Almost Periodic Oscillation in a Watt-type Predator-prey Model with Diffusion and Time Delays

Liyan Pang and Tianwei Zhang

Abstract—**In this paper, we consider a Watt-type predator-prey model with diffusion and time delays. Firstly, by means of Mawhin's continuation theorem of coincidence degree theory, some new sufficient conditions are obtained for the existence of at least one positive almost periodic solution for the above model. Secondly, by using the comparison theorem, we give a permanence result for the model. Thirdly, by constructing a suitable Lyapunov functional, the global asymptotical stability of the model is also investigated. To the best of the author's knowledge, so far, the results in this paper are completely new. Finally, an example and numerical simulations are employed to illustrate the main result of this paper.**

Index Terms—**Almost periodicity, Coincidence degree, Predator-prey, Watt-type, Diffusion.**

I. Introduction

IT is well-known that the theoretical study of predator-prey systems in mathematical ecology has a long history starting with the pioneering work of Lotka and Volterra $[1, 2]$. The principles of Lotka-Volterra model, conservation of mass and decomposition of the rates of change in birth and death processes, have remained valid until today and many theoretical ecologists adhere to these principles. This general approach has been applied to many biological systems in particular with functional response. In population dynamics, a functional response of the predator to the prey density refers to the change in the density of prey attached per unit time per predator as the prey density changes. During the last ten years, there has been extensively investigation on the dynamics of predator-prey models with the different functional responses in the literature, (see [3-13] and references therein).

In the last few years, mathematicians and ecologists have been actively investigating the dispersal of populations, a ubiquitous phenomenon in population dynamics. Levin (1976) showed that both spatial dispersal of populations and population dynamics are much affected by spatial heterogeneity. In real life, dispersal often occurs among patches in ecological environments; because of the ecological effects of human activities and industries, such as the location of manufacturing industries and the pollution of the atmosphere, soil and rivers, reproduction- and population-based territories and other habitats have been broken into patches. Thus, realistic models should include dispersal processes that take into consideration the effects of spatial heterogeneity. A large number of predator-prey models for diffusion have been studied by many researchers (see [14-22] and references therein).

In [22], Lin et al. considered the following T-periodic Watt-type predator-prey system [22-26] with diffusion:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left[r_1(t) - b_1(t)x_1(t)\right] \\ \qquad -a_1(t)y(t)\left[1 - \exp\left(\frac{-c(t)x_1(t)}{y^m(t)}\right)\right] \\ \qquad +D_1(t)\left[x_2(t) - x_1(t)\right], \\ \dot{x}_2(t) = x_2(t)\left[r_2(t) - b_2(t)x_2(t)\right] \\ \qquad +D_2(t)\left[x_1(t) - x_2(t)\right], \\ \dot{y}(t) = y(t)\left[r_3(t) - b_3(t)y(t)\right] \\ \qquad +a_2(t)y(t)\left[1 - \exp\left(\frac{-c(t)x_1(t)}{y^m(t)}\right)\right], \end{cases} \tag{1.1}$$

where $x_1(t)$ and $y(t)$ are the densities of prey species and predator species in patch I at time t, and $x_2(t)$ is the density of prey species in patch II, prey species $x_1(t)$ and $x_(t)$ can diffuse between two patch while the predator species $y(t)$ is confined to patch I. Under the assumption of periodicity of the coefficients of system (1.1), Lin et al. studied the existence of at least one positive periodic solution of system (1.1) by means of Mawhin's continuation theorem of coincidence degree theory.

Since biological and environmental parameters are naturally subject to fluctuation in time, the effects of a periodically varying environment are considered as important selective forces on systems in a fluctuating environment. Therefore, models should take into account the seasonality of the periodically changing environment. However, in applications, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples, see Example 1) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity. In recent years, the almost periodic solution of the continuous models in biological populations has been studied by many authors (see [27-34] and the references cited therein).

Example 1. Consider the following simple predator-prey model:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left[1 - [2 + \sin(\sqrt{2}t)]x_1(t)\right], \\ \dot{x}_2(t) = x_2(t)\left[1 - [2 + \sin(\sqrt{3}t)]x_2(t)\right], \\ \dot{y}(t) = y(t)\left[1 - [2 + \sin(\sqrt{5}t)]y(t)\right] \\ \qquad +y(t)\left[1 - \exp\left(\frac{-x_1(t)}{y(t)}\right)\right]. \end{cases} \tag{1.2}$$

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In system (1.2), corresponding to system (1.1), $b_1(t) = 2+\sin(\sqrt{2}t)$ is $\sqrt{2}\pi$-periodic function, $b_2(t) = 2+\sin(\sqrt{3}t)$ is $\frac{2\sqrt{3}}{3}\pi$-periodic function and $b_3(t) = 2+\sin(\sqrt{5}t)$ is $\frac{2\sqrt{5}}{5}\pi$-periodic function, which imply that system (1.2) is with incommensurable periods. Then there is no a priori reason to expect the existence of positive periodic solutions of system (1.2). Thus, it is significant to study the existence of positive almost periodic solutions of system (1.2).

In addition, to reflect that the dynamical behavior of the models depends on the past history of the system, it is often necessary to incorporate time delays into the models. Therefore, a more realistic model should be described by delayed differential equations [8,14-18]. Motivated by the above reasons, the aim of this paper is to consider the following almost periodic Watt-type predator-prey model with diffusion and time delays:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left[r_1(t) - b_1(t)x_1(t) - d_1(t)x_2(\delta(t))\right] \\ \qquad -a_1(t)y(\tau(t))\left[1-\exp\left(\frac{-c(t)x_1(t)}{y^m(\tau(t))}\right)\right] \\ \qquad +D_1(t)\left[x_2(t)-x_1(t)\right], \\ \dot{x}_2(t) = x_2(t)\left[r_2(t) - b_2(t)x_2(t) - d_2(t)x_1(\zeta(t))\right] \\ \qquad -a_2(t)y(\sigma(t))\left[1-\exp\left(\frac{-c(t)x_2(t)}{y^m(\sigma(t))}\right)\right] \\ \qquad +D_2(t)\left[x_1(t)-x_2(t)\right], \\ \dot{y}(t) = y(t)\left[r_3(t) - b_3(t)y(t)\right] \\ \qquad +a_3(t)y(t)\left[1-\exp\left(\frac{-c(t)x_1(t)}{y^m(t)}\right)\right] \\ \qquad +a_4(t)y(t)\left[1-\exp\left(\frac{-c(t)x_2(t)}{y^m(t)}\right)\right], \end{cases} \tag{1.3}$$

where $r_i\,(1,2,3)$ are intrinsic growth rate, $b_i\,(1,2,3)$ are the rate of intra-specific competition, $d_i\,(i=1,2)$ are parameters representing competitive effects between two prey, $a_i\,(i=1,2)$ are coefficients of decrease of prey species due to predation, $a_i\,(i=3,4)$ are equal to the transformation rate of predator, $D_i\,(i=1,2)$ are the dispersal rate of prey species, δ, ζ, τ and σ are time delays satisfying $\delta(t) \le t$, $\zeta(t) \le t$, $\tau(t) \le t$, $\sigma(t) \le t$, $\lim_{t\to\infty}\delta(t) = \infty$, $\lim_{t\to\infty}\zeta(t) = \infty$, $\lim_{t\to\infty}\tau(t) = \infty$ and $\lim_{t\to\infty}\sigma(t) = \infty$, m is a nonnegative constant. Throughout this paper, we assume that all coefficients in system (1.3) are nonnegative almost periodic functions.

It is well known that Mawhin's continuation theorem of coincidence degree theory is an important method to investigate the existence of positive periodic solutions of some kinds of non-linear ecosystems (see [35-43]). However, it is difficult to be used to investigate the existence of positive almost periodic solutions of non-linear ecosystems. Therefore, to the best of the author's knowledge, so far, there are scarcely any papers concerning with the existence of positive almost periodic solutions of system (1.3) by using Mawhin's continuation theorem. Therefore, the main purpose of this paper is to establish some new sufficient conditions on the existence of positive almost periodic solutions of system (1.3) by using Mawhin's continuous theorem of coincidence degree theory.

