Analysis of Stochastic Gilpin-Ayala Model in Polluted Environments

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***Abstract*—In this paper, we consider a stochastic Gilpin-Ayala model in polluted environments. Firstly, sufficient conditions for extinction, non-persistence in the mean, weak persistence and stochastic permanence of the species are established. The threshold between extinction and weak persistence is obtained. Then global attractivity of the model is studied. Finally, several numerical figures are introduced to validate the results.**

***Index Terms*—environmental pollution, stochastic noises, permanence, extinction.**

I. Introduction

WITH the rapid development of industries and agriculture, many toxins are emitted into the environment. These toxins have let lots of species go to extinction and let many be on the verge of extinction. This motivates scholars to investigate the effects of toxins on populations and to establish theoretical persistence-extinction thresholds of the species.

In recent years, many authors have investigated the effects of toxins on species by using mathematical models. Hallam and his colleagues did pioneering work in [1], [2], [3], where the authors proposed some deterministic population models with toxin effect and established the theoretical persistence-extinction thresholds for their models. From then on, many interesting and important population models with toxin effect were proposed and analyzed. The authors in [4], [5], [6], [7], [8], [9], [10], [11] considered single-species population models in a polluted environment; The authors in [12], [13], [14] investigated the effected of toxins on the persistence and extinction of multi-species models; The studies [15], [16] analyzed the population models with impulsive toxicant input; Stage-structured population models in a polluted environment were studied by [17], [18].

However, the growth of species in the natural world is inevitably affected by environmental noises (May [19]). Therefore several authors considered stochastic population models in a polluted environment, see e.g. [20]-[26]. Especially, Liu and Wang [21] have investigated the following stochastic single-species model with toxin effect:

$$\begin{cases} dx(t) = x(t)[r_0 - r_1C_0(t) - ax(t)]dt + \sigma_1 x(t)dB_1(t) \\ dC_0(t) = [kC_e(t) - (g+m)C_0(t)]dt \\ dC_e(t) = [-hC_e(t) + u(t)]dt \end{cases} \quad (1)$$

where $x(t)$ is the size of the population; $r_0 > 0$ stands for the intrinsic growth rate of the population without toxicant; $r_1 > 0$ denotes the population response to the pollutant present in the organism; $C_0(t)$ and $C_e(t)$ represent the concentration of toxicant in the organism and in the environment, respectively; $B_1(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$; σ_1^2 is the intensity of the white noise; $k > 0$ stands for the organism's net uptake rate of toxicant from the environment; $g > 0$ and $m > 0$ represent the egestion and depuration rates of the toxicant in the organism, respectively; $h > 0$ denotes the toxicant loss rate from the environment by volatilization and so on; $u(t)$ is a non-negative bounded continuous function defined on $[0, +\infty)$ representing the exogenous rate of input of toxicant into the environment. Liu and Wang [21] have obtained the persistence-extinction threshold and established the sufficient conditions for stochastic permanence of the species.

Based on the study [21], some interesting and important questions arise naturally:

(Q1) Model (1) is based on the classical Logistic equation. Gilpin and Ayala [27] have pointed out that Logistic equation has some limitations to describe the reality in some cases and have proposed Gilpin-Ayala model. Then what happens if the model in [21] is replaced by Gilpin-Ayala model?

(Q2) Model (1) assumes that only the growth rate r_0 is affected by random noise. Then what happens if all parameters are affected by random noise? In fact, May [19] have pointed out that due to environmental noise, the growth rate, competition rate and other parameters in population models should be stochastic.

(Q3) The conditions of stochastic permanence in [21] are too restricted, can we improve them?

(Q4) In the study of population models, global attractivity of the solution is one of the most important topics. Then, is the solution of the underlying model globally attractive?

The aims of this paper are to study these problems. Suppose that r_1 and a in model (1) are also affected by random noises, and then by Gilpin-Ayala model, we obtain the following

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stochastic single-species model with toxicant effect:

$$\begin{cases} dx(t) = x(t)[r_0 - r_1 C_0(t) - a x^{\theta}(t)]dt + \sigma_1 x(t) dB_1(t) \\ \qquad + \sigma_2 C_0(t) x(t) dB_2(t) + \sigma_3 x^{1+\theta}(t) dB_3(t), \\ dC_0(t) = [kC_e(t) - (g+m)C_0(t)]dt, \\ dC_e(t) = [-hC_e(t) + u(t)]dt, \end{cases} \tag{2}$$

where $\theta > 0$ is a constant, $B_1(t)$, $B_2(t)$ and $B_3(t)$ are independent standard Brownian motions defined on $(\Omega, \mathcal{F}, \mathcal{P})$. It is easy to see that if $\theta = 1$ and $\sigma_2 = \sigma_3 = 0$, then model (2) becomes model (1).

