Numerical Implementation of Triangular Functions for Solving a Stochastic Nonlinear Volterra-Fredholm Integral Equation

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***Abstract*—This paper presents a numerical method for solving the stochastic nonlinear volterra-fredholm integral equation (SNVFIE) driven by a standard Brownian motion (SBM). The method is illustrated via a stochastic operational matrix (SOM) based on the triangular functions (TFs) in combination with the collocation method. With using this approach, the SNVFIE reduces to a stochastic nonlinear system of $3m+3$ equations and $3m+3$ unknowns. In addition, the error analysis and some numerical examples are provided to demonstrate the applicability and accuracy of this method.**

***Index Terms*—Triangular functions; Standard Brownian motion; Stochastic operational matrix; Stochastic nonlinear volterra-fredholm integral equation; Collocation method.**

I. Introduction

In many fields of science and engineering there are a large number of problems which are intrinsically nonlinear equations, involving stochastic excitations of a Gaussian white noise type. The Gaussian white noise mathematically described as a formal derivative of a Brownian motion process, all such problems are mathematically modeled by stochastic equations, or in more complicated cases, described by stochastic integral equations [5, 7, 9, 10, 11, 12, 13].

In this work, we consider

$$X(t) = X_0 + \int_0^1 \alpha(s, X(s))ds + \int_0^t \beta(s, X(s))ds + \int_0^t \gamma(s, X(s))dB(s), \quad t \in (0, T), \tag{1}$$

where $\alpha(s, X(s)),\ \beta(s, X(s)),\ \gamma(s, X(s)) : (0, T) \times \mathbb{R} \longrightarrow \mathbb{R}$ and $X(s)$ are the unknown stochastic processes defined on a complete probability space $(\Omega, F, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Also, $B(s)$ be the SBM defined on same probability space.

These kinds of equations can not be solved analytically. Also, there is the numerical method for solving the stochastic volterra-fredholm integral equations by using properties of the block pulse functions (BPFs) [10]. Hence, it is important to provide their numerical solutions. Our main motivation for considering Eq. (1) is that it has played important role in mathematical finance, biology, medical and social [2, 12].

In this paper, we use from the SOM based on properties of the TFs without integration. The advantage of this method is simple calculations by using conversion of the SNVFIE to the stochastic nonlinear system of $3m+3$ equations and $3m+3$ unknowns.

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The result of the paper is organized as follows: In Section 2, we state some essential preliminaries which play fundamental role in our method. In Section 3, we solve Eq. (1) by the SOM based on the TFs in combination with the collocation technique. In Sections 4 and 5, we provide the error analysis and some numerical examples to demonstrate the applicability and accuracy of presented method. Finally, in Section 6, is given a brief conclusion.

II. Basic definitions

A. Stochastic concepts

In this section, we review the basic properties of the SBM that are essential for this work. For more details see [2, 12].

Let the functions $\alpha(t, X)$, $\beta(t, X)$ and $\gamma(t, X)$ hold in lipschitz conditions and linear growth, i.e. there are constants $k_1,\ k_2,\ k_3,\ k_4,\ k_5 > 0$ and $k_6 > 0$ such that:

$$\mathcal{A}_1.\ \begin{cases} |\alpha(t,X) - \alpha(t,Y)| \leq k_1|X - Y|,\ (Lipschitz\ continuity), \\ |\alpha(t,X)| < k_2(1 + |X|),\ (Linear\ growth). \end{cases}$$

$$\mathcal{A}_2.\ \begin{cases} |\beta(t,X) - \beta(t,Y)| \leq k_3|X - Y|,\ (Lipschitz\ continuity), \\ |\beta(t,X)| < k_4(1 + |X|),\ (Linear\ growth). \end{cases}$$

$$\mathcal{A}_3.\ \begin{cases} |\gamma(t,X) - \gamma(t,Y)| < k_5|X - Y|,\ (Lipschitz\ continuity), \\ |\gamma(t,X)| < k_6(1 + |X|),\ (Linear\ growth). \end{cases}$$

For $X,\ Y \in R$ and $t \in (0, T)$.

Theorem II.1. *Let $\alpha(t, X(t))$, $\beta(t, X(t))$ and $\gamma(t, X(t))$ hold in conditions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $E \mid X_0 \mid^2 < \infty$. Then, there exists a unique solution for Eq. (1).*

Proof. See [2].