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}^+$ denote the sets of real numbers, integers and positive integers, respectively, $C(\mathbb{X},\mathbb{Y})$ and $C^1(\mathbb{X},\mathbb{Y})$ be the space of continuous functions and continuously differential functions which map $\mathbb{X}$ into $\mathbb{Y}$, respectively. Especially, $C(\mathbb{X}) := C(\mathbb{X},\mathbb{X})$, $C^1(\mathbb{X}) := C^1(\mathbb{X},\mathbb{X})$. Related to a continuous bounded function f, we use the following notations:

$$f^- = \inf_{s\in\mathbb{R}} f(s), \quad f^+ = \sup_{s\in\mathbb{R}} f(s), \quad |f|_\infty = \sup_{s\in\mathbb{R}} |f(s)|.$$

The organization of this Letter is as follows. In Section 2, we make some preparations. In Section 3, by using Mawhin's continuation theorem of coincidence degree theory, we establish sufficient conditions for the existence of at least one positive almost periodic solution to system (1.3). In Sections 4-5, the uniform persistence and global asymptotical stability of system (1.3) are considered. Finally, an example and numerical simulations are also given to illustrate the main result in this paper.

II. Preliminaries

Definition 1. ([44, 45]) $x \in C(\mathbb{R},\mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $\|x(t+\tau) - x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$, where $\|\cdot\|$ is arbitrary norm of $\mathbb{R}^n$. τ is called to the ϵ-almost period of x, $T(x,\epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x,\epsilon)$. The collection of those functions is denoted by $AP(\mathbb{R},\mathbb{R}^n)$. Let $AP(\mathbb{R}) := AP(\mathbb{R},\mathbb{R})$.

Lemma 1. ([44, 45]) *If $x \in AP(\mathbb{R})$, then x is bounded and uniformly continuous on $\mathbb{R}$.*

Lemma 2. [29] *Assume that $x \in AP(\mathbb{R}) \cap C^1(\mathbb{R})$ with $\dot{x} \in C(\mathbb{R})$, for $\forall\epsilon > 0$, we have the following conclusions:*

(I) *there is a point $\xi_\epsilon \in [0,+\infty)$ such that $x(\xi_\epsilon) \in [x^* - \epsilon, x^*]$ and $\dot{x}(\xi_\epsilon) = 0$;*

(II) *there is a point $\eta_\epsilon \in [0,+\infty)$ such that $x(\eta_\epsilon) \in [x_*, x_* + \epsilon]$ and $\dot{x}(\eta_\epsilon) = 0$.*

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin [46].

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\mathrm{dimKer}L = \mathrm{codimIm}L < +\infty$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I-Q)$. It follows that $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : (I-P)\mathbb{X} \to \mathrm{Im}L$ is invertible and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Lemma 3. ([46]) *Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. If all the following conditions hold*:

(a) $Lx \ne \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0,1)$;

(b) $QNx \ne 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;

(c) $\deg\{JQN, \Omega\cap\mathrm{Ker}L, 0\} \ne 0$, *where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is an isomorphism.*

Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

Under the invariant transformation $(x_1,x_2,x_3)^T = (e^u,e^v,e^w)^T$, system (1.3) reduces to

$$\begin{cases} \dot{u}(t) = & r_1(t)-b_1(t)e^{u(t)}-d_1(t)e^{v(\delta(t))} \\ & -a_1(t)e^{w(\tau(t))-u(t)}\Big[1 \\ & -\exp\left(-c(t)e^{u(t)-mw(\tau(t))}\right)\Big] \\ & +D_1(t)\left[e^{v(t)-u(t)}-1\right] := F_1(t), \\ \dot{v}(t) = & r_2(t)-b_2(t)e^{v(t)}-d_2(t)e^{u(\zeta(t))} \\ & -a_2(t)e^{w(\sigma(t))-v(t)}\Big[1 \\ & -\exp\left(-c(t)e^{v(t)-mw(\sigma(t))}\right)\Big] \\ & +D_2(t)\left[e^{u(t)-v(t)}-1\right] := F_2(t), \\ \dot{w}(t) = & r_3(t)-b_3(t)e^{w(t)} \\ & +a_3(t)\left[1-\exp\left(-c(t)e^{u(t)-mw(t)}\right)\right] \\ & +a_4(t)\left[1-\exp\left(-c(t)e^{v(t)-mw(t)}\right)\right] \\ & := F_3(t). \end{cases} \quad (2.1)$$

For $f\in AP(\mathbb{R})$, we denote by

$$\bar{f}=m(f)=\lim_{T\to\infty}\frac{1}{T}\int_0^T f(s)\,\mathrm{d}s,$$

$$\Lambda(f)=\left\{\varpi\in\mathbb{R}:\lim_{T\to\infty}\frac{1}{T}\int_0^T f(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s\neq 0\right\},$$

$$\mathrm{mod}(f)=\left\{\sum_{j=1}^{m}n_j\varpi_j : n_j\in\mathbb{Z}, m\in\mathbb{N}, \varpi_j\in\Lambda(f)\right\}$$

the mean value, the set of Fourier exponents and the module of f, respectively.

Set $\mathbb{X}=\mathbb{Y}=\mathbb{V}_1\bigoplus\mathbb{V}_2$, where

$$\begin{aligned}\mathbb{V}_1=\Big\{&z=(u,v,w)^T\in AP(\mathbb{R},\mathbb{R}^3): \\ &\mathrm{mod}(u)\subseteq\mathrm{mod}(L_u), \mathrm{mod}(v)\subseteq\mathrm{mod}(L_v), \\ &\mathrm{mod}(w)\subseteq\mathrm{mod}(L_w), \forall\varpi\in\Lambda(u)\cup\Lambda(v), |\varpi|\geq\theta_0\Big\},\end{aligned}$$

$$\mathbb{V}_2=\left\{z=(u,v,w)^T\equiv(k_1,k_2,k_3)^T, k_1,k_2,k_3\in\mathbb{R}\right\},$$

where

$$\begin{cases} L_u = & r_1(t)-b_1(t)e^{\varphi_1(0)}-d_1(t)e^{\varphi_2(\delta(0))} \\ & -a_1(t)e^{\varphi_3(\tau(0))-\varphi_1(0)}\Big[1 \\ & -\exp\left(-c(t)e^{\varphi_1(0)-m\varphi_3(\tau(0))}\right)\Big] \\ & +D_1(t)\left[e^{\varphi_2(0)-\varphi_1(0)}-1\right], \\ L_v = & r_2(t)-b_2(t)e^{\varphi_2(0)}-d_2(t)e^{\varphi_1(\zeta(0))} \\ & -a_2(t)e^{\varphi_3(\sigma(0))-\varphi_2(0)}\Big[1 \\ & -\exp\left(-c(t)e^{\varphi_2(0)-m\varphi_3(\sigma(0))}\right)\Big] \\ & +D_2(t)\left[e^{\varphi_1(0)-\varphi_2(0)}-1\right], \\ L_w = & r_3(t)-b_3(t)e^{\varphi_3(0)} \\ & +a_3(t)\left[1-\exp\left(-c(t)e^{\varphi_1(0)-m\varphi_3(0)}\right)\right] \\ & +a_4(t)\left[1-\exp\left(-c(t)e^{\varphi_2(0)-mw(t)}\right)\right], \end{cases}$$

$\varphi = (\varphi_1,\varphi_2,\varphi_3)^T \in C([-\tau,0],\mathbb{R}^3)$, $\tau = \max_{i=1,2,3;i\neq j}\{\tau^+,\delta^+,\sigma^+,\zeta^+\}$, θ_0 is a given positive constant. Define the norm

$$\|z\|_{\mathbb{X}}=\max\left\{\sup_{s\in\mathbb{R}}|u(s)|,\sup_{s\in\mathbb{R}}|v(s)|,\sup_{s\in\mathbb{R}}|w(s)|\right\},$$

where $z=(u,v,w)^T\in\mathbb{X}=\mathbb{Y}$.

