\lim_{x\to x_0} f(x) = L$,
2) $\lim_{n\to\infty} f(x_n) = L$ for all $(x_n)_{n\in\mathbb{N}} \subset A \setminus \{x_0\}$ such that $\lim_{n\to\infty} x_n = x_0$.

Proof: Let $A \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$, $x_0 \in A'$ and $L \in \mathbb{R}^T$. Suppose that $\lim_{x\to x_0} f(x) = L$. Let $(x_n)_{n\in\mathbb{N}} \subset A \setminus \{x_0\}$ such that $\lim_{n\to\infty} x_n = x_0$. Let V be an arbitrary neighbourhood of L. Then there is a neighbourhood, U, of x_0 such that $f(A \cap U \setminus \{x_0\}) \subset V$. Since $\lim_{n\to\infty} x_n = x_0$ there is an n_U such that $x_n \in U$ for all $n \geq n_U$. Thus $f(x_n) \in f(A \cap U \setminus \{x_0\}) \subset V$ for all $n \geq n_U$.

Now suppose $\lim_{x\to x_0} f(x) \neq L$. Then there is a neighbourhood, V, of L such that, for each $n \in \mathbb{N}$, there is $x_n \in A$ such that $0 < |x_n - x_0| < \frac{1}{n}$ (if $x_0 \in \mathbb{R}$) or $x_n \in (-\infty, -n)$ (if $x_0 = -\infty$) or $x_n \in (n, \infty)$ (if $x_0 = \infty$), and $f(x_n) \notin V$. Hence $(x_n)_{n\in\mathbb{N}} \subset A \setminus \{x_0\}$, $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) \neq L$. ∎

Proposition 27: Let $L, M \in \mathbb{R}^T$, $A \subset \mathbb{R}^T$, with functions $f, g : A \to \mathbb{R}^T$, and $x_0 \in A'$ such that $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$. It follows that:

1) If $L, M \in \{-\infty, \infty\}$ and $L + M = \Phi$ do not occur simultaneously then $\lim_{x\to x_0} (f + g)(x) = L + M$;
2) If $L, M \in \{0, \infty, -\infty\}$ and $LM = \Phi$ do not occur simultaneously then $\lim_{x\to x_0} (fg)(x) = LM$;
3) If $M \neq 0$ then $\lim_{x\to x_0} \left(\frac{1}{f}\right)(x) = \frac{1}{M}$ and
4) If $M = 0$ and there is a neighbourhood, U, of x_0, such that $g(x) < 0$ for all $x \in U \setminus \{x_0\}$, then $\lim_{x\to x_0} \left(\frac{1}{g}\right)(x) = -(M^{-1})$. If $M = 0$ and there is a neighbourhood, U, of x_0, such that $g(x) > 0$ for all $x \in U \setminus \{x_0\}$, then $\lim_{x\to x_0} \left(\frac{1}{g}\right)(x) = M^{-1}$.

We use the usual definition of continuity in a topological space. That is if $A \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$ is a function and $x_0 \in A$, we say that f is continuous in x_0 if and only if, for each neighbourhood V of $f(x_0)$, there is a neighbourhood U of x_0 such that $f(A \cap U) \subset V$.

Observation 28: Notice that given $x_0 \in \mathbb{R}$, f is continuous in x_0 in $\mathbb{R}^T$ if and only if f is continuous in x_0 in the usual sense in $\mathbb{R}$.

Observation 29: Notice that if $\Phi \in \mathrm{Dm}(f)$ ($\mathrm{Dm}(f)$ denote the domain of f) then f is continuous in Φ.

Proposition 30: Let $A \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$ and $x_0 \in A$. The following two statements are equivalent:

1) f is continuous in x_0,
2) $\lim_{n \to \infty} f(x_n) = f(x_0)$ for all $(x_n)_{n \in \mathbb{N}} \subset A$ and $\lim_{n \to \infty} x_n = x_0$.

Proposition 31: Let $A \subset \mathbb{R}^T$; $f, g : A \to \mathbb{R}^T$ and $x_0 \in A$ such that f and g are continuous in x_0. It follows:

1) If $f(x_0), g(x_0) \in \{-\infty, \infty\}$ and $(f + g)(x_0) = \Phi$ do not occur simultaneously then $f + g$ is continuous in x_0;
2) If $f(x_0), g(x_0) \in \{0, \infty, -\infty\}$ and $(fg)(x_0) = \Phi$ do not occur simultaneously then fg is continuous in x_0;
3) If $g(x_0) \neq 0$ then $\dfrac{1}{g}$ is continuous in x_0 and
4) If $g(x_0) = 0$ and a neighbourhood, U, of x_0, has $g(x) \geq 0$ for all $x \in U$, then $\dfrac{1}{g}$ is continuous in x_0.

Notice that if $g(x_0) = 0$ and there is no neighbourhood, U, of x_0, such that $g(x) \geq 0$ for all $x \in U$, then $\dfrac{1}{g}$ is not continuous in x_0.

Proposition 32: Let $A, B \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$ and $g : B \to \mathbb{R}^T$ such that $f(A) \subset B$. If f is continuous in x_0 and g is continuous in $f(x_0)$ then $g \circ f$ is continuous in x_0.

Proposition 33: Let A be an open set such that $A \subset \mathbb{R}^T$ and let $f : A \to \mathbb{R}^T$. It follows that f is continuous in A if and only if $f^{-1}(B)$ is open, for all open $B \subset \mathbb{R}^T$.

IV. Transreal Derivative and Integral

In this section we extend the concepts of derivative and integral from the domain of real numbers, $\mathbb{R}$, to the domain of transreal numbers, $\mathbb{R}^T$, largely replacing earlier work on this topic [1]. We draw heavily on the results in [4][5][8].

