Numerical Solution for Solving a System of Fractional Integro-differential Equations

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Abstract—**In this paper, a new numerical method for solving a linear system of fractional integro-differential equations is presented. The fractional derivative is considered in the Caputo sense. The proposed technique is based on the new operational matrices of triangular functions. The suggested method reduces this type of system to the solution of system of linear algebraic equations. To demonstrate the accuracy and applicability of the presented method some test examples are provided. Numerical results show that this approach is easy to implement and accurate when applied to integro-differential equations. We show that the solutions approach to classical solutions as the order of the fractional derivatives approach 1.**

Index Terms—**Fractional calculus; Operational Matrix; Triangular Functions; System of Integro-differential Equations.**

I. Introduction

Differential and integro-differential equations of fractional order arise in many physical and engineering problems such as fluid mechanics, viscoelasticity, diffusion processes, biology and so on [1-14]. Thus, a especial attention has been devoted to the solution of fractional ordinary differential equations, integral equations, and fractional integro-differential equations of physical interest. Some of these numerical methods are Adomians decomposition method, variation iteration method, homotopy analysis method, differential transform method, operational matrices and nonstandard finite difference scheme [15-31]. In this paper, we present numerical solution of an integro-differential equations with fractional derivative of the type:

$$D^{\alpha_i} y_i(t) = f_i(t) + \sum_{j=1}^{n} (a_{ij}(t) y_j(t) + \int_0^t k_{ij}(t,s) y_j(s) ds), \tag{1}$$

$$i = 1, ..., n,$$

with supplementary conditions

$$y_i^{(k)}(0) = b_{ik}, \quad k = 0, 1, ...\lceil \alpha \rceil - 1.$$

The main purpose of this work is to extend the operational matrices of TFs to solve the system of fractional integro differential equations numerically. This paper is organized as follows,
In Section 2, a brief review of TFs and fractional calculus is presented. In Section 3, operational matrices of TFs for fractional integration are derived. Section 4 is devoted to the formulation of system of fractional integro-differential equations. In Section 5, some numerical examples are provided. Finally, Section 6 gives a brief conclusion.

Manuscript received Aug 18, 2014; revised Feb 21, 2015.

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II. Basic definitions

In the following we present some basic definitions and properties of the fractional calculus [32,33] and TFs [34-37].

Definition II.1. Let $f \in L_1$, $\alpha \in R_+$. The Riemman-Liouville fractional integral of f of order α is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0, \quad t > 0.$$

Definition II.2. The Caputo fractional derivative is given by

$$D^\alpha f(t) = \begin{cases} I^{n-\alpha} f^{(n)}(t) & n-1 < \alpha < n, \\ \frac{d^n}{dt^n} f(t). & \alpha = n. \end{cases}$$

Some properties of the fractional operator are mentioned:
1)$D^\alpha x^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \beta \geq \lceil \alpha \rceil, \\ 0, & \beta < \lceil \alpha \rceil. \end{cases}$
2)$D^\alpha I^\alpha f(t) = f(t)$,
3)$I^\alpha D^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0^+) \frac{t^k}{k!}, \quad t > 0, \quad , n-1 \leq \alpha < n$,
4)$I^\alpha I^\beta f(t) = I^{\alpha+\beta}$,
5)$I^\alpha I^\beta f(t) = I^\beta I^\alpha$.

The triangular functions are defined on the interval [0,1) as follows,

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h} & ih \leq t < (i+1)h, \\ 0 & elsewhere, \end{cases}$$

$$T2_i(t) = \begin{cases} \frac{t-ih}{h} & ih \leq t < (i+1)h, \\ 0 & elsewhere, \end{cases}$$

where, $i = 0, ..., m-1$, $h = \frac{T}{m}$.

m-set TF vectors are defined as,

$$T1(t) = [T1_0(t), ..., T1_{m-1}(t)]^T,$$
$$T2(t) = [T2_0(t), ..., T2_{m-1}(t)]^T,$$

and

$$T(t) = [T1(t), T2(t)]^T.$$

A square integrable function $f(t)$ may be expanded in terms of m-set TF series as,

$$f(t) \simeq F1^T T1(t) + F2^T T2(t) = F^T T(t), \tag{2}$$

where, $F1_i = f(ih)$ and $F2_i = f((i+1)h)$ for $i = 0, ..., m-1$. The vectors $F1$ and $F2$ are called the 1D-TF coefficient vectors and $2m$-vector F is defined as:

$$F = [F1, F2]^T.$$

Let X be a $2m-$vector and B be a $2m \times 2m$ matrix, it can be concluded that

$$T(t) T^T(t) X = \tilde{X} T(t), \tag{3}$$

and

$$T^T(t)BT(t) = \hat{B}^T T(t), \tag{4}$$

in which $\tilde{X} = diag(X)$ and $\hat{B}$ is a $2m$ vector with elements equal to the diagonal entries of B. In addition, the integral of $f(t)$ can be approximated as follows

$$\int_0^t f(s)ds \simeq \int_0^t F^T T(s)ds \simeq F^T PT(t),$$

where P ,the operational matrix for integration, is obtained as [28].