Similar to the proof as that in articles [29], it follows that

Lemma 4. *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with $\|\cdot\|_{\mathbb{X}}$.*

Lemma 5. *Let $L:\mathbb{X}\to\mathbb{Y}$, $Lz=L(u,v,w)^T=(\dot{u},\dot{v},\dot{w})^T$, then L is a Fredholm mapping of index zero.*

Lemma 6. *Define $N:\mathbb{X}\to\mathbb{Y}$, $P:\mathbb{X}\to\mathbb{X}$ and $Q:\mathbb{Y}\to\mathbb{Y}$ by*

$$Nz=N\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}F_1(t)\\F_2(t)\\F_3(t)\end{pmatrix},$$

$$Pz=P\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}\bar{u}\\\bar{v}\\\bar{w}\end{pmatrix}=Qz.$$

Then N is L-compact on $\bar{\Omega}$(Ω is an open and bounded subset of $\mathbb{X}$).

III. Main results

Now we are in the position to present and prove our result on the existence of at least one positive almost periodic solution for system (1.3).

(H_1) $b_i^->0$, $i=1,2,3$.

And define

$$\rho_1^+=\rho_2^+=\max\left\{\ln\left[\frac{r_1^+}{b_1^-}\right],\ln\left[\frac{r_2^+}{b_2^-}\right]\right\},\quad \rho_3^-=\ln\left[\frac{r_3^-}{b_3^+}\right],$$

$$\rho_3^+=\ln\left[\frac{r_3^++a_3^++a_4^+}{b_3^-}\right],$$

$$\varrho=\max\left\{(1-m)\rho_3^-,(1-m)\rho_3^+\right\},$$

$$\mu_1(s)=r_1(s)-d_1(s)e^{\rho_2^+}-a_1(s)c(s)e^{\varrho},$$

$$\mu_2(s)=r_2(s)-d_2(s)e^{\rho_1^+}-a_2(s)c(s)e^{\varrho},\quad\forall s\in\mathbb{R}.$$

Theorem 1. *Assume that (H_1) and the following condition hold:*

(H_2) $\mu_j^->0$, $j=1,2$,

then system (1.3) admits at least one positive almost periodic solution.

Proof: It is easy to see that if system (2.1) has one almost periodic solution $(\bar{u},\bar{v},\bar{w})^T$, then $(e^{\bar{u}},e^{\bar{v}},e^{\bar{w}})^T$ is a positive almost periodic solution of system (1.3). Therefore, to completes the proof it suffices to show that system (2.1) has one almost periodic solution.

In order to use Lemma 3, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ as those in Lemma 4 and L,N,P,Q the same as those defined in Lemmas 5 and 6, respectively. It remains to search for an appropriate open and bounded subset $\Omega\subseteq\mathbb{X}$.

Corresponding to the operator equation $Lz = \lambda z$, $\lambda \in (0,1)$, we have

$$\begin{cases} \dot{u}(t) = & \lambda\Big\{ r_1(t) - b_1(t)e^{u(t)} - d_1(t)e^{v(\delta(t))} \\ & -a_1(t)e^{w(\tau(t))-u(t)}\Big[1 \\ & -\exp\left(-c(t)e^{u(t)-mw(\tau(t))}\right)\Big] \\ & +D_1(t)\left[e^{v(t)-u(t)}-1\right]\Big\}, \\ \dot{v}(t) = & \lambda\Big\{ r_2(t) - b_2(t)e^{v(t)} - d_2(t)e^{u(\zeta(t))} \\ & -a_2(t)e^{w(\sigma(t))-v(t)}\Big[1 \\ & -\exp\left(-c(t)e^{v(t)-mw(\sigma(t))}\right)\Big] \\ & +D_2(t)\left[e^{u(t)-v(t)}-1\right]\Big\}, \\ \dot{w}(t) = & \lambda\Big\{ r_3(t) - b_3(t)e^{w(t)} \\ & +a_3(t)\left[1-\exp\left(-c(t)e^{u(t)-mw(t)}\right)\right] \\ & +a_4(t)\left[1-\exp\left(-c(t)e^{v(t)-mw(t)}\right)\right]\Big\}. \end{cases} \tag{3.1}$$

Suppose that $z = (u,v,w)^T \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of system (3.1) for some $\lambda \in (0,1)$, where $\mathrm{Dom}L = \{z = (u,v,w)^T \in \mathbb{X} : u,v,w \in C^1(\mathbb{R}), \dot{u},\dot{v},\dot{w} \in C(\mathbb{R})\}$.

By Lemma 2, for $\forall\epsilon \in (0,1)$, there are four points $\xi_i = \xi_i(\epsilon)$, $\eta_i = \eta_i(\epsilon)$ $(i = 1,2,3)$ such that

$$\begin{cases} \dot{u}(\xi_1) = 0, \\ u(\xi_1) \in [u^*-\epsilon, u^*], \end{cases} \quad \begin{cases} \dot{v}(\xi_2) = 0, \\ v(\xi_2) \in [v^*-\epsilon, v^*], \end{cases} \tag{3.2}$$

$$\begin{cases} \dot{w}(\xi_3) = 0, \\ w(\xi_3) \in [w^*-\epsilon, w^*], \end{cases} \quad \begin{cases} \dot{u}(\eta_1) = 0, \\ u(\eta_1) \in [u_*, u_*+\epsilon], \end{cases} \tag{3.3}$$

$$\begin{cases} \dot{v}(\eta_2) = 0, \\ v(\eta_2) \in [v_*, v_*+\epsilon], \end{cases} \quad \begin{cases} \dot{w}(\eta_3) = 0, \\ w(\eta_3) \in [w_*, w^*+\epsilon], \end{cases} \tag{3.4}$$

where $u^* = \sup_{s\in\mathbb{R}} u(s)$, $v^* = \sup_{s\in\mathbb{R}} v(s)$, $w^* = \sup_{s\in\mathbb{R}} w(s)$, $u_* = \inf_{s\in\mathbb{R}} u(s)$, $v_* = \inf_{s\in\mathbb{R}} v(s)$, $w_* = \inf_{s\in\mathbb{R}} w(s)$.

From system (3.1), it follows from (3.2)-(3.4) that

$$\begin{cases} 0 = r_1(\xi_1) - b_1(\xi_1)e^{u(\xi_1)} - d_1(\xi_1)e^{v(\delta(\xi_1))} \\ \quad -a_1(\xi_1)e^{w(\tau(\xi_1))-u(\xi_1)} \\ \quad \left[1-\exp\left(-c(\xi_1)e^{u(\xi_1)-mw(\tau(\xi_1))}\right)\right] \\ \quad +D_1(\xi_1)\left[e^{v(\xi_1)-u(\xi_1)}-1\right], \\ 0 = r_2(\xi_2) - b_2(\xi_2)e^{v(\xi_2)} - d_2(\xi_2)e^{u(\zeta(\xi_2))} \\ \quad -a_2(\xi_2)e^{w(\sigma(\xi_2))-v(\xi_2)} \\ \quad \left[1-\exp\left(-c(\xi_2)e^{v(\xi_2)-mw(\sigma(\xi_2))}\right)\right] \\ \quad +D_2(\xi_2)\left[e^{u(\xi_2)-v(\xi_2)}-1\right], \\ 0 = r_3(\xi_3) - b_3(\xi_3)e^{w(\xi_3)} \\ \quad +a_3(\xi_3)\left[1-\exp\left(-c(\xi_3)e^{u(\xi_3)-mw(\xi_3)}\right)\right] \\ \quad +a_4(\xi_3)\left[1-\exp\left(-c(\xi_3)e^{v(\xi_3)-mw(\xi_3)}\right)\right] \end{cases} \tag{3.5}$$

and

$$\begin{cases} 0 = r_1(\eta_1) - b_1(\eta_1)e^{u(\eta_1)} - d_1(\eta_1)e^{v(\delta(\eta_1))} \\ \quad -a_1(\eta_1)e^{w(\tau(\eta_1))-u(\eta_1)} \\ \quad \left[1-\exp\left(-c(\eta_1)e^{u(\eta_1)-mw(\tau(\eta_1))}\right)\right] \\ \quad +D_1(\eta_1)\left[e^{v(\eta_1)-u(\eta_1)}-1\right], \\ 0 = r_2(\eta_2) - b_2(\eta_2)e^{v(\eta_2)} - d_2(\eta_2)e^{u(\zeta(\eta_2))} \\ \quad -a_2(\eta_2)e^{w(\sigma(\eta_2))-v(\eta_2)} \\ \quad \left[1-\exp\left(-c(\eta_2)e^{v(\eta_2)-mw(\sigma(\eta_2))}\right)\right] \\ \quad +D_2(\eta_2)\left[e^{u(\eta_2)-v(\eta_2)}-1\right], \\ 0 = r_3(\eta_3) - b_3(\eta_3)e^{w(\eta_3)} \\ \quad +a_3(\eta_3)\left[1-\exp\left(-c(\eta_3)e^{u(\eta_3)-mw(\eta_3)}\right)\right] \\ \quad +a_4(\eta_3)\left[1-\exp\left(-c(\eta_3)e^{v(\eta_3)-mw(\eta_3)}\right)\right]. \end{cases} \tag{3.6}$$