A. Transreal Derivative

Definition 34: Let $A \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$ and $x_0 \in A$. Here A' denotes the set of limit points of A.

i) If $x_0 \in \mathbb{R} \cap A'$, we say f is *differentiable at x_0 on $\mathbb{R}^T$* if and only if f is differentiable at x_0 in the usual sense. And in this case, $f'(x_0)$ is called *the derivative of f at x_0 on $\mathbb{R}^T$* and it is denoted as $f'_{\mathbb{R}^T}(x_0)$.

ii) If $x_0 \in \{-\infty, \infty\} \cap D'$ (where D denotes the set of points in A at which f is differentiable in the usual sense), we say f is *differentiable at x_0 on $\mathbb{R}^T$* if and only if the following limit exists

$$\lim_{x \to x_0} f'(x).$$

And if this limit exists then it is called the *derivative of f at x_0 on $\mathbb{R}^T$* and it is denoted as $f'_{\mathbb{R}^T}(x_0)$.

iii) If $x_0 \notin A'$ we define the *derivative of f at x_0 on $\mathbb{R}^T$* as $f'_{\mathbb{R}^T}(x_0) := \Phi$.

Note that we choose an intuitive way of defining the derivative on $\mathbb{R}^T$ as the slope of the tangent line. So the derivative, on $\mathbb{R}^T$, at a real number is the usual derivative at that number. And we have the intuitive view that if the limit of slopes of the tangent lines at x, when x tends to

∞ (or $-\infty$), is $L \in \mathbb{R}^T$ then the slope of the tangent line at ∞ (or $-\infty$) is L. Because of this we choose to define $f'_{\mathbb{R}^T}(\infty) = \lim_{x \to \infty} f'(x) \left(f'_{\mathbb{R}^T}(-\infty) = \lim_{x \to -\infty} f'(x) \right)$.

Observe that it is not possible to define the derivative at $x_0 \notin A'$ by way of a limit because, as is known, if we try to apply the limit definition at $x_0 \notin A'$, any $L \in \mathbb{R}^T$ could be the limit $\lim_{x \to x_0} f(x)$. In fact, since $x_0 \notin A'$, there is a neighbourhood U of x_0 such that $A \cap U = \emptyset$, hence for any neighbourhood V of L, $f(x) \in V$ for all $x \in A \cap U$. Because, vacuously, there is no $x \in A \cap U$ such that $f(x) \notin V$. Rather than accept the indeterminacy of the derivative at $x_0 \notin A'$, we choose to define $f'_{\mathbb{R}^T}(x_0) = \Phi$. This will presently lead us to the position where the exponential is identically its own derivative with $e'(x) = e(x)$ so that the usual properties of this important function hold when extended to $\mathbb{R}^T$.

Example 35: Let $f(x) = e^x$. It follows from Definition 34 that $f'_{\mathbb{R}^T}(x) = e^x$ for all $x \in \mathbb{R}^T$. Particularly, $f'_{\mathbb{R}^T}(-\infty) = 0$, $f'_{\mathbb{R}^T}(\infty) = \infty$ and $f'_{\mathbb{R}^T}(\Phi) = \Phi$.

Observation 36: Note that differentiability on $\mathbb{R}^T$ does not imply continuity. For example let $f : \mathbb{R}^T \to \mathbb{R}^T$, where

$$f(x) = \begin{cases} e^x & \text{, if } x \neq \infty \\ 1 & \text{, if } x = \infty \end{cases}.$$

Clearly f is not continuous at ∞ but $\lim_{x \to \infty} f'(x) = \infty$, whence f is differentiable at ∞ on $\mathbb{R}^T$. For the definition of e^x in $\mathbb{R}^T$ see [1].

The usual derivative is generally defined as a rate of change of a function. That is, if $x_0 \in \mathbb{R}$ then $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$. Our definition may not show it explicitly, but the derivative, on $\mathbb{R}^T$, at ∞ or $-\infty$, is also a rate of change of a function. This is shown in propositions 38 and 39. First we need the following definition.

Definition 37: Let $A \subset \mathbb{R}^T$, $f : A \times A \to \mathbb{R}^T$, $x_0 \in A'$ and $L \in \mathbb{R}^T$. We say that

$$\lim_{\substack{x \to x_0 \\ y \to x_0}} f(x, y) = L$$

if and only if, given an arbitrary neighbourhood V of L there is a neighbourhood U of x_0 such that $f(x, y) \in V$ whenever $x \neq y$ and $x, y \in A \cap U \setminus \{x_0\}$.

Note that $\lim_{\substack{x \to x_0 \\ y \to x_0}} f(x, y) \neq \lim_{(x,y) \to (x_0, x_0)} f(x, y)$, where $\lim_{(x,y) \to (x_0, x_0)} f(x, y)$ denotes the limit, in the usual sense, of a function of two variables. In other words, these are different limiting processes.

Proposition 38: Let $a \in \mathbb{R}$ and $f : (a, \infty] \to \mathbb{R}^T$ such that f is differentiable in (a, ∞). It follows that f is differentiable at ∞ if and only if there exists $\lim_{\substack{x \to \infty \\ y \to \infty}} \dfrac{f(x) - f(y)}{x - y}$. And in this case,

$$f'_{\mathbb{R}^T}(\infty) = \lim_{\substack{x \to \infty \\ y \to \infty}} \dfrac{f(x) - f(y)}{x - y}.$$

Proof: Let $a \in \mathbb{R}$ and $f : (a, \infty] \to \mathbb{R}^T$ such that f is differentiable in (a, ∞). Observe that f is continuous in (a, ∞).

First let us suppose that $f'_{\mathbb{R}^T}(\infty) = L \in \mathbb{R}^T$, that is $\lim_{z \to \infty} f'_{\mathbb{R}^T}(z) = L$. Let V be an arbitrary neighbourhood of L. Then there is $M > a$ such that $f'_{\mathbb{R}^T}(z) \in V$ for all $z \in (M, \infty)$. Let $x, y \in (M, \infty)$ such that $x \neq y$. Say

$x < y$. Since f is continuous in $[x, y]$ and differentiable in (x, y), by the Mean Value Theorem, there is $z \in (x, y)$ such that $\dfrac{f(x) - f(y)}{x - y} = f'_{\mathbb{R}^T}(z)$. Since $z \in (x, y) \subset (M, \infty)$ we have

$$\frac{f(x) - f(y)}{x - y} = f'_{\mathbb{R}^T}(z) \in V.$$

Thus $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} \dfrac{f(x) - f(y)}{x - y} = L$.