III. Operational matrix of fractional integration

In this section, we expand integration operational matrix of TFs to operational matrix of fractional integration.

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds = \frac{1}{\Gamma(\alpha)}\{t^{\alpha-1} * f(t)\},$$

where $0 \leq t < T$. Also $t^{\alpha-1} * f(t)$ denotes the convolution product of $t^{\alpha-1}$ and $f(t)$. From Eq. (2) we get,

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\{t^{\alpha-1} * f(t)\} \simeq F^T \frac{1}{\Gamma(\alpha)}\{t^{\alpha-1} * T(t)\}.$$

We now compute $\frac{1}{\Gamma(\alpha)}\{t^{\alpha-1} * T(t)\}$ as,

$$I^\alpha T(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\begin{pmatrix}T1(s)\\T2(s)\end{pmatrix}ds = \begin{pmatrix}I^\alpha T1(t)\\I^\alpha T2(t)\end{pmatrix}.$$

First we obtain

$$\frac{1}{\Gamma(\alpha)}\{t^{\alpha-1} * T1_i(t)\},$$

in which $i = 0, ..., m-1$. By using the Laplace transform, we get

$$Ł\{I^\alpha T1_i(t)\} = \frac{1}{\Gamma(\alpha)}Ł\{t^{\alpha-1}\}Ł\{T1_i(t)\},$$

where,

$$Ł\{t^{\alpha-1}\} = \frac{\Gamma(\alpha)}{s^\alpha},$$

and

$$\begin{aligned}Ł\{T1_i(t)\} &= Ł\big\{u(t-ih) - \frac{t-ih}{h}u(t-ih)\\ &\quad + \frac{t-(i+1)h}{h}u(t-(i+1)h)\big\}\\ &= [\frac{e^{-ihs}}{s} - \frac{e^{-ihs}}{hs^2} + \frac{e^{-(i+1)hs}}{hs^2}],\end{aligned}$$

then,

$$Ł\{I^\alpha T1_i(t)\} = \frac{e^{-ihs}}{s^{\alpha+1}} - \frac{e^{-ihs}}{hs^{\alpha+2}} + \frac{e^{-(i+1)hs}}{hs^{\alpha+2}}. \tag{5}$$

Inverse Laplace transform of Eq. (5), yields

$$I^\alpha T1_i(t) = \frac{1}{\Gamma(\alpha+2)}\big((\alpha+1)(t-ih)^\alpha u(t-ih) - \frac{(t-ih)^{\alpha+1}}{h}u(t-ih) + \frac{(t-(i+1)h)^{\alpha+1}}{h}u(t-(i+1)h)\big). \tag{6}$$

Also, the fractional integration of $T2_i(t)$ is

$$\begin{aligned}I^\alpha T2_i(t) = \frac{1}{\Gamma(\alpha+2)}\big(&(t-ih)^{\alpha+1}u(t-ih)\\ &-(t-(i+1)h)^{\alpha+1}u(t-(i+1)h)\\ &-(\alpha+1)(t-(i+1)h)^\alpha u(t-(i+1)h)\big).\end{aligned} \tag{7}$$

Expansion of $I^\alpha T1_i(t)$ with respect to TFs is

$$I^\alpha T1_i(t) \simeq [c_{i0}, ..., c_{im-1}]T1(t) + [d_{i0}, ..., d_{im-1}]T2(t),$$

where $c_{ij} = I^\alpha T1_i(jh)$ and $d_{ij} = I^\alpha T1_i((j+1)h)$, $j = 0, ..., m-1$, from Eq.(6), we get

$$\begin{aligned}&c_{ij} = 0, \quad j \leq i,\\ &c_{ij} = \frac{h^\alpha}{\Gamma(\alpha+2)}\big((\alpha+1)(j-i)^\alpha - (j-i)^{\alpha+1} +\\ &(j-i-1)^{\alpha+1}\big), \quad i < j,\\ &d_{ij} = c_{ij+1},\end{aligned}$$