By the third equation of system (3.5), it follows that

$$\begin{aligned} & b_3(\xi_3)e^{w(\xi_3)} \\ = & \ r_3(\xi_3) + a_3(\xi_3)\left[1-\exp\left(-c(\xi_3)e^{u(\xi_3)-mw(\xi_3)}\right)\right] \\ & +a_4(\xi_3)\left[1-\exp\left(-c(\xi_3)e^{v(\xi_3)-mw(\xi_3)}\right)\right] \\ \le & \ r_3^+ + a_3^+ + a_4^+, \end{aligned}$$

which yields

$$w(\xi_3) \le \ln\left[\frac{r_3^+ + a_3^+ + a_4^+}{b_3^-}\right] := \rho_3^+.$$

From (3.3), we have

$$w^* \le w(\xi_3) + \epsilon \le \rho_3^+ + \epsilon.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$w^* \le \rho_3^+. \tag{3.7}$$

From the third equation of system (3.6), it follows that

$$\begin{aligned} & b_3^+ e^{w(\eta_3)} \\ \ge & \ b_3(\eta_3)e^{w(\eta_3)} \\ = & \ r_3(\eta_3) + a_3(\eta_3)\left[1-\exp\left(-c(\eta_3)e^{u(\eta_3)-mw(\eta_3)}\right)\right] \\ & +a_4(\eta_3)\left[1-\exp\left(-c(\eta_3)e^{v(\eta_3)-mw(\eta_3)}\right)\right] \\ \ge & \ r_3^-, \end{aligned}$$

which yields

$$w(\eta_3) \ge \ln\left[\frac{r_3^-}{b_3^+}\right].$$

From (3.4), we have

$$w_* \ge w(\eta_3) - \epsilon \ge \ln\left[\frac{r_3^-}{b_3^+}\right] - \epsilon.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$w_* \ge \ln\left[\frac{r_3^-}{b_3^+}\right] := \rho_3^-. \tag{3.8}$$

Further, from the first equation of system (3.5), it follows that

$$b_1(\xi_1)e^{u(\xi_1)} \le r_1(\xi_1) + D_1(\xi_1)\left[e^{v(\xi_1)-u(\xi_1)}-1\right],$$

which implies from (3.2) that

$$b_1^- e^{u^*-\epsilon} \le r_1^+ + D_1(\xi_1)\left[e^{v^*-(u^*-\epsilon)}-1\right]. \tag{3.9}$$

Similarly, we have from the second equation of system (3.5) that

$$b_2^- e^{v^*-\epsilon} \le r_2^+ + D_2(\xi_2)\left[e^{u^*-(v^*-\epsilon)} - 1\right]. \quad (3.10)$$

Obviously, u^* and v^* only have the following two cases.

(C_1) $u^* \le v^*$. From (3.10), we get

$$b_2^- e^{v^*-\epsilon} \le r_2^+ + D_2(\xi_2)\left[e^{\epsilon} - 1\right]$$
$$\Longrightarrow v^* \le \ln \frac{e^{\epsilon}\left[r_2^+ + D_2^+(e^{\epsilon}-1)\right]}{b_2^-}.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$u^* \le v^* \le \ln\left[\frac{r_2^+}{b_2^-}\right]. \quad (3.11)$$

(C_2) $u^* > v^*$. From (3.9), we get

$$b_1^- e^{u^*-\epsilon} \le r_1^+ + D_1(\xi_1)\left[e^{\epsilon} - 1\right]$$
$$\Longrightarrow u^* \le \ln \frac{e^{\epsilon}\left[r_1^+ + D_1^+(e^{\epsilon}-1)\right]}{b_1^-}.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$v^* < u^* \le \ln\left[\frac{r_1^+}{b_1^-}\right]. \quad (3.12)$$

Let $\rho_1^+ = \rho_2^+ = \max\left\{\ln\left[\frac{r_1^+}{b_1^-}\right], \ln\left[\frac{r_2^+}{b_2^-}\right]\right\}$. Then we have from (3.11)-(3.12) that

$$\max\{u^*, v^*\} \le \rho_1^+ = \rho_2^+. \quad (3.13)$$

On the other hand, by the inequality $1 - e^{-cx} \le cx\,(x > 0, c > 0)$, it follows from the first equation of system (3.6) that

$$\begin{aligned} b_1(\eta_1)e^{u(\eta_1)} \ge\ & r_1(\eta_1) - d_1(\eta_1)e^{v(\delta(\eta_1))} \\ & - a_1(\eta_1)c(\eta_1)e^{(1-m)w(\tau(\eta_1))} \\ & + D_1(\eta_1)\left[e^{v(\eta_1)-u(\eta_1)} - 1\right], \end{aligned}$$

which implies from (3.3) that

$$\begin{aligned} b_1^+ e^{u_*+\epsilon} \ge\ & r_1(\eta_1) - d_1(\eta_1)e^{\rho_2^+} - a_1(\eta_1)c(\eta_1)e^{\varrho} \\ & + D_1(\eta_1)\left[e^{v_*-(u_*+\epsilon)} - 1\right]. \end{aligned} \quad (3.14)$$

Similarly, we have from the second equation of system (3.6) that

$$\begin{aligned} b_2^+ e^{v_*+\epsilon} \ge\ & r_2(\eta_2) - d_2(\eta_2)e^{\rho_1^+} - a_2(\eta_2)c(\eta_2)e^{\varrho} \\ & + D_2(\eta_2)\left[e^{u_*-(v_*+\epsilon)} - 1\right]. \end{aligned} \quad (3.15)$$

Obviously, u_* and v_* only have the following two cases.

(C_1) $u_* \ge v_*$. From (3.15), we get

$$b_2^+ e^{v_*+\epsilon} \ge \mu_2^- + D_2(\eta_2)\left[e^{-\epsilon} - 1\right].$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$b_2^+ e^{v_*} \ge \mu_2^-.$$

Then

$$u^* \ge v^* \ge \ln\left[\frac{\mu_2^-}{b_2^+}\right]. \quad (3.16)$$

(C_2) $u_* < v_*$. From (3.14), we get

$$b_1^+ e^{u_*+\epsilon} \ge \mu_1^- + D_1(\eta_1)\left[e^{-\epsilon} - 1\right].$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$b_1^+ e^{u_*} \ge \mu_1^-.$$

Then

$$v_* \ge u_* \ge \ln\left[\frac{\mu_1^-}{b_1^+}\right]. \quad (3.17)$$

Let $\rho_1^- = \rho_2^- = \min\left\{\ln\left[\frac{\mu_1^-}{b_1^+}\right], \ln\left[\frac{\mu_2^-}{b_2^+}\right]\right\}$. Then we have from (3.16)-(3.17) that

$$\min\{u_*, v_*\} \ge \rho_1^- = \rho_2^-. \quad (3.18)$$

Obviously, $\rho_i^{\pm}$ is independent of λ, $i = 1, 2, 3$. Let

$$\Omega = \left\{ z = (u, v)^T \in \mathbb{X} \;\middle|\; \begin{array}{l} \rho_1^- - 1 < u_* \le u^* < \rho_1^+ + 1, \\ \rho_2^- - 1 < v_* \le v^* < \rho_2^+ + 1, \\ \rho_3^- - 1 < w_* \le w^* < \rho_3^+ + 1 \end{array} \right\}.$$

Then Ω is bounded open subset of $\mathbb{X}$. Therefore, Ω satisfies condition (a) of Lemma 3.