Definition II.2. $\{B(t),\ t \geq 0\}$ be the SBM with the main properties as follows:

1. The SBM has independent increments for $0 \leq t_0 \leq t_1 \leq ... \leq t_n \leq T$.
2. $B(t+h) - B(t)$ be normally distribution with mean 0 and variance h, for all $t \geq 0$, $h > 0$.
3. $B(t)$ be the continuous function.

Definition II.3. Let $\nu = \nu(S, T)$ be the class of functions $\alpha(t, \omega) : [0, \infty) \times \Omega \longrightarrow R$ such that:

1. The function $\alpha(t, \omega)$ be $\beta \times F$ measurable.
2. The function $\alpha(t, \omega)$ is $\mathcal{F}_t$-adapted.
3. $E\big[\int_S^T \alpha^2(t, \omega)dt\big] < \infty$.

Theorem II.4. *(The Itô isometry). Let $\beta \in \nu(S,T)$, then*

$$E\big[(\int_S^T \beta(t,\omega)dB(t)(\omega))^2\big] = E\big[\int_S^T \beta^2(t,\omega)dt\big].$$

Proof. See [2].

Lemma II.5. *(The Gronwall inequality) Let $\alpha, \beta \in [t_0,T] \to R$ be integral with*

$$0 \le \alpha(t) \le \beta(t) + L\int_{t_0}^t \alpha(s)ds,\ t\in[t_0,T],\ L>0,$$

then

$$\alpha(t) \le \beta(t)\big(1 + L\int_{t_0}^t e^{L(t-s)}ds\big), \quad t\in[t_0,T].$$

Proof. see [6].

B. Triangular functions

In this section, we introduce the basic properties of the TFs that are essential for this paper. For more details see [1, 3, 4, 8, 13].

1. Two m-sets of TFs are defined as follows:

$$T_i^1(t) = \begin{cases} 1-\frac{t-ih}{h} & ih \le t < (i+1))h, \\ 0 & otherwise, \end{cases}$$

and

$$T_i^2(t) = \begin{cases} \frac{t-ih}{h} & ih \le t < (i+1))h, \\ 0 & otherwise, \end{cases}$$

where $h = \frac{T}{m}$, $i = 0,\dots,m-1$, $T1(t) = [T_0^1(t),...,T_{m-1}^1(t)]^T$ and $T2(t) = [T_0^2(t),...,T_{m-1}^2(t)]^T$.

2. Let $f(x) \in L^2\big([0,T]\big)$, then

$$f(t) \approx F^T.T(t),$$

where $F = [F1,\ F2]^T$, $F1 = (F1_i)_{1\times m} = (f(ih))_{1\times m}$, $F2 = (F2_i)_{1\times m} = (f(i+1)h)_{1\times m}$ and $T(t) = \begin{pmatrix} T1(t) \\ T2(t) \end{pmatrix}$.

3.

$$\int_0^t T(s)ds \approx P_T.T(t),$$

with

$$P_T = \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix},$$

$$P1 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{m\times m},$$

and

$$P2 = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}_{m\times m}.$$

4.

$$\int_0^1 T(s)ds \approx P_T.T(t).$$

5.

$$\int_0^t T(s)dB(s) \approx P_s.T(t),$$

with

$$P_s = \begin{pmatrix} P1_s & P1_s \\ P2_s & P2_s \end{pmatrix},$$

$$P1_s = \begin{pmatrix} \alpha(0) & \beta(0) & \beta(0) & \dots & \beta(0) \\ 0 & \alpha(1) & \beta(1) & \dots & \beta(1) \\ 0 & 0 & \alpha(2) & \dots & \beta(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \beta(m-2) \\ 0 & 0 & 0 & \dots & \alpha(m-1) \end{pmatrix}_{m\times m},$$

$$P2_s = \begin{pmatrix} \gamma(0) & \rho(0) & \rho(0) & \dots & \rho(0) \\ 0 & \gamma(1) & \rho(1) & \dots & \rho(1) \\ 0 & 0 & \gamma(2) & \dots & \rho(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \rho(m-2) \\ 0 & 0 & 0 & \dots & \gamma(m-1) \end{pmatrix}_{m\times m},$$

and

$$\begin{cases} \alpha(i) = (i+1)[B((i+0.5)h) - B(ih)] - \int_{ih}^{(i+0.5)h} \frac{s}{h}dB(s), \\ \beta(i) = (i+1)[B((i+1)h) - B(ih)] - \int_{ih}^{(i+1)h} \frac{s}{h}dB(s), \\ \gamma(i) = -i[B((i+0.5)h) - B(ih)] + \int_{ih}^{(i+0.5)h} \frac{s}{h}dB(s), \\ \rho(i) = -i[B((i+1)h) - B(ih)] + \int_{ih}^{(i+1)h} \frac{s}{h}dB(s). \end{cases}$$