Finally, for $i = 0, ..., m-1$, $j = 0, ..., m-1$, we can write

$$I^\alpha T1(t) = P1_\alpha T1(t) + P2_\alpha T2(t), \tag{8}$$

where $P1_\alpha$ and $P2_\alpha$ are $m \times m$ operational matrices of fractional integration in TF domain. These matrices can be computed as follow,

$$P1_\alpha = \begin{pmatrix}0 & \xi_1 & \xi_2 & \cdots & \xi_{m-1}\\ 0 & 0 & \xi_1 & \cdots & \xi_{m-2}\\ 0 & 0 & 0 & \cdots & \xi_{m-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 0\end{pmatrix},$$

$$P2_\alpha = \begin{pmatrix}\xi_1 & \xi_2 & \xi_3 & \cdots & \xi_m\\ 0 & \xi_1 & \xi_2 & \cdots & \xi_{m-1}\\ 0 & 0 & \xi_1 & \cdots & \xi_{m-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & \xi_1\end{pmatrix},$$

where, $\xi_r = \frac{h^\alpha}{\Gamma(\alpha+2)}\big((\alpha+1)r^\alpha - r^{\alpha+1} + (r-1)^{\alpha+1}\big)$.
In the same way, the following approximation can be achieved for $T2$.

$$I^\alpha T2(t) \simeq P3_\alpha T1(t) + P4_\alpha T2(t), \tag{9}$$

where,

$$P3_\alpha = \begin{pmatrix}0 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m-1}\\ 0 & 0 & \zeta_1 & \cdots & \zeta_{m-2}\\ 0 & 0 & 0 & \cdots & \zeta_{m-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 0\end{pmatrix},$$

$$P4_\alpha = \begin{pmatrix}\zeta_1 & \zeta_2 & \zeta_3 & \cdots & \xi_m\\ 0 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m-1}\\ 0 & 0 & \zeta_1 & \cdots & \zeta_{m-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & \zeta_1\end{pmatrix},$$

and $\zeta_r = \frac{h^\alpha}{\Gamma(\alpha+2)}\big(r^{\alpha+1} - (r-1)^{\alpha+1} - (\alpha+1)(r-1)^\alpha\big)$.
By using Eqs. (8-9) fractional integration of $T(t)$ can be obtained as,

$$\begin{aligned}I^\alpha T(t) = \begin{pmatrix}I^\alpha T1(t)\\ I^\alpha T2(t)\end{pmatrix} &\simeq \begin{pmatrix}P1_\alpha T1(t) + P2_\alpha T2(t)\\ P3_\alpha T1(t) + P4_\alpha T2(t)\end{pmatrix}\\ &= \begin{pmatrix}P1_\alpha & P2_\alpha\\ P3_\alpha & P4_\alpha\end{pmatrix}\begin{pmatrix}T1(t)\\ T2(t)\end{pmatrix},\end{aligned}$$

so,

$$I^{\alpha}T(t) = P_{\alpha}T(t), \tag{10}$$

where P_{α}, fractional integration operational matrix of T(t), is

$$\begin{pmatrix} P1_{\alpha} & P2_{\alpha} \\ P3_{\alpha} & P4_{\alpha} \end{pmatrix}.$$

The fractional integration of $f(t)$ can be approximated as,

$$I^{\alpha}f(t) \simeq F^{T}P_{\alpha}T(t). \tag{11}$$

IV. Solving linear system of fractional integro-differential equation

In this section, we use the obtained operational matrix of fractional integration to solve Eq. (1). Applying the operator I^{α_i}, the inverse operator of D^{α_i}, to both sides of Eq. (1) yields

$$y_i(t) = g_i(t) + I^{\alpha_i}\big(f_i(t) + \sum_{j=1}^{n}\big(a_{ij}(t)y_j(t) + \int_0^t k_{ij}(t,s)y_j(s)ds\big)\big), \quad i = 1,..,n, \tag{12}$$

where $g_i(t) = \sum_{j=0}^{p-1} y_i^{(j)}(0^+)\frac{t^j}{j!},\ \alpha - 1 \le p < \alpha$. We can approximate the functions as follows,

$$g_i(t) \simeq G_i^T T(t) = T^T(t)G_i, \tag{13}$$

$$f_i(t) \simeq F_i^T T(t) = T^T(t)F_i, \tag{14}$$

$$a_{ij}(t) \simeq A_{ij}^T T(t) = T^T(t)A_{ij}, \tag{15}$$

$$y_i(t) \simeq Y_i^T T(t) = T^T(t)Y_i, \tag{16}$$

$$k_{ij}(t,s) \simeq T^T(t)K_{ij}T(s), \tag{17}$$

where 2m-vectors $G_i,\ Y_i,\ F_i,\ A_{ij}$ and $2m \times 2m$ matrix $K_{ij},\ i,j = 1,2,..,n$ are TF coefficients. Integral term in Eq. (12) can be approximated as

$$\int_0^t k_{ij}(t,s)y_j(s)ds \simeq T^T(t)K_{ij}\int_0^t T(s)T^T(s)Y_j ds = T^T(t)K_{ij}\int_0^t \tilde{Y_j}T(s)ds = T^T(t)K_{ij}\tilde{Y_j}PT(t) \simeq \hat{B_{ij}}^T T(t),$$