Now suppose that $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} \dfrac{f(x) - f(y)}{x - y} = L$. Note that $L \neq \Phi$ for $f'_{\mathbb{R}^T}(z) \in \mathbb{R}$ for all $z \in (a, \infty)$. If $L \in \mathbb{R}$, let there be an arbitrary $\varepsilon \in \mathbb{R}^+$. Then there is $M \geq a$ such that $-\dfrac{\varepsilon}{2} < \dfrac{f(x) - f(y)}{x - y} - L < \dfrac{\varepsilon}{2}$ whenever $x, y \in (M, \infty)$ and $x \neq y$. For each $x \in (M, \infty)$, taking the limit in the inequality with y tending to x, we obtain $-\varepsilon < -\dfrac{\varepsilon}{2} \leq \lim\limits_{y \to x} \dfrac{f(y) - f(x)}{y - x} - L \leq \dfrac{\varepsilon}{2} < \varepsilon$, whence $-\varepsilon < f'_{\mathbb{R}^T}(x) - L < \varepsilon$, therefore $\lim\limits_{x \to \infty} f'_{\mathbb{R}^T}(x) = L$. If $L = \infty$, let there be an arbitrary $N \in \mathbb{R}^+$. Then there is $M \geq a$ such that $2N < \dfrac{f(x) - f(y)}{x - y}$ whenever $x, y \in (M, \infty)$ and $x \neq y$. For each $x \in (M, \infty)$, taking the limit in the inequality with y tending to x, we obtain $N < 2N \leq \lim\limits_{y \to x} \dfrac{f(y) - f(x)}{y - x}$, whence $N < f'(x)$, therefore $\lim\limits_{x \to \infty} f'_{\mathbb{R}^T}(x) = \infty$. If $L = -\infty$ the result follows similarly. $\blacksquare$

Proposition 39: Let $a \in \mathbb{R}$ and $[-\infty, a) \to \mathbb{R}^T$ such that f is differentiable in $[-\infty, a)$. It follows that f is differentiable at $-\infty$ if and only if there exists $\lim\limits_{\substack{x \to -\infty \\ y \to -\infty}} \dfrac{f(x) - f(y)}{x - y}$. And in this case,

$$f'_{\mathbb{R}^T}(-\infty) = \lim_{\substack{x \to -\infty \\ y \to -\infty}} \frac{f(x) - f(y)}{x - y}.$$

Proof: The proof is similar to the proof of Proposition 38. $\blacksquare$

What about the limit $\lim\limits_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y}$ for $x_0 \in \mathbb{R}$? Propositions 40 and 42 show that it is still the case that $f'_{\mathbb{R}^T}(x_0) = \lim\limits_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y}$ but additional conditions must be satisfied.

Proposition 40: Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. If f is continuous at x_0 and there exists the limit $\lim\limits_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y}$ then f is differentiable at x_0 and

$$f'_{\mathbb{R}^T}(x_0) = \lim_{\substack{x \to x_0 \\ y \to x_0}} \frac{f(x) - f(y)}{x - y}.$$

Proof: Let f be continuous at x_0 such that there exists a limit $\lim\limits_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y}$, say $\lim\limits_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y} = a$. Since f is continuous at x_0, $\lim\limits_{y \to x_0} f(y) = f(x_0)$. Let there be an arbitrary $\varepsilon \in \mathbb{R}^+$. Then there is a $\delta \in \mathbb{R}^+$ such that for each $x \in A \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$, it follows that

$$-\frac{\varepsilon}{2} < \frac{f(x) - f(y)}{x - y} - a < \frac{\varepsilon}{2}$$

for all $y \in A \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$.

Taking the limit in the above inequality with y tending to x_0, we obtain $-\dfrac{\varepsilon}{2} \leq \dfrac{f(x) - f(x_0)}{x - x_0} - a \leq \dfrac{\varepsilon}{2}$. Thus

$$-\varepsilon < -\frac{\varepsilon}{2} \leq \frac{f(x) - f(x_0)}{x - x_0} - a \leq \frac{\varepsilon}{2} < \varepsilon$$

for all $x \in A \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$, whence $f'_{\mathbb{R}^T}(x_0) = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = a$. $\blacksquare$

Observation 41: Notice that in Proposition 40, the hypothesis of the continuity of f is, in fact, needed. For instance let $f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \begin{cases} x & , \text{ if } x \neq 0 \\ 1 & , \text{ if } x = 0 \end{cases}.$$

Clearly f is not differentiable at 0, but $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{f(x) - f(y)}{x - y} = 1$.

Proposition 42: Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. If f is continuously differentiable in x_0 (which means f is differentiable in x_0 and $f'_{\mathbb{R}^T}$ is continuous in x_0), then there exists $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{f(x) - f(y)}{x - y}$ and

$$\lim_{\substack{x \to x_0 \\ y \to x_0}} \frac{f(x) - f(y)}{x - y} = f'_{\mathbb{R}^T}(x_0).$$