III. Using of the TFs for solving the SNVFIE

Let

$$\begin{cases} f(s) = \alpha(s,X(s)), \\ g(s) = \beta(s,X(s)), \\ k(s) = \gamma(s,X(s)), \end{cases} \tag{2}$$

with substituting (2) in Eq. (1), we get

$$X(t) = X_0 + \int_0^1 f(s)ds + \int_0^t g(s)ds + \int_0^t k(s)dB(s). \tag{3}$$

By using properties of the TFs, we can write

$$\begin{cases} f(s) \approx F^T.T(s), \\ g(s) \approx G^T.T(s), \\ k(s) \approx K^T.T(s), \end{cases} \tag{4}$$

where

$$\begin{cases} F^T = (f_i)_{2m\times1} = \big(f(0), f(h), \dots, f((m-1)h),\\ f(h), f(2h), \dots, f(mh)\big)_{2m\times1},\\ \\ G^T = (g_i)_{2m\times1} = \big(g(0), g(h), \dots, g((m-1)h),\\ g(h), g(2h), \dots, g(mh)\big)_{2m\times1},\\ \\ K^T = (k_i)_{2m\times1} = \big(k(0), k(h), \dots, k((m-1)h),\\ k(h), k(2h), \dots, k(mh)\big)_{2m\times1}. \end{cases}$$

With substituting (4) in Eq. (3), we get

$$X(t) \approx X_0 + \int_0^1 F^T.T(s)ds + \int_0^t G^T.T(s)ds + \int_0^t K^T.T(s)dB(s), \quad (5)$$

or

$$X(t) \approx X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t), \quad (6)$$

by substituting Eq. (6) in Eq. (2), we obtain

$$\begin{cases} f(t) \approx \alpha(t, X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t)),\\ \\ g(t) \approx \beta(t, X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t)),\\ \\ k(t) \approx \gamma(t, X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t)). \end{cases} \quad (7)$$

Now, with replacing $\approx$ by $=$, the relation (7) is approximated via the collocation method in $m+1$ nodes $t_j = \frac{j}{\frac{1}{T}m+1}$ $(j = 0, 1, \dots, m)$, as follows:

$$\begin{cases} f(t_j) = \alpha(t_j, X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j)),\\ \\ g(t_j) = \beta(t_j, X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j)),\\ \\ k(t_j) = \gamma(t_j, X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j)), \end{cases} \quad (8)$$

or

$$\begin{cases} F^T T(t_j) = \alpha(t_j, X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j)),\\ \\ G^T T(t_j) = \beta(t_j, X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j)),\\ \\ K^T T(t_j) = \gamma(t_j, X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j)), \end{cases} \quad (9)$$

where be the nonlinear system of $3m+3$ equations and $3m+3$ unknowns. After solving Eq. (9), we conclude that

$$X(t) \approx X_m(t) = X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t). \quad (10)$$

IV. Error analysis

Theorem IV.1. *Let $f(t)$ be a continuous function, twice differentiable and $|f''(t)| < M$ on $(0,1)$. Also, let $\hat{f}(t)$ be the TFs and $e(t) = f(t) - \hat{f}(t)$, then*

$$|e(t)|^2 \le O(h^2), \quad t \in (0,1). \quad (11)$$

Proof. See [13].

Let

$$\begin{cases} f^m(t) = \alpha(t, X_m(t)),\\ g^m(t) = \beta(t, X_m(t)),\\ k^m(t) = \gamma(t, X_m(t)), \end{cases} \quad (12)$$

and

$$\begin{cases} \hat{f}(t) = \hat{\alpha}(t, X_m(t)),\\ \hat{g}(t) = \hat{\beta}(t, X_m(t)),\\ \hat{k}(t) = \hat{\gamma}(t, X_m(t)), \end{cases} \quad (13)$$

where $\hat{f}(t)$, $\hat{g}(t)$ and $\hat{k}(t)$ are approximated by using the properties of the TFs. Also, let $X_m(t)$ be numerical solution of Eq. (1) defined in Eq. (10).