so

$$I^{\alpha_i}(\hat{B_{ij}}^T T(t)) \simeq \hat{B_{ij}}^T P_{\alpha_i}T(t), \tag{18}$$

and

$$I^{\alpha_i}(a_{ij}(t)y_j(t)) \simeq I^{\alpha_i}(T^T(t)A_{ij}Y_j^T T(t)) \simeq I^{\alpha_i}(\hat{A_{ij}}^T T(t)) \simeq \hat{A_{ij}}^T P_{\alpha_i}T(t). \tag{19}$$

where $\tilde{Y_j} = diag(Y_j)$ and $\hat{B_{ij}}, \hat{A_{ij}}$ are defined in (4). By substituting Eqs. (13-19) in Eq. (12), we get

$$Y_i^T T(t) = G_i^T T(t) + F_i^T P_{\alpha_i}T(t) + \sum_{j=1}^{n}\big(\hat{A_{ij}}^T P_{\alpha_i}T(t) + \hat{B_{ij}}^T P_{\alpha_i}T(t)\big), \tag{20}$$

therefore, problem (20) reduces to the following problem:

$$Y_i - P_{\alpha_i}^T F_i - \sum_{j=1}^{n}\big(P_{\alpha_i}^T\hat{A_{ij}} + P_{\alpha_i}^T\hat{B_{ij}}\big) = G_i, \quad i = 1,2,...,n. \tag{21}$$

A linear system of algebraic equations is achieved in Eq. (21). Components of unknown vectors Y_i can be obtained by solving this system, using an iterative method.

V. Numerical examples

In order to illustrate the applicability of the proposed method, we apply the presented method for the following examples.

Example 1. Consider the following linear system of fractional integro-differential equations [38]

$$\begin{cases} D^{\alpha}y_1(t) = 1 + t + t^2 - y_1(t) - \int_0^t (y_1(s) + y_2(s))ds, \\ D^{\alpha}y_2(t) = -1 - t + y_1(t) - \int_0^t (y_1(s) - y_2(s))ds, \\ 0 < \alpha \le 1, \end{cases} \tag{22}$$

with these supplementary conditions

$$y_1(0) = 1,\ y_2(0) = -1.$$

The exact solution is $y_1(t) = t + e^t,\ y_2(t) = t - e^t$. We implemented the suggested method with $m = 16$ and $m = 32$. The obtained numerical results are shown in Table I and Figs 1-4. In Table I, the absolute error between the exact solution and the approximate solution, at $m = 16$ (in columns 2,3) and $m = 32$ (in columns 4,5) respectively, are given. Figs. 1 and 2 show the evolution results for the system of fractional integro differential Eqs. (22) at $m = 32$ when $\alpha = 1$. And Figs. 2 and 4 show the behavior of obtained approximate solution for the proposed system (22) at $m = 32$ with different values of α. From Table I and Figs. 1-2 we can conclude that our approximate solutions are in good agreement with the exact values and with high accuracy in comparison with the approximate solution obtained in [38].

Example 2. The following linear system of fractional integro differential equation is considered

$$\begin{cases} D^{\alpha}y_1(t) = 1 + t^2 + sint - \int_0^t (y_1(s) + y_2(s))ds, \\ D^{\alpha}y_2(t) = -1 + t + sint + cost - \int_0^t (y_2(s) - y_1(s))ds, \\ 0 < \alpha \le 2, \end{cases} \tag{23}$$

with these supplementary conditions

$$y_1(0) = 1,\ y_2(0) = 0,\ y_1'(0) = 1,\ y_2'(0) = 2.$$

The exact solution for $\alpha = 2$ is $y_1(t) = t + cos(t),\ y_2(t) = t + sin(t)$. The errors for $\alpha = 2$ are obtained in Table II ($m = 16$ in columns 2,3 and $m = 32$ in columns 4,5). Fig.5 and Fig.6 show numerical results for different values of α with $m = 32$.

Example 3. Consider the following system of fractional integro differential equations [39],

$$\begin{cases} D^{\alpha}y_1(t) = 2 + e^t - 3e^{2t} + e^{3t} + \int_0^t (6y_2(s) - 3y_3(s))ds, \\ D^{\alpha}y_2(t) = e^t + 2e^{2t} - e^{3t} + \int_0^t (3y_3(s) - y_1(s))ds, \\ D^{\alpha}y_3(t) = -e^t + e^{2t} + 3e^{3t} + \int_0^t (y_1(s) - 2y_2(s))ds, \\ 0 < \alpha \le 1, \end{cases} \tag{24}$$

subject to the initial conditions

$$y_1(0) = y_2(0) = y_3(0) = 1,$$

The exact solution of this system, when $\alpha = 1$, is

$$y_1(t) = e^t, \quad y_2(t) = e^{2t}, \quad y_3(t) = e^{3t}.$$

In Table III the absolute error between the exact solution and the approximate solution, at $m = 16$ (in columns 2,3,4) and $m = 32$ (in columns 5,6,7) are presented respectively. Figs. 7-12 show the evolution results for the system of integro-differential equation (24) with $m = 32$. It is easy to conclude that the solution continuously depends on the space-fractional derivative.