Now we show that condition (b) of Lemma 3 holds, i.e., we prove that $QNz \ne 0$ for all $z = (u, v, w)^T \in \partial\Omega \cap \mathrm{Ker}L = \partial\Omega \cap \mathbb{R}^3$. If it is not true, then there exists at least one constant vector $z_0 = (u_0, v_0, w_0)^T \in \partial\Omega$ such that

$$\begin{cases} 0 = m\Big(r_1(t) - b_1(t)e^{u_0} - d_1(t)e^{v_0} \\ \quad -a_1(t)e^{w_0-u_0}\left[1 - \exp\left(-c(t)e^{u_0-mw_0}\right)\right] \\ \quad +D_1(t)\left[e^{v_0-u_0} - 1\right]\Big), \\ 0 = m\Big(r_2(t) - b_2(t)e^{v_0} - d_2(t)e^{u_0} \\ \quad -a_2(t)e^{w_0-v_0}\left[1 - \exp\left(-c(t)e^{v_0-mw_0}\right)\right] \\ \quad +D_2(t)\left[e^{u_0-v_0} - 1\right]\Big), \\ 0 = m\Big(r_3(t) - b_3(t)e^{w_0} \\ \quad +a_3(t)\left[1 - \exp\left(-c(t)e^{u_0-mw_0}\right)\right] \\ \quad +a_4(t)\left[1 - \exp\left(-c(t)e^{v_0-mw_0}\right)\right]\Big). \end{cases}$$

Similar to the arguments as that in (3.7), (3.8), (3.13) and (3.18), it follows that

$$\rho_1^- \le u_0 \le \rho_1^+, \quad \rho_2^- \le v_0 \le \rho_2^+, \quad \rho_3^- \le w_0 \le \rho_3^+.$$

Then $z_0 \in \Omega \cap \mathbb{R}^3$. This contradicts the fact that $z_0 \in \partial\Omega$. This proves that condition (b) of Lemma 3 holds.

Finally, we will show that condition (c) of Lemma 3 is satisfied. Let us consider the homotopy

$$H(\iota, z) = \iota QNz + (1-\iota)\Phi z, \ \ (\iota, z) \in [0, 1] \times \mathbb{R}^3,$$

where

$$\Phi z = \Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \bar{r}_1 - \bar{b}_1 e^u \\ \bar{r}_2 - \bar{b}_2 e^v \\ \bar{r}_3 - \bar{b}_3 e^w \end{pmatrix}.$$

From the above discussion it is easy to verify that $H(\iota, z) \ne 0$ on $\partial\Omega \cap \mathrm{Ker}L$, $\forall \iota \in [0, 1]$. Further, $\Phi z = 0$ has a solution:

$$(u^*, v^*, w^*)^T = \left(\ln\frac{\bar{r}_1}{\bar{b}_1}, \ln\frac{\bar{r}_2}{\bar{b}_2}, \ln\frac{\bar{r}_3}{\bar{b}_3}\right)^T \in \Omega.$$

A direct computation yields

$$\begin{aligned}&\deg\big(\Phi,\Omega\cap\mathrm{Ker}L,0\big)\\ &=\mathrm{sign}\begin{vmatrix}-\bar{b}_1e^{u^*} & 0 & 0\\ 0 & -\bar{b}_2e^{v^*} & 0\\ 0 & 0 & -\bar{b}_3e^{w^*}\end{vmatrix}\\ &=-1.\end{aligned}$$

By the invariance property of homotopy, we have

$$\begin{aligned}\deg\big(JQN,\Omega\cap\mathrm{Ker}L,0\big)&=\deg\big(QN,\Omega\cap\mathrm{Ker}L,0\big)\\ &=\deg\big(\Phi,\Omega\cap\mathrm{Ker}L,0\big)\neq 0,\end{aligned}$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q=\mathrm{Ker}L$. Obviously, all the conditions of Lemma 3 are satisfied. Therefore, system (2.1) has at least one almost periodic solution, that is, system (1.3) has at least one positive almost periodic solution. This completes the proof. ■

Corollary 1. *Assume that* (H_1)*-*(H_2) *hold. Suppose further that* r_i*,* b_i*,* a_j*,* d_k*,* D_k*,* c*,* τ*,* σ*,* ζ *and* δ *of system* (1.3) *are continuous nonnegative periodic functions with periods* α_i*,* β_i*,* η_j*,* κ_k*,* ς_k*,* ω_1*,* ω_2*,* ω_3*,* ω_4 *and* ω_5*, respectively, then system* (1.3) *has at least one positive almost periodic solution,* $i=1,2,3,\ j=1,2,3,4,\ k=1,2$.

Remark 1. By Corollary 1, it is easy to obtain the existence of at least one positive almost periodic solution of system (1.2) in Example 1, although the positive periodic solution of system (1.2) is nonexistent.

In Corollary 1, let $\alpha_i=\beta_i=\eta_j=\kappa_k=\varsigma_k=\omega_1=\omega_2=\omega_3=\omega_4=\omega_5=\omega$, $i=1,2,3,\ j=1,2,3,4,\ k=1,2$, then we obtain that

Corollary 2. *Assume that* (H_1)*-*(H_2) *hold. Suppose further that* r_i*,* b_i*,* a_j*,* d_k*,* D_k*,* c*,* τ*,* σ*,* ζ *and* δ *of system* (1.3) *are continuous nonnegative* ω*-periodic functions, then system* (1.3) *has at least one positive* ω*-periodic solution.*

IV. Uniform persistence

Our object in this section is to prove the uniform persistence of system (1.3).

Theorem 2. *Assume that*

(H_3) $r_1^->D_1^++d_1^+M_2+a_1^+c^+\max\{M_3^{1-m},N_3^{1-m}\}$, $r_2^->D_2^++d_2^+M_1+a_2^+c^+\max\{M_3^{1-m},N_3^{1-m}\}$,

then for any positive solution $(x_1,x_2,y)^T$ *of system* (1.3) *satisfies*

$$N_i\le x_i(t)\le M_i,\quad i=1,2,\quad N_3\le y(t)\le M_3,$$

where N_i *and* M_i *are defined as those in* (4.2)*-*(4.7)*,* $i=1,2,3$*. That is, system* (1.3) *is uniformly persistent.*

Proof:

(1) If $x_1(t)\ge x_2(t)$, then we have from the first equation of system (1.3) that

$$\dot{x}_1(t)\le x_1(t)\left[r_1^+-b_1^-x_1(t)\right]. \tag{4.1}$$

By Lemmas 2.3 and 2.4 in [22], we have from (4.1) that

$$x_1(t)\le\frac{r_1^+}{b_1^-}\le\max\left\{\frac{r_1^+}{b_1^-},\frac{r_2^+}{b_2^-}\right\}:=M_1. \tag{4.2}$$

(2) If $x_2(t)\ge x_1(t)$, then we have from the second equation of system (1.3) that

$$\dot{x}_2(t)\le x_2(t)\left[r_2^+-b_2^-x_2(t)\right].$$

By Lemmas 2.3 and 2.4 in [22], we have

$$x_2(t)\le\frac{r_2^+}{b_2^-}\le\max\left\{\frac{r_1^+}{b_1^-},\frac{r_2^+}{b_2^-}\right\}:=M_2. \tag{4.3}$$

By the third equation of system (1.3), ones have

$$\dot{y}(t)\ge y(t)\left[r_3^--b_3^+y(t)\right],$$

by Lemmas 2.3 and 2.4 in [22], we have

$$y(t)\ge\frac{r_3^-}{b_3^+}:=N_3. \tag{4.4}$$

Using the inequality $1-e^{-x}\le x,\ x\ge 0$, we have from (4.4) that

$$\begin{aligned}\dot{y}(t)&\le y(t)\left[r_3^+-b_3^-y(t)\right]\\ &\quad+a_3^+c^+x_1(t)y^{1-m}(t)+a_4^+c^+x_2(t)y^{1-m}(t)\\ &\le y(t)\Big[r_3^++a_3^+c^+M_1N_3^{-m}+a_4^+c^+M_2N_3^{-m}\\ &\qquad -b_3^-y(t)\Big],\end{aligned}$$

which implies that

$$y(t)\le\frac{r_3^++a_3^+c^+M_1N_3^{-m}+a_4^+c^+M_2N_3^{-m}}{b_3^-}:=M_3 \tag{4.5}$$

In view of the first equation of system (1.3), it follows that

$$\begin{aligned}\dot{x}_1(t)&\ge x_1(t)\Big[r_1^--D_1^+-d_1^+M_2\\ &\qquad -a_1^+c^+\max\{M_3^{1-m},N_3^{1-m}\}-b_1^+x_1(t)\Big],\end{aligned}$$

which implies that

$$\begin{aligned}x_1(t)&\ge\frac{r_1^--D_1^+-d_1^+M_2-a_1^+c^+\max\{M_3^{1-m},N_3^{1-m}\}}{b_1^+}\\ &:=N_1.\end{aligned} \tag{4.6}$$

Similar to the argument as that in (4.6), we obtain from the second equation of system (1.3) that

$$\begin{aligned}x_2(t)&\ge\frac{r_2^--D_2^+-d_2^+M_1-a_2^+c^+\max\{M_3^{1-m},N_3^{1-m}\}}{b_2^+}\\ &:=N_2.\end{aligned} \tag{4.7}$$

The proof is completed. ■

V. Global asymptotical stability

The main result of this section concerns the global asymptotical stability of system (1.3).