Theorem IV.2. *Let $X_m(t)$ be the approximation solution of Eq. (1) defined in Eq. (10) and conditions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $E \mid X_0 \mid^2 < \infty$ hold. Then,*

$$\| X(t) - X_m(t) \|^2 \le O(h^2), \qquad t \in (0,1), \quad (14)$$

where $\| X \|^2 = E[X^2]$.

Proof

$$X(t) - X_m(t) = \int_0^1 (f(s) - \hat{f}(s))ds + \int_0^t (g(s) - \hat{g}(s))ds + \int_0^t (k(s) - \hat{k}(s))dB(s), \quad (15)$$

via $\big(x+y+z\big)^2 \le 3\big(x^2+y^2+z^2\big)$ and the property of the Itô isometry for the SBM, we can write

$$\| X(t) - X_m(t) \|^2 \le 3\big(\| \int_0^1 (f(s) - \hat{f}(s))ds \|^2 + \| \int_0^t (g(s) - \hat{g}(s))ds \|^2 + \| \int_0^t (k(s) - \hat{k}(s))dB(s) \|^2 \big) \le 3\big(\int_0^1 \| f(s) - \hat{f}(s) \|^2 ds + \int_0^t \| g(s) - \hat{g}(s) \|^2 ds + \| \int_0^t (k(s) - \hat{k}(s))ds \|^2 \big) \le 3[\int_0^1 \| f(s) - \hat{f}(s) \|^2 ds + \int_0^t \| g(s) - \hat{g}(s) \|^2 ds + \int_0^t \| k(s) - \hat{k}(s) \|^2 ds] \le 3\big(2\int_0^1 \| f(s) - f^m(s) \|^2 ds + 2\int_0^1 \| f^m(s) - \hat{f}(s) \|^2 ds + 2\int_0^t \| g(s) - g^m(s) \|^2 ds + 2\int_0^t \| g^m(s) - \hat{g}(s) \|^2 ds 2\int_0^t \| k(s) - k^m(s) \|^2 ds + 2\int_0^t \| k^m(s) - \hat{k}(s) \|^2 ds\big) \le 6\big(\int_0^1 \| f(s) - f^m(s) \|^2 ds + \int_0^1 \| f^m(s) - \hat{f}(s) \|^2 ds + \int_0^t \| g(s) - g^m(s) \|^2 ds + \int_0^t \| g^m(s) - \hat{g}(s) \|^2 ds + \int_0^t \| k(s) - k^m(s) \|^2 ds + \int_0^t \| k^m(s) - \hat{k}(s) \|^2 ds\big). \quad (16)$$

By using Theorem ($IV.1$), we have

$$\begin{cases} \| f^m(s) - \hat{f}(s) \|^2 \leq l_1 h^2, \\ \| g^m(s) - \hat{g}(s) \|^2 \leq l_2 h^2, \\ \| k^m(s) - \hat{k}(s) \|^2 \leq l_3 h^2. \end{cases} \quad (17)$$

Also, by using conditions $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$, we have

$$\begin{cases} \int_0^1 \| f(s) - f^m(s) \|^2 \, ds \leq k_1 \int_0^1 \| X(s) - X_m(s) \|^2 \, ds, \\ \\ \int_0^t \| g(s) - g^m(s) \|^2 \, ds \leq k_3 \int_0^t \| X(s) - X_m(s) \|^2 \, ds, \\ \\ \int_0^t \| k(s) - k^m(s) \|^2 \, ds \leq k_5 \int_0^t \| X(s) - X_m(s) \|^2 \, ds, \end{cases}$$

by the mean value theorem

$$\begin{cases} \int_0^1 \| f(s) - f^m(s) \|^2 \, ds \leq k_1 \int_0^1 \| X(s) - X_m(s) \|^2 \, ds \leq k_1 \| X(t) - X_m(t) \|^2, \\ \\ \int_0^t \| g(s) - g^m(s) \|^2 \, ds \leq k_3 \int_0^t \| X(s) - X_m(s) \|^2 \, ds, \\ \\ \int_0^t \| k(s) - k^m(s) \|^2 \, ds \leq k_5 \int_0^t \| X(s) - X_m(s) \|^2 \, ds. \end{cases} \quad (18)$$