Example 4. We consider the following system of integro differential equations of fractional order,

$$\begin{cases} D^{0.25}y_1(t) = f_1(t) + \int_0^t (sinsy_1(s) + y_2(s))ds, \\ D^{0.5}y_2(t) = f_2(t) + \int_0^t ((1 - \frac{3}{4}(y_2(s) + y_3(s))ds, \\ D^{0.75}y_3(t) = f_3(t) - \int_0^t (y_1(s) + y_2(s) + y_3(s))ds, \end{cases} \tag{25}$$

The exact solution of this system is

$$y_1(t) = t, \quad y_2(t) = t^2, \quad y_3(t) = t^3.$$

and $f_1(t) = \frac{1}{\Gamma(1.75)}t^{\frac{3}{4}} + tcost - sint - \frac{t^3}{3}$, $f_2(t) = \frac{2}{\Gamma(2.5)}t^{\frac{3}{2}} - \frac{t^3}{3} + \frac{3}{16}t^5$, $f_3(t) = \frac{6}{\Gamma(3.25)}t^{\frac{9}{4}} + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4}$.
In Table IV the absolute error between the exact solution and the approximate solution, at $m = 16$ (in columns 2,3,4) and $m = 32$ (in columns 5,6,7) are presented respectively.

VI. Conclusion

In this paper, the application of TF operational matrix of fractional order has been successfully employed to obtain the approximate solutions for linear system of fractional order integro-differential equations. The fractional derivative is considered in the Caputo sense. From the obtained numerical results we can see that the obtained solution using the suggested method are in good agreement with the exact solution and with the presented method in [38]. It is easy to conclude that the solution continuously depends on the fractional derivative. The presented method provides a technique that requires less computational work. Also the last problems show efficiency and accuracy of the method.

Acknowledgements

The authors are extending their heartfelt thanks to the reviewers for their valuable suggestions for the improvement of the article.

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TABLE I: The absolute error between the exact solution and the approximate solution at $m = 16$ and $m = 32$.

t_i	$\lvert y_{1ex} - y_{1app}\rvert$	$\lvert y_{2ex} - y_{2app}\rvert$	$\lvert y_{1ex} - y_{1app}\rvert$	$\lvert y_{2ex} - y_{2app}\rvert$
0	0	0	0	0
0.1	5.5×10^{-4}	5.5×10^{-4}	9.6×10^{-5}	9.6×10^{-5}
0.2	4.7×10^{-4}	4.7×10^{-4}	1.6×10^{-4}	1.6×10^{-4}
0.3	5.5×10^{-4}	5.5×10^{-4}	1.9×10^{-4}	1.9×10^{-4}
0.4	9.0×10^{-4}	9.0×10^{-4}	1.6×10^{-4}	1.6×10^{-4}
0.5	2.7×10^{-4}	2.7×10^{-4}	6.7×10^{-5}	6.7×10^{-5}
0.6	1.2×10^{-3}	1.2×10^{-3}	2.3×10^{-4}	2.3×10^{-4}
0.7	1.0×10^{-3}	1.0×10^{-3}	3.5×10^{-4}	3.5×10^{-4}
0.8	1.3×10^{-3}	1.3×10^{-3}	4.0×10^{-4}	4.0×10^{-4}
0.9	1.9×10^{-3}	1.9×10^{-3}	4.0×10^{-4}	4.0×10^{-4}

TABLE II: The absolute error between the exact solution and the approximate solution at $m = 16$ and $m = 32$.

t_i	$\lvert y_{1ex} - y_{1app}\rvert$	$\lvert y_{2ex} - y_{2app}\rvert$	$\lvert y_{1ex} - y_{1app}\rvert$	$\lvert y_{2ex} - y_{2app}\rvert$
0	0	0	0	0
0.1	4.6×10^{-4}	4.5×10^{-5}	7.7×10^{-5}	8.3×10^{-6}
0.2	2.9×10^{-4}	8.9×10^{-5}	1.1×10^{-4}	2.6×10^{-5}
0.3	3.0×10^{-4}	1.1×10^{-4}	1.0×10^{-4}	5.5×10^{-5}
0.4	4.0×10^{-4}	2.7×10^{-4}	6.6×10^{-5}	1.1×10^{-4}
0.5	3.2×10^{-5}	2.6×10^{-4}	8.0×10^{-6}	2.6×10^{-4}
0.6	3.4×10^{-4}	9.3×10^{-4}	5.3×10^{-5}	7.0×10^{-4}
0.7	1.8×10^{-4}	1.6×10^{-3}	7.7×10^{-5}	1.4×10^{-3}
0.8	1.5×10^{-3}	2.9×10^{-3}	7.0×10^{-5}	2.7×10^{-3}
0.9	2.2×10^{-3}	5.2×10^{-3}	4.0×10^{-5}	5.0×10^{-3}