Theorem 3. *Assume that* (H_1) *and* (H_3) *hold. Suppose further that*

(H_4) $\tau,\delta,\sigma,\zeta\in C^1(\mathbb{R})$, $\inf_{t\in\mathbb{R}}\{\dot{\tau}(t),\dot{\delta}(t),\dot{\zeta}(t),\dot{\sigma}(t)\}>0$, *and there exist positive constants* $\lambda_i (i=1,2,3)$ *such that*

$$\inf_{t\in\mathbb{R}}\left\{\lambda_1b_1(t)-\frac{\lambda_1M_3a_1(t)c(t)}{N_3^m}\right.$$

$$-\frac{\lambda_3 M_3 a_3(t)c(t)}{N_3^m} - \frac{\lambda_2 d_2(\zeta^{-1}(t))}{\dot{\zeta}(\zeta^{-1}(t))}\Bigg\} > 0,$$

$$\inf_{t\in\mathbb{R}} \Bigg\{ \lambda_2 b_2(t) - \frac{\lambda_2 M_3 a_2(t)c(t)}{N_3^m} - \frac{\lambda_3 M_3 a_4(t)c(t)}{N_3^m} - \frac{\lambda_1 d_1(\delta^{-1}(t))}{\dot{\delta}(\delta^{-1}(t))}\Bigg\} > 0,$$

$$\inf_{t\in\mathbb{R}} \Bigg\{ \lambda_3 b_3(t) - \lambda_3 a_3(t)\Big[1 + \frac{mM_1M_3c(t)}{N_3^{m+1}}\Big] -\lambda_3 a_4(t)\Big[1 + \frac{mM_2M_3c(t)}{N_3^{m+1}}\Big] -\lambda_1 \frac{a_1(\tau^{-1}(t))}{\dot{\tau}(\tau^{-1}(t))}\Big[1 + \frac{mM_1M_3c(\tau^{-1}(t))}{N_3^{m+1}}\Big] -\lambda_2 \frac{a_2(\sigma^{-1}(t))}{\dot{\sigma}(\sigma^{-1}(t))}\Big[1 + \frac{mM_2M_3c(\sigma^{-1}(t))}{N_3^{m+1}}\Big]\Bigg\} > 0,$$

where f^{-1} is the inverse function of $f \in C^1(\mathbb{R})$ and $\inf_{t\in\mathbb{R}} \dot{f}(t) > 0$.

Then system (1.3) is globally asymptotically stable.

Proof: Suppose that $X(t) = (x_1(t), x_2(t), y(t))^T$ and $X^*(t) = (x_1^*(t), x_2^*(t), y^*(t))^T$ are any two solutions of system (1.3).

By (H_4), there exists s positive constant Θ such that

$$\inf_{t\in\mathbb{R}} \Bigg\{ \lambda_1 b_1(t) - \frac{\lambda_1 M_3 a_1(t)c(t)}{N_3^m} - \frac{\lambda_3 M_3 a_3(t)c(t)}{N_3^m} - \frac{\lambda_2 d_2(\zeta^{-1}(t))}{\dot{\zeta}(\zeta^{-1}(t))}\Bigg\} > \Theta,$$

$$\inf_{t\in\mathbb{R}} \Bigg\{ \lambda_2 b_2(t) - \frac{\lambda_2 M_3 a_2(t)c(t)}{N_3^m} - \frac{\lambda_3 M_3 a_4(t)c(t)}{N_3^m} - \frac{\lambda_1 d_1(\delta^{-1}(t))}{\dot{\delta}(\delta^{-1}(t))}\Bigg\} > \Theta,$$

$$\inf_{t\in\mathbb{R}} \Bigg\{ \lambda_3 b_3(t) - \lambda_3 a_3(t)\Big[1 + \frac{mM_1M_3c(t)}{N_3^{m+1}}\Big] -\lambda_3 a_4(t)\Big[1 + \frac{mM_2M_3c(t)}{N_3^{m+1}}\Big] -\lambda_1 \frac{a_1(\tau^{-1}(t))}{\dot{\tau}(\tau^{-1}(t))}\Big[1 + \frac{mM_1M_3c(\tau^{-1}(t))}{N_3^{m+1}}\Big] -\lambda_2 \frac{a_2(\sigma^{-1}(t))}{\dot{\sigma}(\sigma^{-1}(t))}\Big[1 + \frac{mM_2M_3c(\sigma^{-1}(t))}{N_3^{m+1}}\Big]\Bigg\} > \Theta.$$

Construct a Lyapunov functional as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t), \quad \forall t \geq T_5,$$

where

$$V_1(t) = \sum_{i=1}^{2} \lambda_i |\ln x_i(t) - \ln x_i^*(t)| + \lambda_3 |\ln y(t) - \ln y^*(t)|,$$

$$V_2(t) = \lambda_2 \int_{\zeta(t)}^{t} \frac{d_2(\zeta^{-1}(s))}{\dot{\zeta}(\zeta^{-1}(s))}|x_1(s) - x_1^*(s)|\,\mathrm{d}s + \lambda_1 \int_{\delta(t)}^{t} \frac{d_1(\delta^{-1}(s))}{\dot{\delta}(\delta^{-1}(s))}|x_2(s) - x_2^*(s)|\,\mathrm{d}s,$$

$$V_3(t) = \lambda_1 \int_{\tau(t)}^{t} \frac{a_1(\tau^{-1}(s))}{\dot{\tau}(\tau^{-1}(s))}\Big[1 + \frac{mM_1M_3c(\tau^{-1}(s))}{N_3^{m+1}}\Big] |y(s) - y^*(s)|\,\mathrm{d}s + \lambda_2 \int_{\sigma(t)}^{t} \frac{a_2(\sigma^{-1}(s))}{\dot{\sigma}(\sigma^{-1}(s))}\Big[1 + \frac{mM_2M_3c(\sigma^{-1}(s))}{N_3^{m+1}}\Big] |y(s) - y^*(s)|\,\mathrm{d}s.$$

Calculating the upper right derivative of $V_1(t)$ along the solution of system(1.3), it follows that