With substituting (17) and (18) in (16), we obtain

$$\| X(t) - X_m(t) \|^2 \leq 6\big(l_1 h^2 + k_1 \| X(t) - X_m(t) \|^2 + l_2 h^2 + k_3 \int_0^t \| X(s) - X_m(s) \|^2 \, ds + l_3 h^2 + k_5 \int_0^t \| X(s) - X_m(s) \|^2 \, ds\big), \quad (19)$$

or

$$\eta(t) \leq \mu + \lambda \int_0^t \eta(s) ds,$$

where $\mu = \frac{6l_1h^2 + 6l_2h^2 + 6l_3h^2}{1-6k_1}$, $\lambda = \frac{6k_3 + 6k_5}{1-6k_1}$ and $\eta(s) = \| X(s) - X_m(s) \|^2$. Furthermore, from Gronwall inequality, we get

$$\eta(t) \leq \mu\big(1 + \lambda \int_0^t \exp\big(\lambda(t-s)\big) ds\big), \quad t \in (0,1),$$

so

$$\| X(t) - X_m(t) \|^2 \leq O(h^2). \qquad \square$$

V. Numerical examples

Example 1. *Consider the SNVFIE as follows:*

$$X(t) = \frac{-1}{3} + \int_0^1 s^4 (X(s))^2 ds + \int_0^t s^2 (X(s))^2 ds + \frac{-1}{300} \int_0^t s^2 X(s) dB(s). \quad (20)$$

Let $m = 16$, $T = 0.25$ *and*

$$\begin{cases} f(s) = s^4 (X(s))^2, \\ g(s) = s^2 (X(s))^2, \\ k(s) = \frac{-s^2 X(s)}{300}, \end{cases} \quad (21)$$

with substituting (21) in Eq. (20), we get

$$X(t) = X_0 + \int_0^1 f(s) ds + \int_0^t g(s) ds + \int_0^t k(s) dB(s). \quad (22)$$

By using properties of the TFs and the presented method in the section 3, we can write

$$X(t) \approx X_0 + \int_0^1 F^T.T(s) ds + \int_0^t G^T.T(s) ds + \int_0^t K^T.T(s) dB(s), \quad (23)$$

or

$$X(t) \approx X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t), \quad (24)$$

by substituting Eq. (24) in Eq. (21), we obtain

$$\begin{cases} f(t) \approx t^4 (X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t))^2, \\ \\ g(t) \approx t^2 (X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t))^2, \\ \\ k(t) \approx \frac{-t^2 (X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t))}{300}. \end{cases} \quad (25)$$

Now, with replacing $\approx$ *by* $=$, *the relation (25) is approximated via the collocation method in* 17 *nodes* $t_j = \frac{j}{65}$ $(j = 0, 1, \ldots, 16)$, *as follows:*

$$\begin{cases} F^T T(t_j) = t_j^4 (X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j))^2, \\ \\ G^T T(t_j) = t_j^2 (X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j))^2, \\ \\ K^T T(t_j) = \frac{-t_j^2 (X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j))}{300}, \end{cases}$$

where be the nonlinear system of 51 *equations and* 51 *unknowns.*
Results have been shown in Figures (1-3) via a comparison between numerical solution of stochastic model and numerical solution of deterministic. Also, numerical solution of deterministic has been approximated via properties of the TFs and the BPFs in Figures (2-3).

Example 2. *Consider the SNVFIE as follows:*

$$\begin{cases} X(t) = X_0 + \int_0^1 sX(s) - \int_0^t sX(s) ds - \frac{1}{5000} \int_0^t s^4 X(s) dB(s), \\ \\ X_0 = -t^2 - 1 + \frac{1}{4} + \frac{1}{4} t^4 - \frac{1}{2} t^2, \end{cases} \quad (26)$$

Let $m = 16$, $T = 0.25$ *and*

$$\begin{cases} f(s) = sX(s), \\ g(s) = -sX(s), \\ k(s) = \frac{-s^4 X(s)}{5000}, \end{cases} \quad (27)$$

with substituting (27) in Eq. (26), we get

$$X(t) = X_0 + \int_0^1 f(s) ds + \int_0^t g(s) ds + \int_0^t k(s) dB(s). \quad (28)$$