TABLE III: The absolute error between the exact solution and the approximate solution at $m = 16$ and $m = 32$.

t_i	$\lvert y_{1ex} - y_{1app}\rvert$	$\lvert y_{2ex} - y_{2app}\rvert$	$\lvert y_{3ex} - y_{3app}\rvert$	$\lvert y_{1ex} - y_{1app}\rvert$	$\lvert y_{2ex} - y_{2app}\rvert$	$\lvert y_{3ex} - y_{3app}\rvert$
0	0	0	0	0	0	0
0.1	5.4×10^{-4}	2.6×10^{-3}	6.6×10^{-3}	9.3×10^{-5}	4.7×10^{-4}	1.2×10^{-3}
0.2	4.0×10^{-4}	2.8×10^{-3}	7.6×10^{-3}	1.5×10^{-4}	9.3×10^{-4}	2.5×10^{-3}
0.3	3.8×10^{-4}	3.9×10^{-3}	1.0×10^{-2}	1.5×10^{-4}	1.3×10^{-3}	3.6×10^{-3}
0.4	5.0×10^{-4}	7.3×10^{-3}	2.0×10^{-2}	6.8×10^{-5}	1.4×10^{-3}	3.9×10^{-3}
0.5	4.6×10^{-4}	4.9×10^{-3}	1.0×10^{-2}	1.1×10^{-4}	1.2×10^{-3}	2.4×10^{-3}
0.6	5.8×10^{-5}	1.3×10^{-2}	3.8×10^{-2}	8.3×10^{-5}	2.9×10^{-3}	7.8×10^{-3}
0.7	9.2×10^{-4}	1.7×10^{-2}	4.3×10^{-2}	1.5×10^{-4}	4.8×10^{-3}	1.3×10^{-2}
0.8	1.8×10^{-3}	2.3×10^{-2}	5.6×10^{-2}	3.7×10^{-4}	6.5×10^{-3}	1.8×10^{-2}
0.9	2.6×10^{-3}	3.5×10^{-2}	9.8×10^{-2}	7.6×10^{-4}	8.0×10^{-4}	1.9×10^{-2}

TABLE IV: The absolute error between the exact solution and the approximate solution at $m = 16$ and $m = 32$.

t_i	$\lvert y_{1ex} - y_{1app}\rvert$	$\lvert y_{2ex} - y_{2app}\rvert$	$\lvert y_{3ex} - y_{3app}\rvert$	$\lvert y_{1ex} - y_{1app}\rvert$	$\lvert y_{2ex} - y_{2app}\rvert$	$\lvert y_{3ex} - y_{3app}\rvert$
0	0	0	0	0	0	0
0.1	1.6×10^{-3}	2.6×10^{-3}	6.5×10^{-4}	4.3×10^{-4}	7.9×10^{-4}	1.8×10^{-4}
0.2	7.4×10^{-4}	1.8×10^{-3}	8.9×10^{-4}	3.6×10^{-4}	5.4×10^{-4}	3.0×10^{-4}
0.3	8.1×10^{-4}	1.1×10^{-3}	1.1×10^{-3}	9.3×10^{-4}	2.1×10^{-4}	3.9×10^{-4}
0.4	1.8×10^{-3}	1.2×10^{-4}	2.0×10^{-3}	2.4×10^{-3}	1.8×10^{-3}	4.2×10^{-4}
0.5	4.1×10^{-3}	3.1×10^{-3}	8.7×10^{-4}	5.5×10^{-3}	4.7×10^{-3}	3.7×10^{-4}

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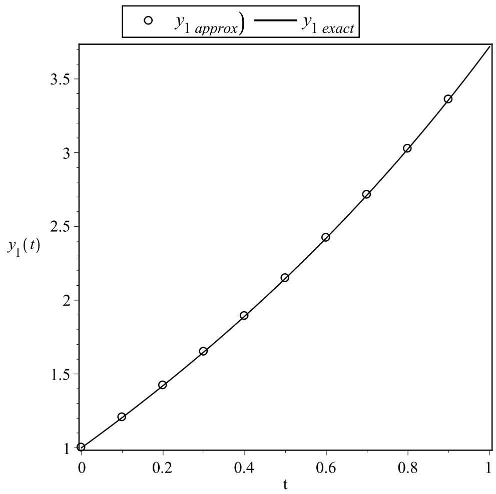

Fig.1 results for Example 1 .