Proof: Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$ such that f is continuously differentiable in x_0. Let us denote, as a, the derivative of f at x_0, that is, $f'_{\mathbb{R}^T}(x_0) = a$. Let there be an arbitrary $\varepsilon \in \mathbb{R}^+$. Since $f'_{\mathbb{R}^T}$ is continuous at x_0, there is a $\delta \in \mathbb{R}^+$ such that $f'_{\mathbb{R}^T}(z) \in (a - \varepsilon, a + \varepsilon)$ whenever $z \in A \cap (x_0 - \delta, x_0 + \delta)$. Now let $x, y \in A \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ such that $x \neq y$. Say $x < y$. Since f is continuous in $[x, y]$ and differentiable in (x, y), by the Mean Value Theorem, there is $z \in (x, y)$ such that $\dfrac{f(x) - f(y)}{x - y} = f'_{\mathbb{R}^T}(z)$. Since $z \in (x, y) \subset A \cap (x_0 - \delta, x_0 + \delta)$, we have

$$\frac{f(x) - f(y)}{x - y} = f'_{\mathbb{R}^T}(z) \in (a - \varepsilon, a + \varepsilon).$$

Thus $\lim\limits_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y} = a$. $\blacksquare$

Observation 43: Notice that in Proposition 42, the hypothesis of the continuity of $f'_{\mathbb{R}^T}$ is, in fact, needed. Let $f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}.$$

Note that $f'_{\mathbb{R}^T}(0) = 0$ but $\lim\limits_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y}$ does not exist. Indeed given an arbitrary $\delta \in \mathbb{R}^+$, let us take a positive, even integer, n, that is sufficiently large that $\dfrac{1}{n\pi} \in (-\delta, \delta)$. Denoting $x = \dfrac{1}{n\pi}$, $y = \dfrac{1}{n\pi + \frac{\pi}{2}}$ and $z = \dfrac{1}{(n+1)\pi + \frac{\pi}{2}}$, we have $x, y, z \in (-\delta, \delta)$ and $\dfrac{f(x) - f(y)}{x - y} = -\dfrac{4n}{2n\pi + \pi}$ and $\dfrac{f(x) - f(z)}{x - z} = \dfrac{4n}{6n\pi + 9\pi}$.

If we make some changes to the definition of $\lim_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y}$ then, under suitable conditions, we can withdraw the hypothesis of the continuity of $f'_{\mathbb{R}^T}$ in Proposition 42. This is explained in the following proposition.

Proposition 44: Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'_- \cap A'_+$. If f is differentiable at x_0 then, given an arbitrary neighbourhood V of $f'_{\mathbb{R}^T}(x_0)$, there is a neighbourhood U of x_0 such that $\dfrac{f(x) - f(y)}{x - y} \in V$, whenever $x, y \in A \cap U$ and $x < x_0 < y$.

Proof: Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'_- \cap A'_+$ such that f is differentiable at x_0. Let us denote as a the derivative of f at x_0, that is $f'_{\mathbb{R}^T}(x_0) = a$.

Let $V = (a - \varepsilon, a + \varepsilon)$ for some $\varepsilon \in \mathbb{R}^+$. Then there is a $\delta \in \mathbb{R}^+$ such that $\left| \dfrac{f(x) - f(x_0)}{x - x_0} - a \right| < \dfrac{\varepsilon}{3}$ whenever $x \in A \cap (x_0 - \delta, x_0)$ and $\left| \dfrac{f(y) - f(x_0)}{y - x_0} - a \right| < \dfrac{\varepsilon}{3}$ whenever $y \in A \cap (x_0, x_0 + \delta)$. Now let $x, y \in A \cap (x_0 - \delta, x_0 + \delta)$ such that $x < x_0 < y$. Observe that $\dfrac{f(x) - f(y)}{x - y} - a = \dfrac{y - x_0}{y - x} \left(\dfrac{f(y) - f(x_0)}{y - x_0} - a \right) - \dfrac{y - x_0}{y - x} \left(\dfrac{f(x) - f(x_0)}{x - x_0} - a \right) + \left(\dfrac{f(x) - f(x_0)}{x - x_0} - a \right)$ and that $\left| \dfrac{y - x_0}{y - x} \right| < 1$. Hence

$$\left| \dfrac{f(x) - f(y)}{x - y} - a \right| \leq \left| \dfrac{y - x_0}{y - x} \right| \left| \dfrac{f(y) - f(x_0)}{y - x_0} - a \right| + \left| \dfrac{y - x_0}{y - x} \right| \left| \dfrac{f(x) - f(x_0)}{x - x_0} - a \right| + \left| \dfrac{f(x) - f(x_0)}{x - x_0} - a \right| < \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} = \varepsilon.$$

Thus $\dfrac{f(x) - f(y)}{x - y} \in V$. ∎

Observation 45: Notice that the converse of Proposition 44 is false. Indeed, let f be the function in Observation 41. For all neighbourhood V of 1, there is a neighbourhood U of 0 such that $\dfrac{f(x) - f(y)}{x - y} \in V$, whenever $x, y \in A \cap U$ and $x < 0 < y$, but f is not differentiable at 0. This means that, even with the change to the definition of $\lim_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y}$, the hypothesis of the continuity of f in Proposition 40 cannot be withdrawn.

B. Transreal Integral

Definition 46: Let $a, b \in \mathbb{R}^T$. We define:

a) $(a, b) := \{x \in \mathbb{R}^T; \ a < x < b\}$, $(a, b] := (a, b) \cup \{b\}$, $[a, b) := \{a\} \cup (a, b)$ and $[a, b] := \{a\} \cup (a, b) \cup \{b\}$. We say that A, with $A \subset \mathbb{R}^T$, is an interval if and only if A is one of these four types of sets.