In Section 2, we carry out the survival analysis for model (2). Sufficient conditions for extinction, non-persistence in the mean, weak persistence and stochastic permanence of the species are established. The threshold between extinction and weak persistence is obtained. The results in [21] are improved and extended. In Section 3, we show that model (2) is globally attractive. In Section 4, we extend these results to a $n-$species model. In Section 5, some numerical simulations are given to illustrate the main results. In the last section, we give conclusions.

II. Persistence and Extinction

Both $C_0(t)$ and $C_e(t)$ in model (2) are concentrations, so we should give some conditions under which $0 \le C_0(t) < 1, \ 0 \le C_e(t) < 1$. In fact, we have

Lemma 1. *([21]) If $0 < k \le g+m, \ \limsup_{t\to+\infty} u(t) \le h$, then $0 \le C_0(t) < 1, \ 0 \le C_e(t) < 1$ for all $t \ge 0$.*

From now on, we always suppose that $0 < k \le g + m, \ \limsup_{t\to+\infty} u(t) \le h$. Note that the last two equations in model (2) are linear with respect to $C_0(t)$ and $C_e(t)$, it is easy to obtain their explicit solutions. So in the following study, we need only to consider the first equation in model (2), that is

$$\begin{aligned} dx(t) &= x(t)[r_0 - r_1 C_0(t) - a x^{\theta}(t)]dt + \sigma_1 x(t) dB_1(t) \\ &\quad + \sigma_2 C_0(t) x(t) dB_2(t) + \sigma_3 x^{1+\theta}(t) dB_3(t). \end{aligned} \tag{3}$$

Note that $x(t)$ in Eq.(3) represents the population size, then $x(t)$ should be nonnegative. So first of all, we must show that for any given positive initial value, Eq. (3) has a unique global positive solution.

Lemma 2. *For any initial data $x(0) = x_0 > 0$, Eq. (3) has a unique global positive solution $x(t)$ almost surely (a.s.).*

Proof: The proof is similar to that of Theorem 4.1 in [28] and hence is omitted. ■

Before we state and prove our main results, we recall some important definitions.

Definition 1. *(i) $x(t)$ is said to go to extinction if $\lim_{t\to+\infty} x(t) = 0$.*
(ii) $x(t)$ is said to be non-persistent in the mean if there is a positive constant β such that

$$\lim_{t\to+\infty} t^{-1} \int_0^t x^{\beta}(s) ds = 0.$$

(iii) $x(t)$ is said to be weakly persistent ([4]) if $\limsup_{t\to+\infty} x(t) > 0$.
(iv) $x(t)$ is said to be stochastically permanent ([29]), if for every $0 < \varepsilon < 1$, there are positive constants β and M such that $\liminf_{t\to+\infty} P\{x(t) \ge \beta\} \ge 1 - \varepsilon$ and $\liminf_{t\to+\infty} P\{x(t) \le M\} \ge 1 - \varepsilon$.

Theorem 1. *If $\limsup_{t\to+\infty} t^{-1} \int_0^t b(s) ds < 0$, then $x(t)$ goes to extinction a.s., where*

$$b(t) = r_0 - 0.5\sigma_1^2 - r_1 C_0(t) - 0.5\sigma_2^2 C_0^2(t).$$

Proof: By Itô's formula

$$\begin{aligned} \ln[x(t)/x_0] &= \int_0^t \left[b(s) - a x^{\theta}(s) - 0.5\sigma_3^2(s) x^{2\theta}(s) \right] ds \\ &\quad + \sigma_1 B_1(t) + M_2(t) + M_3(t), \end{aligned} \tag{4}$$

where

$$M_2(t) = \int_0^t \sigma_2 C_0(s) dB_2(s), M_3(t) = \int_0^t \sigma_3 x^{\theta}(s) dB_3(s).$$

The quadratic variation of $M_2(t)$ is

$$\langle M_2(t), M_2(t) \rangle = \int_0^t \sigma_2^2 C_0^2(s) ds \le \sigma_2^2 t.$$

It then follows from the strong law of large numbers for martingales(see e.g., [30], P.16) that

$$\lim_{t\to+\infty} M_2(t)/t = 0, \quad a.s. \tag{5}$$

The quadratic variation of $M_3(t)$ is

$$\langle M_3, M_3 \rangle = \int_0^t \sigma_3^2 x^{2\theta}(s) ds.$$

In view of the exponential martingale inequality,

$$\mathcal{P}\left\{ \sup_{0\le t\le k} \left[M_3(t) - \frac{1}{2}\langle M_3, M_3 \rangle \right] > 2\ln k \right\} \le 1/k^2.$$

An application of the Borel-Cantelli lemma (see e.g. [30], P.10), for almost all $\omega \in \Omega$, there is a stochastic integer $k_0 = k_0(\omega)$ such that for $k \ge k_0$,

$$\sup_{0\le t\le k} \left[M_3(t) - \frac{1}{2}\langle M_3, M_3 \rangle \right] \le 2\ln k.$$