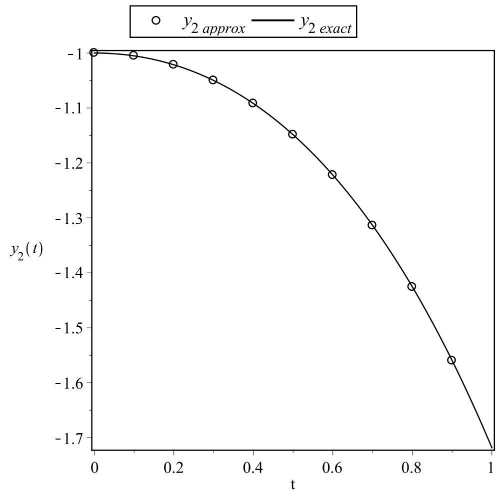

Fig.2 results for Example 1.

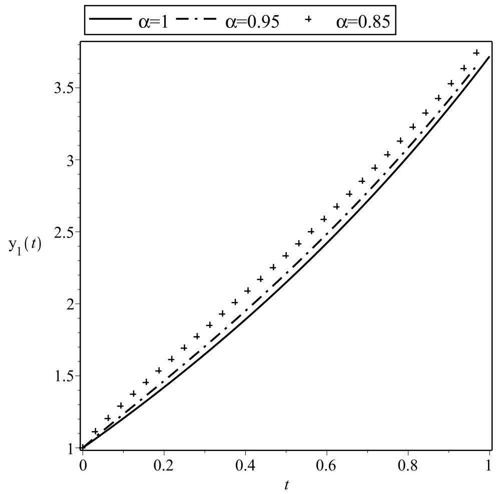

Fig.3 results for Example 1.

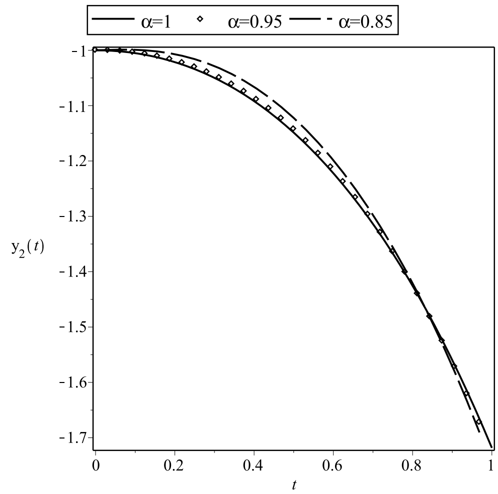

Fig.4 results for Example 1.

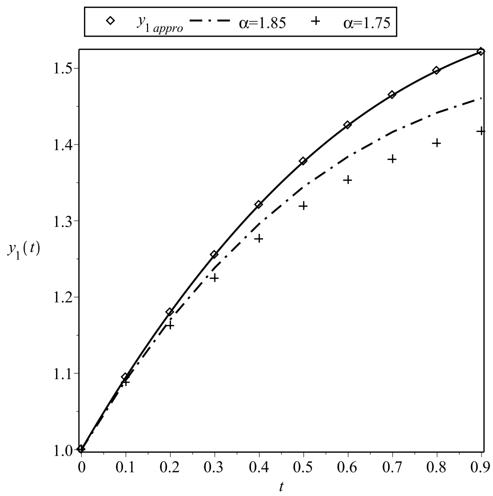

Fig.5 results for Example 2.

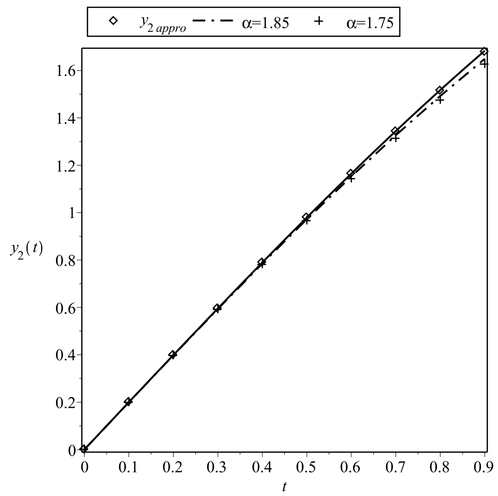

Fig.6 results for Example 2.