$$\begin{aligned}
&D^+V_1(t)\\
&= \sum_{i=1}^{2} \lambda_i \Big[\frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{x}_i^*(t)}{x_i^*(t)}\Big]\mathrm{sgn}(x_i(t) - x_i^*(t))\\
&\quad +\lambda_3 \Big[\frac{\dot{y}(t)}{y(t)} - \frac{\dot{y}^*(t)}{y^*(t)}\Big]\mathrm{sgn}(y(t) - y^*(t))\\
&\leq \lambda_1 \Big[- b_1(t)|x_1(t) - x_1^*(t)|\\
&\quad +d_1(t)|x_2(\delta(t)) - x_2^*(\delta(t))|\\
&\quad +a_1(t)\Big|y(\tau(t))\Big[1 - \exp\Big(\frac{-c(t)x_1(t)}{y^m(\tau(t))}\Big)\Big]\\
&\quad -y^*(\tau(t))\Big[1 - \exp\Big(\frac{-c(t)x_1^*(t)}{y^{*m}(\tau(t))}\Big)\Big]\Big|\Big]\\
&\quad +\lambda_1 \mathrm{sgn}(x_1(t) - x_1^*(t))\frac{D_1(t)[x_2(t)x_1^*(t) - x_1(t)x_2^*(t)]}{x_1(t)x_1^*(t)}\\
&\quad +\lambda_2 \Big[- b_2(t)|x_2(t) - x_2^*(t)|\\
&\quad +d_2(t)|x_1(\zeta(t)) - x_1^*(\zeta(t))|\\
&\quad +a_2(t)\Big|y(\sigma(t))\Big[1 - \exp\Big(\frac{-c(t)x_2(t)}{y^m(\sigma(t))}\Big)\Big]\\
&\quad -y^*(\sigma(t))\Big[1 - \exp\Big(\frac{-c(t)x_2^*(t)}{y^{*m}(\sigma(t))}\Big)\Big]\Big|\Big]\\
&\quad +\lambda_2 \mathrm{sgn}(x_2(t) - x_2^*(t))\frac{D_2(t)[x_1(t)x_2^*(t) - x_2(t)x_1^*(t)]}{x_2(t)x_2^*(t)}\\
&\quad -\lambda_3 b_3(t)|y(t) - y^*(t)|\\
&\quad +\lambda_3 a_3(t)\Big|y(t)\Big[1 - \exp\Big(\frac{-c(t)x_1(t)}{y^m(t)}\Big)\Big]\\
&\quad -y^*(t)\Big[1 - \exp\Big(\frac{-c(t)x_1^*(t)}{y^{*m}(t)}\Big)\Big]\Big|\\
&\quad +\lambda_3 a_4(t)\Big|y(t)\Big[1 - \exp\Big(\frac{-c(t)x_2(t)}{y^m(t)}\Big)\Big]\\
&\quad -y^*(t)\Big[1 - \exp\Big(\frac{-c(t)x_2^*(t)}{y^{*m}(t)}\Big)\Big]\Big|\\
&\leq \lambda_1 \Big\{ - b_1(t)|x_1(t) - x_1^*(t)|\\
&\quad +d_1(t)|x_2(\delta(t)) - x_2^*(\delta(t))|\\
&\quad +a_1(t)\Big[1 + \frac{mM_1M_3c(t)}{N_3^{m+1}}\Big]|y(\tau(t)) - y^*(\tau(t))|\\
&\quad +a_1(t)\frac{M_3c(t)}{N_3^m}|x_1(t) - x_1^*(t)|\Big\}\\
&\quad +\lambda_1 \frac{D_1(t)|x_2(t) - x_2^*(t)|}{N_1} + \lambda_2 \Big\{ - b_2(t)|x_2(t) - x_2^*(t)|\\
&\quad +d_2(t)|x_1(\zeta(t)) - x_1^*(\zeta(t))|\\
&\quad +a_2(t)\Big[1 + \frac{mM_2M_3c(t)}{N_3^{m+1}}\Big]|y(\sigma(t)) - y^*(\sigma(t))|
\end{aligned}$$

$$+a_2(t)\frac{M_3c(t)}{N_3^m}|x_2(t)-x_2^*(t)|\Big\}$$
$$+\lambda_2\frac{D_2(t)|x_1(t)-x_1^*(t)|}{N_2}-\lambda_3b_3(t)|y(t)-y^*(t)|$$
$$+\lambda_3a_3(t)\Big\{\Big[1+\frac{mM_1M_3c(t)}{N_3^{m+1}}\Big]|y(t)-y^*(t)|$$
$$+\frac{M_3c(t)}{N_3^m}|x_1(t)-x_1^*(t)|\Big\}$$
$$+\lambda_3a_4(t)\Big\{\Big[1+\frac{mM_2M_3c(t)}{N_3^{m+1}}\Big]|y(t)-y^*(t)|$$
$$+\frac{M_3c(t)}{N_3^m}|x_2(t)-x_2^*(t)|\Big\}. \quad (5.1)$$

Here we use the following inequality:

$$\mathrm{sgn}(x_i(t)-x_i^*(t))\sum_{j=1}^n D_{ji}(t)\frac{[x_j(t)x_i^*(t)-x_i(t)x_j^*(t)]}{x_i(t)x_i^*(t)}$$
$$\le\sum_{j=1}^n\frac{D_{ji}(t)}{N_i-\epsilon_3}|x_j(t)-x_j^*(t)|.$$

Moreover, we obtain that

$$\begin{aligned}D^+V_2(t) \;=\; &\lambda_2\frac{d_2(\zeta^{-1}(t))}{\dot\zeta(\zeta^{-1}(t))}|x_1(t)-x_1^*(t)|\\ &-\lambda_2d_2(t)|x_1(\zeta(t))-x_1^*(\zeta(t))|\\ &+\lambda_1\frac{d_1(\delta^{-1}(t))}{\dot\delta(\delta^{-1}(t))}|x_2(t)-x_2^*(t)|\\ &-\lambda_1d_1(t)|x_2(\delta(t))-x_2^*(\delta(t))|,\end{aligned} \quad (5.2)$$

$$\begin{aligned}D^+V_3(t) \;=\; &\lambda_1\frac{a_1(\tau^{-1}(t))}{\dot\tau(\tau^{-1}(t))}\Big[1+\frac{mM_1M_3c(\tau^{-1}(t))}{N_3^{m+1}}\Big]\\ &|y(t)-y^*(t)|\\ &-\lambda_1a_1(t)\Big[1+\frac{mM_1M_3c(t)}{N_3^{m+1}}\Big]\\ &|y(\tau(t))-y^*(\tau(t))|\\ &+\lambda_2\frac{a_2(\sigma^{-1}(t))}{\dot\sigma(\sigma^{-1}(t))}\Big[1+\frac{mM_2M_3c(\sigma^{-1}(t))}{N_3^{m+1}}\Big]\\ &|y(t)-y^*(t)|\\ &-\lambda_2a_2(t)\Big[1+\frac{mM_2M_3c(t)}{N_3^{m+1}}\Big]\\ &|y(\sigma(t))-y^*(\sigma(t))|.\end{aligned} \quad (5.3)$$

From (5.1)-(5.3), one has

$$\begin{aligned}D^+V(t) \le &-\Big[\lambda_1b_1(t)-\frac{\lambda_1M_3a_1(t)c(t)}{N_3^m}\\ &-\frac{\lambda_3M_3a_3(t)c(t)}{N_3^m}-\frac{\lambda_2d_2(\zeta^{-1}(t))}{\dot\zeta(\zeta^{-1}(t))}\Big]\\ &|x_1(t)-x_1^*(t)|\\ &-\Big[\lambda_2b_2(t)-\frac{\lambda_2M_3a_2(t)c(t)}{N_3^m}\\ &-\frac{\lambda_3M_3a_4(t)c(t)}{N_3^m}-\frac{\lambda_1d_1(\delta^{-1}(t))}{\dot\delta(\delta^{-1}(t))}\Big]\\ &|x_2(t)-x_2^*(t)|\\ &-\Big\{\lambda_3b_3(t)-\lambda_3a_3(t)\Big[1+\frac{mM_1M_3c(t)}{N_3^{m+1}}\Big]\\ &-\lambda_3a_4(t)\Big[1+\frac{mM_2M_3c(t)}{N_3^{m+1}}\Big]\\ &-\lambda_1\frac{a_1(\tau^{-1}(t))}{\dot\tau(\tau^{-1}(t))}\Big[1+\frac{mM_1M_3c(\tau^{-1}(t))}{N_3^{m+1}}\Big]\\ &-\lambda_2\frac{a_2(\sigma^{-1}(t))}{\dot\sigma(\sigma^{-1}(t))}\Big[1+\frac{mM_2M_3c(\sigma^{-1}(t))}{N_3^{m+1}}\Big]\Big\}\\ &|y(t)-y^*(t)|\\ \le &-\Theta\Big[\sum_{i=1}^2|x_i(t)-x_i^*(t)|+|y(t)-y^*(t)|\Big]\end{aligned} \quad (5.4)$$

Therefore, V is non-increasing. Integrating (5.4) from 0 to t leads to

$$V(t)+\Theta\int_0^t\Big[\sum_{i=1}^2|x_i(s)-x_i^*(s)|+|y(s)-y^*(s)|\Big]\,\mathrm{d}s$$
$$\le V(0)<+\infty,\quad \forall t\ge 0,$$

that is,

$$\int_0^{+\infty}\Big[\sum_{i=1}^2|x_i(s)-x_i^*(s)|+|y(s)-y^*(s)|\Big]\,\mathrm{d}s<+\infty,$$

which implies that

$$\lim_{s\to+\infty}|x_i(s)-x_i^*(s)|=0,\quad \lim_{s\to+\infty}|y(s)-y^*(s)|=0,$$

where $i=1,2$. Thus, system (1.3) is globally asymptotically stable. This completes the proof. ■