Notice that we could define $[a, b] = \{x \in \mathbb{R}^T; \ a \leq x \leq b\}$, but, in this way, we would have $[a, \Phi] = \emptyset$. However, from our definition, we have that $[a, \Phi] = \{a, \Phi\}$. Note also that $(a, \Phi) = \emptyset = (\Phi, a)$, $(a, \Phi] = \{\Phi\} = [\Phi, a)$, $[a, \Phi) = \{a\} = (\Phi, a]$ and $[\Phi, a] = \{\Phi, a\}$ for all $a \in \mathbb{R}^T$.

b) If $I \in \{(a, b), (a, b], [a, b), [a, b]\}$, we define the *length* of I as

$$|I| := \begin{cases} 0 & , \text{ if } I = \emptyset \\ k - k & , \text{ if } I = \{k\} \text{ for some } k \in \mathbb{R}^T \\ b - a & , \text{ otherwise} \end{cases}.$$

Notice that we could define, simply, $|I| = b - a$. But $|I|$ would not be well defined. Because we would have, for $a \in \mathbb{R}$, $|[a, \Phi]| = \Phi - a = \Phi$ and $|[a, a]| = a - a = 0$, but $[a, \Phi) = \{a\} = [a, a)$. And, thus, we would have the absurdity $\Phi = |[a, \Phi)| = |[a, a)| = 0$.

c) Let $A \subset \mathbb{R}^T$. We say that $\mathcal{X}_A$ is the *characteristic function* of A if and only if

$$\mathcal{X}_A(x) = \begin{cases} 1 & , \text{ if } x \in A \\ 0 & , \text{ if } x \notin A \end{cases}.$$

d) Let $[a, b]$ be an interval. A set P is said to be a *partition* of $[a, b]$ if and only if there are $n \in \mathbb{N}$, $x_0, \ldots, x_n \in [a, b]$ such that $P = (x_0, \ldots, x_n)$ where $x_0 = a$, $x_n = b$ and, furthermore, if $n = 2$, $x_0 \leq x_1$ and if $n > 2$, $x_0 < x_1 < \cdots < x_{n-1} < x_n$.

Notice that we could require, simply, $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$ but then the interval $[a, \Phi]$, for $a \in \mathbb{R}$ would not have any partitions because $a \leq \Phi$ does not hold. Note also that, in the case $n = 2$, we require $x_0 \leq x_1$ in order for $P = (a, a, \Phi)$ to be a partitions of $[a, \Phi]$. This allows us to define step functions on $[a, \Phi]$, $\varphi = \varphi(a) \mathcal{X}_{(a,a]} + \varphi(\Phi) \mathcal{X}_{(a,\Phi]}$.

e) We say that $\varphi : [a, b] \to \mathbb{R}^T$ is a *step function on $[a, b]$* if and only if there is a partition $P = (x_0, \ldots, x_n)$ of $[a, b]$ and $c_1, \ldots, c_n \in \mathbb{R}^T$ such that

$$\varphi = \sum_{j=1}^{n} c_j \mathcal{X}_{I_j},$$

where $I_j = (x_{j-1}, x_j]$ for all $j \in \{1, \ldots, n\}$.

We denote as $\mathcal{S}([a, b])$ the set of step functions on $[a, b]$ and note that the description of a step function is not unique.

Definition 47: Let $a, b \in \mathbb{R}^T$ and $\varphi = \sum_{j=1}^{n} c_j \mathcal{X}_{I_j}$ be a step function on $[a, b]$. We define the *integral in $\mathbb{R}^T$ of φ on $[a, b]$* as

$$\int_{\mathbb{R}^T}^{b}{}_{a} \varphi(x) \ dx := \sum_{\substack{j=1 \\ j; \ c_j \neq 0}}^{n} c_j |I_j|.$$

Notice that the integral of a step function is independent of the particular description used.

Definition 48: Let $a, b \in \mathbb{R}^T$ and let there be a function $f : [a, b] \to \mathbb{R}^T$. We say that f is *integrable in $\mathbb{R}^T$ on $[a, b]$* if and only if

$$\inf \left\{ \int_{\mathbb{R}^T}^{b}{}_{a} \varphi(x) \ dx; \ \varphi \in \mathcal{S}([a, b]) \text{ and } \varphi \not< f \right\} = \sup \left\{ \int_{\mathbb{R}^T}^{b}{}_{a} \sigma(x) \ dx; \ \sigma \in \mathcal{S}([a, b]) \text{ and } f \not< \sigma \right\}.$$

And in this case the *integral of f in $\mathbb{R}^T$ on $[a, b]$* is defined as

$$\int_{\mathbb{R}^T}^{b}{}_{a} f(x) \ dx := \inf \left\{ \int_{\mathbb{R}^T}^{b}{}_{a} \varphi(x) \ dx; \ \varphi \in \mathcal{S}([a, b]) \text{ and } \varphi \not< f \right\}.$$

See Definition 21 to recall sup and inf.

Notice that if φ is a step function on $[a, b]$ then definitions 47 and 48 give the same result.

Proposition 49: a) Let $a, b \in \mathbb{R}$ and let there be a bounded function $f : [a, b] \to \mathbb{R}$. It follows that f is Riemann integrable in $\mathbb{R}$ if and only if f is integrable in $\mathbb{R}^T$. And in this case, $\int_a^b f(x) \ dx = \int_{a \atop \mathbb{R}^T}^b f(x) \ dx$.

b) Let $a \in \mathbb{R}$ and let $f : [a, \infty] \to \mathbb{R}$ be a function that is Riemann integrable on every closed subinterval of $[a, \infty)$. The improper Riemann integral $\int_a^\infty |f|(x) \ dx$ exists if and only if f is integrable in $\mathbb{R}^T$. And in this case, $\int_a^\infty f(x) \ dx = \int_{a \atop \mathbb{R}^T}^\infty f(x) \ dx$.

c) Let $b \in \mathbb{R}$ and let $f : [-\infty, b] \to \mathbb{R}$ be a function that is Riemann integrable on every closed subinterval of $(-\infty, b]$. The improper Riemann integral $\int_{-\infty}^b |f|(x) \ dx$ exists if and only if f is integrable in $\mathbb{R}^T$. And in this case, $\int_{-\infty}^b f(x) \ dx = \int_{-\infty \atop \mathbb{R}^T}^b f(x) \ dx$.

d) Let $f : [-\infty, \infty] \to \mathbb{R}$ be a function that is Riemann integrable on every closed subinterval of $(-\infty, \infty)$. The improper Riemann integral $\int_{-\infty}^\infty |f|(x) \ dx$ exists if and only if f is integrable in $\mathbb{R}^T$. And in this case, $\int_{-\infty}^\infty f(x) \ dx = \int_{-\infty \atop \mathbb{R}^T}^\infty f(x) \ dx$.

e) Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}^T$ be a function such that $f((a, b]) \subset \mathbb{R}$, $f(a) = \infty$ and f is Riemann integrable on any subinterval in $(a, b]$. The improper Riemann integral $\int_a^b |f|(x) \ dx$ exists if and only if f is integrable in $\mathbb{R}^T$. And in this case $\int_a^b f(x) \ dx = \int_{a \atop \mathbb{R}^T}^b f(x) \ dx$.

f) Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}^T$ be a function such that $f([a, b)) \subset \mathbb{R}$, $f(b) = \infty$ and f is Riemann integrable on any subinterval in $[a, b)$. The improper Riemann integral $\int_a^b |f|(x) \ dx$ exists if and only if f is integrable in $\mathbb{R}^T$. And in this case $\int_a^b f(x) \ dx = \int_{a \atop \mathbb{R}^T}^b f(x) \ dx$.

Proof: a) It is sufficient to observe that, since $[a, b] \subset \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$, the integral $\int_{a \atop \mathbb{R}^T}^b f(x) \ dx$ is precisely the Darboux integral, which is known to be equivalent to the Riemann integral.

b), c), d), e) and f) It is sufficient to note that if $[a, b] \subset [-\infty, \infty]$ and $f : [a, b] \to [-\infty, \infty]$ is a non-negative function that is Lebesgue integrable then the integral $\int_{a \atop \mathbb{R}^T}^b f(x) \ dx$ is equal the Lebesgue integral of f on (a, b). See [12], Section 2.1 and use the Theorems 37, 38, 45 and 46 in [11]. ∎

Example 50: Let $f : \mathbb{R}^T \to \mathbb{R}^T$ and let there be an aribtrary $a \in \mathbb{R}^T$. It follows that:

a) If $a \in \mathbb{R}$ and $f(a) \in \mathbb{R}$ then $\int_{a \atop \mathbb{R}^T}^a f(x) \ dx = 0$. Because $\int_{a \atop \mathbb{R}^T}^a f(x) \ dx = f(a) \, |(a, a]| = f(a) \times 0 = 0$;

b) If $a \in \{-\infty, \infty, \Phi\}$ then $\int_{a \atop \mathbb{R}^T}^a f(x) \ dx = \Phi$. Because $\int_{a \atop \mathbb{R}^T}^a f(x) \ dx = f(a) |(a, a]| = f(a) \times \Phi = \Phi$;

c) $\int_{a \atop \mathbb{R}^T}^\Phi f(x) \ dx = \int_{\Phi \atop \mathbb{R}^T}^a f(x) \ dx = \Phi$. In order to see this, let $\varphi \in \mathcal{S}([a, \Phi])$. Whence $\int_{a \atop \mathbb{R}^T}^\Phi \varphi(x) \ dx = \varphi(a)|(a, a]| + \varphi(\Phi)|(a, \Phi]| = \varphi(a)|(a, a]| + \varphi(\Phi)\Phi = \varphi(a)|(a, a]| + \Phi = \Phi$. Thus $\int_{a \atop \mathbb{R}^T}^\Phi f(x) \ dx = \Phi$.

The reader will appreciate that it would be possible to define the integral in $\mathbb{R}^T$ in a more general way, for example by defining it in a manner analogous to the Lebesgue integral, if we replace step functions with single functions. However, in this paper, we had the more modest aim of giving the first, detailed definition of the integral in $\mathbb{R}^T$. We choose a definition that extends the concept of the integral to $\mathbb{R}^T$ in a simple way. We then found that it is totally coincident with the Riemann integral when the domain and codomain of a function are subsets of the real numbers: $\text{Dm}(f) \subset \mathbb{R}$ and $\text{CDm}(f) \subset \mathbb{R}$.

V. DISCUSSION

The transintegral, as described above, is the first mathematical structure that has been defined for which the trans version is less general than the usual one. (We now know that there is a more general definition of the transintegral that contains the real integral. A paper on that subject has been submitted for publication elsewhere but we press on, here, with a notational device that admits all of the results of real calculus to transreal calculus. We mention it because this notational approach may be of more widespread use.)

One possibility for defining the transintegral, so that it contains the usual integral, is that we should define the transintegral asymptotically toward the infinities, as usual, and then observe that the the infinities are singleton points which make no additional contribution to the transintegral. The resulting transintegral differing from the usual one only in that it is defined over functions of transnumbers.

While a difference in integrals remains, we may handle the difference notationally. Consider the symbols for the usual integral: $\int_a^b f(x) \ \mathrm{d}x$. We describe a notation to indicate whether a limit of integration, say a, is exact, $x = a$, or asymptotic, $x \to a$, for transreal a, x. We specify the reading of an isolated symbol, a, so that a is a shorthand for $x = a$ when $a \in \mathbb{R} \cup \{\Phi\}$ and a is a shorthand for $x \to a$ when $a \in \{-\infty, \infty\}$. When the shorthand does not apply we write the limit explicitly. For example the fragment $\int_0^\infty$ indicates the integral from exactly zero, asymptotically toward infinity,

as usual, and the new fragment $\int_{x\to 0}^{x=\infty}$ indicates the integral asymptotically from zero, exactly to infinity. This notation preserves the whole of the usual notation for integrals, preserves all of the results of real integration and introduces new, non-finite results.

We believe it is important to examine many possible definitions of the transintegral and their uses before coming to a judgement on what the standard definition should be. This is entirely normal in a new area of mathematics, as recapitulated in the various revisions of the transmathematical structures developed to date.

The transreal derivative is and, in future, the transreal integral will be, supersets of their real counterparts. They differ from their real counterparts only in being more powerful: they give solutions at singularities where real calculus fails. Hence software that implements transreal calculus is more competent than software that implements real calculus.