By using properties of the TFs and the presented method in the section 3, we can write

$$X(t) \approx X_0 + \int_0^1 F^T.T(s) ds + \int_0^t G^T.T(s) ds + \int_0^t K^T.T(s) dB(s), \quad (29)$$

or

$$X(t) \approx X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t), \quad (30)$$

by substituting Eq. (30) in Eq. (27), we obtain

$$\begin{cases} f(t) \approx t(X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t)), \\ g(t) \approx -t(X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t)), \\ k(t) \approx \frac{-t^4(X_0 + F^T P_T T(t) + G^T P_T T(t) + K^T P_s T(t))}{5000}. \end{cases} \quad (31)$$

Now, with replacing $\approx$ by $=$, the relation (31) is approximated via the collocation method in 17 *nodes* $t_j = \frac{j}{65}$ $(j = 0, 1, \ldots, 16)$*, as follows:*

$$\begin{cases} F^T T(t_j) = t_j(X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j)), \\ G^T T(t_j) = -t_j(X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j)), \\ K^T T(t_j) = \frac{-t_j^4(X_0 + F^T P_T T(t_j) + G^T P_T T(t_j) + K^T P_s T(t_j))}{5000}, \end{cases} \quad (32)$$

where be the nonlinear system of 51 *equations and* 51 *unknowns.*

Results have been shown in Figures (4-5) via a comparison between numerical solution of stochastic model and numerical solution of deterministic. Also, numerical solution of deterministic has been approximated via properties of the TFs.

VI. Conclusion

The purpose of this paper is to present a numerical method for solving the SNVFIE driven by the SBM and comparison between numerical solution of deterministic and numerical solution of stochastic model. The advantages of this method are simple calculations, conversion of the SNVFIE to the stochastic nonlinear system and convergence faster than the other methods. Also, the numerical results demonstrate accuracy of presented method.

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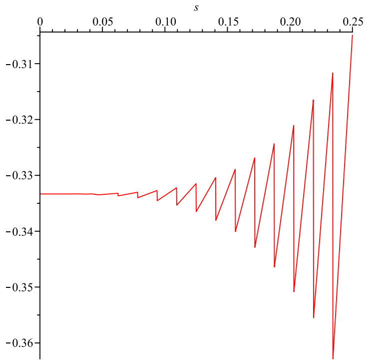

Fig.1 results for stochastic model.

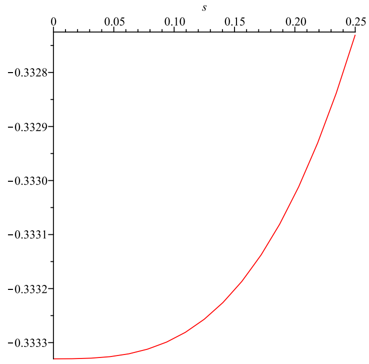

Fig.2 results for deterministic model (TFs).

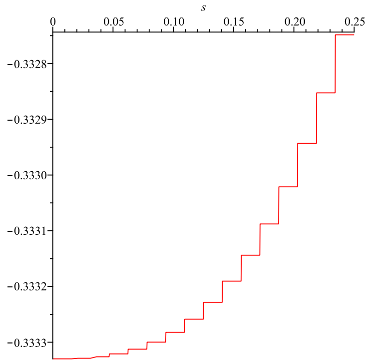

Fig.3 results for deterministic model (BPFs).

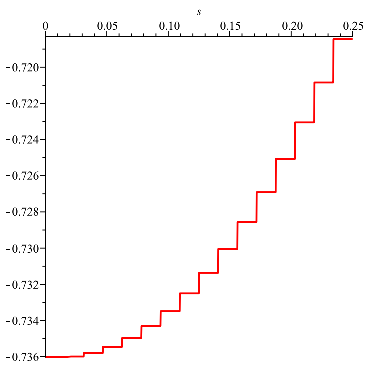

Fig.4 results for stochastic model.

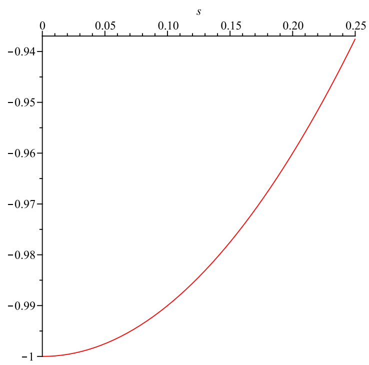

Fig.5 results for deterministic model (TFs).