Together with Theorem 1, we obtain

Theorem 4. *Assume that (H_1)-(H_4) hold, then system (1.3) has at least one positive almost periodic solution, which is globally asymptotically stable.*

Corollary 3. *Assume that (H_1)-(H_4) hold. Suppose further that all the coefficients in system (1.3) are ω-periodic functions, then system (1.3) has at least one positive ω-periodic solution, which is globally asymptotically stable.*

VI. An example and numerical simulations

Example 2. Consider the following Watt-type predator-prey model with diffusion and different periods:

$$\begin{cases}\dot x_1(t)= x_1(t)[2+|\sin(\sqrt2t)|\\ \quad -b_1(t)x_1(t)-0.1x_2(t-1)]\\ \quad -0.1y(t-1)\Big[1-\exp\Big(\frac{-x_1(t)}{y^{0.5}(t-1)}\Big)\Big]\\ \quad +D_1(t)\,[x_2(t)-x_1(t)]\,,\\ \dot x_2(t)= x_2(t)[2+|\cos(\sqrt{13}t)|\\ \quad -b_2(t)x_2(t)-0.1x_1(t-2)]\\ \quad -0.1y(t-2)\Big[1-\exp\Big(\frac{-x_2(t)}{y^{0.5}(t-2)}\Big)\Big]\\ \quad +D_2(t)\,[x_1(t)-x_2(t)]\,,\\ \dot y(t)= y(t)\,[1+|\cos(\sqrt5t)|-b_3(t)y(t)]\\ \quad +0.1\cos^2(\sqrt3t)y(t)\Big[1-\exp\Big(\frac{-x_1(t)}{y^{0.5}(t)}\Big)\Big]\\ \quad +0.1\cos^2(\sqrt3t)y(t)\Big[1-\exp\Big(\frac{-x_2(t)}{y^{0.5}(t)}\Big)\Big],\end{cases} \quad (6.1)$$

where

$$\begin{pmatrix}b_1(s)\\ b_2(s)\\ b_3(s)\end{pmatrix}=\begin{pmatrix}5+|\sin(\sqrt2s)|\\ 5+|\cos(\sqrt2s)|\\ 5+|\cos(\sqrt3s)|\end{pmatrix},$$

$$\begin{pmatrix}D_1(s)\\ D_2(s)\end{pmatrix}=0.1\begin{pmatrix}2+|\sin(\sqrt3s)|\\ 1+|\cos(\sqrt3s)|\end{pmatrix},\quad \forall s\in\mathbb{R}.$$

Then system (6.1) is uniformly persistent and has at least one positive almost periodic solution which is globally asymptotically stable.

Proof: Corresponding system (1.3), we have $m = 0.5$, $d_1 = d_2 \equiv 0.1$, $c \equiv 1$,

$$\begin{pmatrix} r_1(s) \\ r_2(s) \\ r_3(s) \end{pmatrix} = \begin{pmatrix} 2 + |\sin(\sqrt{2}s)| \\ 2 + |\cos(\sqrt{13}s)| \\ 1 + |\cos(\sqrt{5}s)| \end{pmatrix},$$

$$\begin{pmatrix} a_1(s) = a_2(s) \\ a_3(s) = a_4(s) \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.1\cos^2(\sqrt{3}s) \end{pmatrix}, \quad \forall s \in \mathbb{R}.$$

Obviously, $b_1^- = b_2^- = b_3^- = 5$. By a easy calculation, we obtain that

$$\rho_1^+ = \rho_2^+ \approx \ln 1.5, \quad \varrho \approx \ln 1.2, \quad \mu_1 = \mu_2 > 1.7.$$

So (H_1)-(H_2) in Theorem 1 hold. By Theorem 1, system (6.1) admits at least one positive almost periodic solution (see Figure 1).

Further, it is not difficult to calculate that $M_1 = M_2 = 0.6$, $N_3 \approx 0.17$, $M_3 \approx 0.55$, which implies that (H_3) in Theorem 2 holds and $N_1 \approx 0.32$, $N_2 \approx 0.25$. Taking $\lambda_1 = \lambda_2 = \lambda_3 = 1$, it is easy to verify that (H_4) in Theorem 4 is satisfied. By Theorems 2 and 4, system (6.1) is uniformly persistent and has at least one positive almost periodic solution, which is globally asymptotically stable (see Figures 2-4). This completes the proof. ■

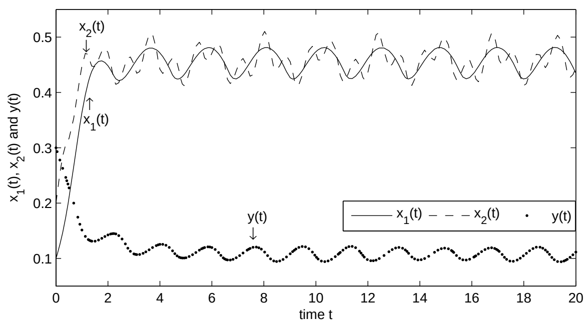

Fig. 1 Almost periodic oscillations of system (6.1)

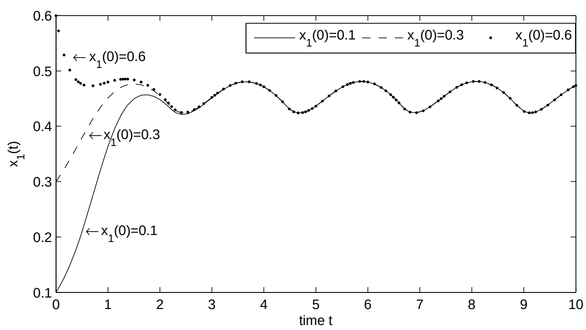

Fig. 2 Global asymptotical stability of state variable x_1 of system (6.1)

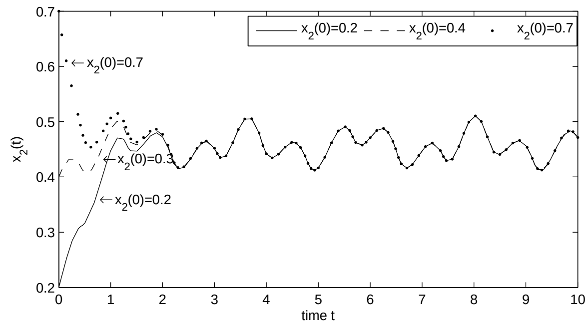

Fig. 3 Global asymptotical stability of state variable x_2 of system (6.1)

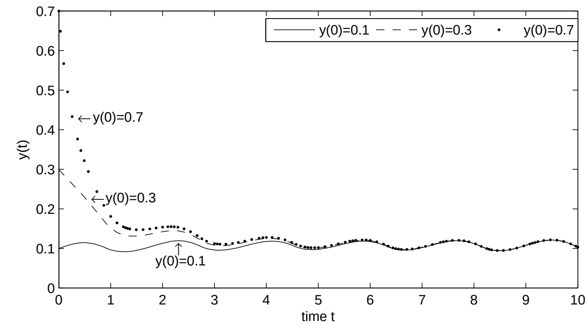

Fig. 4 Global asymptotical stability of state variable y of system (6.1)

Remark 2. Clearly, system (6.1) is with incommensurable periods. Through all the coefficients of system (6.1) are periodic functions, the positive periodic solutions could not possibly exist. However, by the work in this paper, the positive almost periodic solutions of system (6.1) exactly exist.

VII. Conclusion

In this paper we have obtained the uniform permanence and existence of a globally asymptotically stable positive almost periodic solution for a Watt-type almost periodic predator-prey model with diffusion and time delays. The approach is based on the continuation theorem of coincidence degree theory, the comparison theorem and Lyapunov functional. And Lemma 2 in Section 2 and Lemmas 2.3-2.4 in [22] are critical to study the permanence and the existence of positive almost periodic solution of the biological model. It is important to notice that the approach used in this paper can be extended to other types of biological model such as epidemic models, Lotka-Volterra systems and other similar models of first order [47-48]. Future work will include models based on impulsive differential equations and biological dynamic systems on time scales.

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