In other words, $M_3(t) \le 2\ln k + 0.5 \int_0^t \sigma_3^2 x^{2\theta}(s) ds$ for all $0 \le t \le k, \ k \ge k_0$ almost surely. Substituting this inequality into (4) gives

$$\begin{aligned} &\ln x(t) - \ln x_0 \\ &\le \int_0^t (b(s) - a x^{\theta}(s)) ds + \sigma_1 B_1(t) + M_2(t) + 2\ln k \\ &\le \int_0^t b(s) ds + \sigma_1 B_1(t) + M_2(t) + 2\ln k \end{aligned} \tag{6}$$

for all $0 \le t \le k, \ k \ge k_0$ almost surely. Hence for $0 < k-1 \le t \le k, \quad k \ge k_0$, we have

$$\begin{aligned} t^{-1}\{\ln x(t) - \ln x_0\} &\le t^{-1} \textstyle\int_0^t b(s) ds \\ &\quad + \sigma_1 B_1(t)/t + M_2(t)/t + 2(k-1)^{-1} \ln k. \end{aligned}$$

Then by (19) and $\lim_{t\to+\infty} B_1(t)/t = 0$, we get $\limsup_{t\to+\infty} t^{-1} \ln x(t) \le \limsup_{t\to+\infty} t^{-1} \int_0^t b(s) ds < 0$. Therefore $\lim_{t\to+\infty} x(t) = 0$. ■

Theorem 2. *If* $\limsup_{t\to+\infty} t^{-1}\int_0^t b(s)ds = 0$, *then* $x(t)$ *is non-persistent in the mean a.s.*

Proof: It is easy to see that for arbitrarily given $\varepsilon > 0$, we can find a positive constant T_1 such that

$$t^{-1}\int_0^t b(s)ds < \limsup_{t\to+\infty} t^{-1}\int_0^t b(s)ds + \varepsilon/2 = \varepsilon/4$$

for $t > T_1$. Substituting this inequality into (6) gives

$$\begin{aligned}&\ln x(t) - \ln x_0\\&< \varepsilon t/4 - a\int_0^t x^\theta(s)ds + \sigma_1 B_1(t) + M_2(t) + 2\ln k\end{aligned}$$

for all $T_1 \le t \le k,\ k \ge k_0$. Let t be sufficiently large such that $T_1 \le T \le k-1 \le t \le k,\ k \ge k_0$ and

$$\sigma_1 B_1(t)/t \le \varepsilon/4,\ (\ln k)/t \le \varepsilon/8,\ M_2(t)/t \le \varepsilon/4.$$

Consequently for $T \le k-1 \le t \le k$ and $k \ge k_0$,

$$\ln x(t) - \ln x_0 \le \varepsilon t - a\int_0^t x^\theta(s)ds.$$

Denote $\lambda(t) = \int_0^t x^\theta(s)ds$. Therefore,

$$\theta^{-1}\ln(d\lambda/dt) < \varepsilon t - a\lambda(t) + \ln\ x_0.$$

Hence, $e^{\theta a\lambda(t)}(d\lambda/dt) < x_0^\theta e^{\theta\varepsilon t}$. In other words, we have shown that

$$e^{\theta a\lambda(t)} < e^{\theta a\lambda(T)} + x_0^\theta a\varepsilon^{-1}e^{\theta\varepsilon t} - x_0^\theta a\varepsilon^{-1}e^{\theta\varepsilon T}.$$

Taking the logarithm yields

$$\lambda(t) < (\theta a)^{-1}\ln\left\{x_0^\theta a\varepsilon^{-1}e^{\theta\varepsilon t} + e^{\theta a\lambda(T)} - x_0^\theta a\varepsilon^{-1}e^{\theta\varepsilon T}\right\}.$$

Therefore,

$$\begin{aligned}&\limsup_{t\to+\infty}\left\{t^{-1}\int_0^t x^\theta(s)ds\right\} \le \limsup_{t\to+\infty}\theta^{-1}a^{-1}\\&\times\left\{t^{-1}\ln\left\{x_0^\theta a\varepsilon^{-1}e^{\theta\varepsilon t} + e^{\theta a\lambda(T)} - x_0^\theta a\varepsilon^{-1}e^{\theta\varepsilon T}\right\}\right\}.\end{aligned}$$

It then follows from the L'Hospital rule that

$$\limsup_{t\to+\infty}\left\{t^{-1}\int_0^t x^\theta(s)ds\right\} \le \varepsilon/a.$$

By the arbitrariness of ε that $\limsup_{t\to+\infty} t^{-1}\int_0^t x^\theta(s)ds \le 0$. Note that $x(t) \ge 0$, then $\lim_{t\to+\infty} t^{-1}\int_0^t x^\theta(s)ds = 0$. ■