However, both kinds of calculus and software are partial. There are occasions when both a real limit and a transreal limit fail to exist, say where the function oscillates, unboundedly, toward both positive and negative infinity. In these cases a solution can be had mathematically by operating on solution sets. Where the limit, derivative, integral, or whatever does not exist the solution is the empty set. In general it is impractical for a computer to operate on arbitrary sets but it may be feasible simply to return a flag to say that the limit, etc. does not exist.

It is already known that the methods just developed are sufficient to extend Newtonian Physics to a Trans-Newtonian Physics that operates at singularities. We hope the present series of paper will build confidence in transmathematics to the point where such results are accepted for publication.

VI. Conclusion

We add the usual topology of measure theory and integration theory to the space of transreal numbers and prove that this space is a compact, separable, Hausdorff space. Using these results we extend the limit and continuity of real functions to transreal functions. We extend real derivatives and integrals so that they hold over functions of transreal numbers. This gives us a transmathematics which operates at mathematical singularities.

In the following Appendix A we also prove that the space of transreal numbers is metrisable and we derive a real-numbered metric for the transreal numbers. Hence the transreal numbers are a metric space.

Separately we show that the usual geometrical construction of the tangent is defined for infinite values of the function when it is calculated using transreal arithmetic. We describe the transreal tangent and transreal arctangent as total functions of transreal numbers. We show that while the finite values of the transreal tangent have a primitive period of π, the function has a primitive period of 2π when the infinite values are considered because the transreal tangent alternately asymptotes to and arrives at an infinity on alternate periods of π.

Appendix A
Metrics

All of the proofs, given above, are developed in terms of topology but the Urysohn Metrisation Theorem[13][8] shows that metrics exist for the set of transreal numbers which are compatible with their topology. Indeed the set of transreal numbers is compact (Proposition 13), hence regular[9]. Furthermore the class of sets of the form (a, b), $[-\infty, b)$, $(a, \infty]$ and $\{\Phi\}$, where $a, b \in \mathbb{Q}$, make a countable basis (Observation 3) for $\mathbb{R}^T$, hence, by the Urysohn Metrisation Theorem, it is metrisable.

In addition to the existential assurance given by the Urysohn Metrisation Theorem, we can explicitly construct a metric in $\mathbb{R}^T$. This construction follows below. We note, in passing, the earlier definition of a transmetric which gives the distance between any two transreal numbers in terms of a transreal number [2].

Let $\varphi : [-\infty, \infty] \to [-1, 1]$ be defined as

$$\varphi(x) = \begin{cases} -1 & , \text{ if } x = -\infty \\ \dfrac{x}{1 + |x|} & , \text{ if } x \in \mathbb{R} \\ 1 & , \text{ if } x = \infty \end{cases}.$$

Notice that φ is a increasing function and φ is a homeomorphism[10]. Now let $d : \mathbb{R}^T \times \mathbb{R}^T \to \mathbb{R}^T$ be defined as

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 2, & \text{if } x = \Phi \text{ or else } y = \Phi. \\ |\varphi(x) - \varphi(y)|, & \text{otherwise} \end{cases}$$

Proposition 51: $\mathbb{R}^T$ is metrisable. More specifically, d, as defined above, is a metric on $\mathbb{R}^T$ which induces the topology of $\mathbb{R}^T$.

Proof: Let us see that d is, in fact, a metric on $\mathbb{R}^T$. Clearly, for all $x, y \in \mathbb{R}^T$, $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ and $d(x, y) \geq 0$. If $x, y, z \in [-\infty, \infty]$ then $d(x, z) = |\varphi(x) - \varphi(z)| = |\varphi(x) - \varphi(y) + \varphi(y) - \varphi(z)| \leq |\varphi(x) - \varphi(y)| + |\varphi(y) - \varphi(z)| = d(x, y) + d(y, z)$. The reader can verify that triangular inequality is also true when $x, y, z \in [-\infty, \infty]$ does not hold.

Now let us see that the topology induced by d and the transreal topology are the same. Recall that, in metric spaces, $B(x, r)$ denotes the ball of centre x and radius r, that is, $B(x, r) = \{y \in \mathbb{R}^T; \ d(y, x) < r\}$. Let us see that every open set in the transreal topology is also open in the topology induced by d. Let U be an arbitrary open set in the transreal topology and $x \in U$. If $x = \Phi$ then we take $B(x, 1)$ and so $B(x, 1) = \{\Phi\} \subset U$. If $x \neq \Phi$, since φ^{-1} is continuous, there is $\delta \in \mathbb{R}$ with $0 < \delta < 2$ such that $y \in U$ whenever $|\varphi(y) - \varphi(x)| < \delta$. So $B(x, \delta) \subset U$.

Now let us see that every ball on metric d is an open set in the transreal topology. Let x be an arbitrary transreal number, $x \in \mathbb{R}^T$, and let r be an aribitrary, positive, real number,

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[8]Urysohn Metrisation Theorem: Every regular space X with a countable basis is metrisable ([10], Theorem 34.1).

A topological space is regular if and only if all singleton sets are closed and, for each pair consisting of a point x and a closed set B disjoint from $\{x\}$, there exist disjoint open sets containing x and B, respectively.

A topological space is metrisable if and only if it possesses a metric which is compatible with their topology.

[9]Every compact space is regular ([10], Theorem 32.3).

[10]A homeomorphism is a continuous bijective function whose inverse is continuous.

$r \in \mathbb{R}$. If $x = \Phi$, we take the neighbourhood $U = \{\Phi\}$ of Φ, hence $U \subset B(x, r)$. If $x \neq \Phi$, since φ is continuous, there is a neighbourhood U of x such that $|\varphi(y) - \varphi(x)| < r$ whenever $y \in U$. So $U \subset B(x, r)$. ∎

As $\mathbb{R}^T$ is metrical, all results on metric spaces hold.

Corollary 52: $\mathbb{R}^T$ is a complete, metric space.