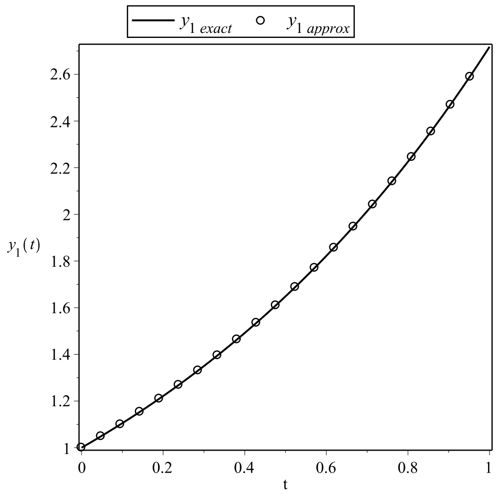

Fig.7 results for Example 3.

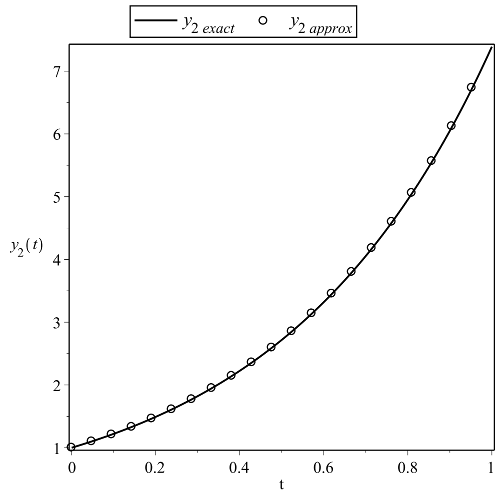

Fig.8 results for Example 3.

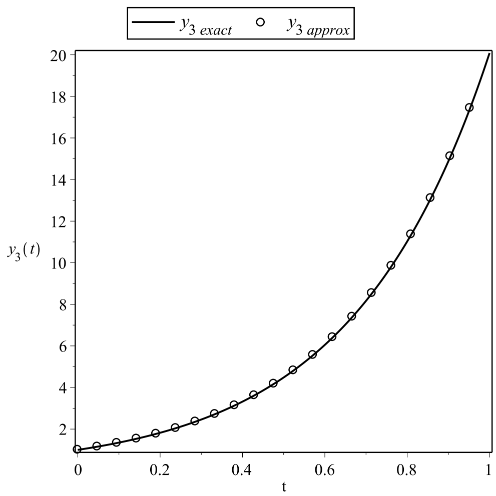

Fig.9 results for Example 3.

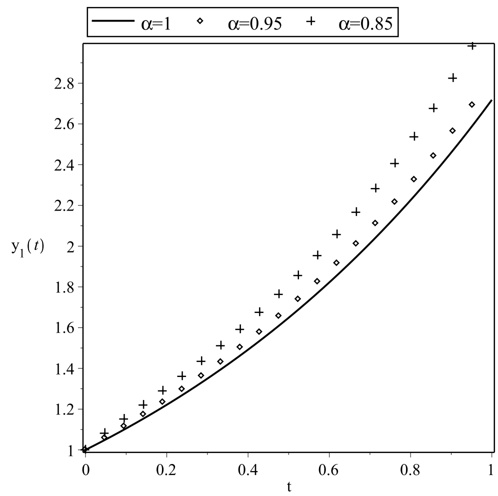

Fig.10 results for Example 3.

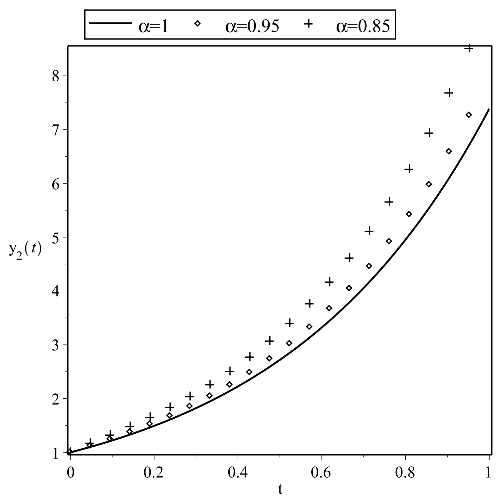

Fig.11 results for Example 3

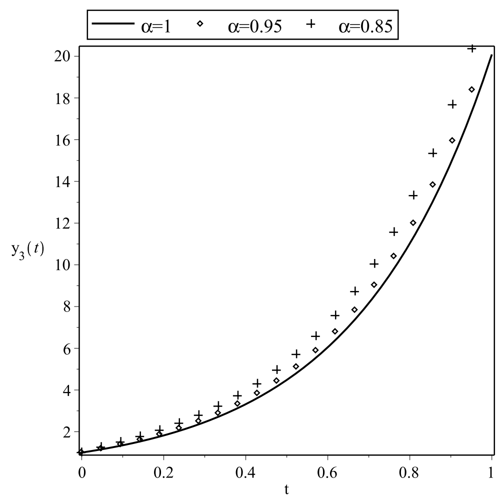

Fig.12 results for Example 3 .