Proof: Every compact, metric space is complete and $\mathbb{R}^T$ is compact and metric. ∎

As $\mathbb{R}^T$ is complete, all results on complete, metric spaces hold.

APPENDIX B
TRANSREAL RELATIONAL OPERATORS

This appendix corrects typographical errors in [4]. As the transreal relational operators are used throughout the present paper so it is essential that they are given correctly. The heading rows of tables now correctly show r_2; in the not-equal table and the not-less-or-equal table the last row correctly shows T, T, T, F; in the not-less-or-greater table the last row correctly shows F, F, F, F.

There are three basic, transreal, relational operators: *less than* $(<)$, *equal to* $(=)$, *grater than*, $(>)$. These operators are mutually exclusive so they can be combined in $2^3 = 8$ ways. All 8 combinations are distinct and meaningful, including the empty operator with no occurrences of the basic operators. All 8 combinations can be combined with the logical negation operator $(!)$. This yields $2 \times 2^3 = 2^4 = 16$ distinct and meaningful operators. The multiplication table for each operator is given here.

The relational operators can be formalised as production rules of the form $a \bullet b \rightarrow c$, where $\bullet$ is the operator. Hence "$a \bullet b$" is replaced by "c". The empty operator is indicated by epsilon (ϵ) so "$a\epsilon b$" is identical to "ab" whence the empty operator implements the identity concatenation $ab \rightarrow ab$. This is shown in the first multiplication table, entitled *Epsilon.* This operator occurs, trivially, in all written languages, including computer languages. Combining the empty operator with the logical negation operator yields "$a!\epsilon b$" which is identical to "$a!b$" and, following custom, we take the operator "$!$" as a unary, right associative operator, so that, for example, "X!T" is replaced by "XF" where T stands for True, F stands for False and X stands for an arbitrary symbol. This is shown in the second multiplication table, entitled *Not Epsilon.* This operator, with a possibly different glyph, occurs in most high-level, computer languages. The remaining multiplication tables are truth tables. The labels on the rows and columns indicate the arguments: negative infinity $(-\infty)$, an arbitrary real number (r_i), positive infinity (∞), nullity (Φ). As usual T stands for True and F stands for False. In a departure from the usual notation, an asterisk (*) stands for a conditional truth value. For example, in the third table, entitled *Less,* the asterisk in the row labeled r_1 and column labeled r_2 is to be replaced by the truth value of $r_1 < r_2$, and similarly in the other tables. This recruitment of the real relation, less than, to define the corresponding transreal relation, is a context-sensitive reading of the symbol $<$. Computer scientists are generally comfortable with context-sensitive readings but many mathematicians regard them as an abuse of notation; even so, such notations are very common and are easily understood.

It can be seen, by inspection, that the multiplication tables are distinct. The labour of inspecting the tables can be reduced by exploiting symmetries. It is sufficient to notice that the first two elements, respectively FT, TF, FF of the first row of the tables *Less, Equal, Greater* are distinct and, similarly, TT, FT, TF of *Less or Equal, Less or Greater, Greater or Equal* are distinct.

Epsilon

ϵ	b
a	ab

Not Epsilon

$!\epsilon$	F	T
a	aT	aF

Less

$<$	$-\infty$	r_2	∞	Φ
$-\infty$	F	T	T	F
r_1	F	*	T	F
∞	F	F	F	F
Φ	F	F	F	F

Equal

$=$	$-\infty$	r_2	∞	Φ
$-\infty$	T	F	F	F
r_1	F	*	F	F
∞	F	F	T	F
Φ	F	F	F	T

Greater

$>$	$-\infty$	r_2	∞	Φ
$-\infty$	F	F	F	F
r_1	T	*	F	F
∞	T	T	F	F
Φ	F	F	F	F

Less or Equal

$<=$	$-\infty$	r_2	∞	Φ
$-\infty$	T	T	T	F
r_1	F	*	T	F
∞	F	F	T	F
Φ	F	F	F	T

Less or Greater

$<>$	$-\infty$	r_2	∞	Φ
$-\infty$	F	T	T	F
r_1	T	*	T	F
∞	T	T	F	F
Φ	F	F	F	F

Greater or Equal

>=	$-\infty$	r_2	∞	Φ
$-\infty$	T	F	F	F
r_1	T	*	F	F
∞	T	T	T	F
Φ	F	F	F	T

Less or Equal or Greater

<=>	$-\infty$	r_2	∞	Φ
$-\infty$	T	T	T	F
r_1	T	T	T	F
∞	T	T	T	F
Φ	F	F	F	T

Not Less

!<	$-\infty$	r_2	∞	Φ
$-\infty$	T	F	F	T
r_1	T	*	F	T
∞	T	T	T	T
Φ	T	T	T	T

Not Equal

!=	$-\infty$	r_2	∞	Φ
$-\infty$	F	T	T	T
r_1	T	*	T	T
∞	T	T	F	T
Φ	T	T	T	F

Not Greater

!>	$-\infty$	r_2	∞	Φ
$-\infty$	T	T	T	T
r_1	F	*	T	T
∞	F	F	T	T
Φ	T	T	T	T

Not Less or Equal

!<=	$-\infty$	r_2	∞	Φ
$-\infty$	F	F	F	T
r_1	T	*	F	T
∞	T	T	F	T
Φ	T	T	T	F

Not Less or Greater

!<>	$-\infty$	r_2	∞	Φ
$-\infty$	T	F	F	T
r_1	F	*	F	T
∞	F	F	T	T
Φ	T	T	T	T

Not Greater or Equal

!>=	$-\infty$	r_2	∞	Φ
$-\infty$	F	T	T	T
r_1	F	*	T	T
∞	F	F	F	T
Φ	T	T	T	F

Not Less or Equal or Greater

!<=>	$-\infty$	r_2	∞	Φ
$-\infty$	F	F	F	T
r_1	F	F	F	T
∞	F	F	F	T
Φ	T	T	T	F

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