Theorem 3. *If* $\limsup_{t\to+\infty} t^{-1}\int_0^t b(s)ds > 0$, *then* $x(t)$ *is weakly persistent a.s.*

Proof: : To begin with, let us prove

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{\ln t} \le 1, \quad a.s. \tag{7}$$

In fact, by Itô's formula,

$$\begin{aligned}&d(e^t\ln x) = e^t\left[\ln x + b(t) - ax^\theta - 0.5\sigma_3^2x^{2\theta}\right]dt\\&+e^t\sigma_1 dB_1(t) + e^t\sigma_2 C_0(t)dB_2(t) + e^t\sigma_3x^\theta dB_3(t).\end{aligned}$$

Consequently,

$$\begin{aligned}&e^t\ln x(t) - \ln x_0\\&= \int_0^t e^s\left[\ln x(s) + b(s) - ax^\theta(s) - 0.5\sigma_3^2x^{2\theta}(s)\right]ds\\&+N_1(t) + N_2(t) + N_3(t),\end{aligned} \tag{8}$$

where

$$N_1(t) = \int_0^t e^s\sigma_1 dB_1(s),\ N_2(t) = \int_0^t e^s\sigma_2 C_0(s)dB_2(s),$$

$$N_3(t) = \int_0^t e^s\sigma_3x^\theta(s)dB_3(s).$$

Let $N(t) = N_1(t)+N_2(t)+N_3(t)$. Note that $N(t)$ is a local martingale, whose quadratic variation is

$$\langle N, N\rangle = \int_0^t e^{2s}[\sigma_1^2 + \sigma_2^2C_0^2(s) + \sigma_3^2x^{2\theta}(s)]ds.$$

By the exponential martingale inequality,

$$\mathcal{P}\left\{\sup_{0\le t\le\mu k}\left[N(t) - 0.5e^{-\mu k}\langle N, N\rangle\right] > \rho e^{\mu k}\ln k\right\} \le k^{-\rho},$$

where $\rho > 1$ and $\mu > 0$ is arbitrary. By virtue of the Borel-Cantelli lemma, for almost all $\omega \in \Omega$, there is a $k_0(\omega)$ such that for every $k \ge k_0(\omega)$,

$$N(t) \le 0.5e^{-\mu k}\langle N(t), N(t)\rangle + \rho e^{\mu k}\ln k, \quad 0 \le t \le \mu k.$$

When this inequality is used in (8), one can see that

$$\begin{aligned}&e^t\ln x(t) - \ln x_0\\&\le \int_0^t e^s\left[\ln x(s) + b(s) - ax^\theta(s) - 0.5\sigma_3^2x^{2\theta}(s)\right]ds\\&+0.5e^{-\mu k}\int_0^t e^{2s}\sigma_1^2ds + 0.5e^{-\mu k}\int_0^t e^{2s}\sigma_2^2C_0^2(s)ds\\&+0.5e^{-\mu k}\int_0^t e^{2s}\sigma_3^2x^{2\theta}(s)ds + \rho e^{\mu k}\ln k\\&= \int_0^t e^s\left[\ln x(s) + 0.5e^{s-\mu k}\sigma_1^2 + 0.5e^{s-\mu k}\sigma_2^2C_0^2(s)\right.\\&\left.+b(s) - ax^\theta - 0.5\sigma_3^2x^{2\theta}[1 - e^{s-\mu k}]\right]ds + \rho e^{\mu k}\ln k.\end{aligned}$$

Note that for arbitrary $0 \le t \le \mu k$ and $x > 0$, there is a constant C independent of k such that

$$\begin{aligned}&\ln x + b(t) + 0.5e^{t-\mu k}\sigma_1^2 + 0.5e^{t-\mu k}\sigma_2^2C_0^2(t)\\&- ax^\theta - 0.5\sigma_3^2x^{2\theta}[1 - e^{t-\mu k}] \le C.\end{aligned}$$

That is to say, for arbitrary $0 \le t \le \mu k$, we have

$$e^t\ln x(t) - \ln x_0 \le C[e^t - 1] + \rho e^{\mu k}\ln k.$$

Thus if $\mu(k-1) \le t \le \mu k$ and $k \ge k_0(\omega)$, then

$$\begin{aligned}\ln x(t)/\ln t &\le e^{-t}\ln x_0/\ln t + C[1 - e^{-t}]/\ln t\\&+\rho e^{-\mu(k-1)}e^{\mu k}\ln k/\ln t.\end{aligned}$$

Letting $k \to +\infty$ yields $\limsup_{t\to+\infty}\frac{\ln x(t)}{\ln t} \le \rho e^\mu$. Letting $\rho \to 1$ and $\mu \to 0$ gives the required assertion (21).

Now suppose that $\limsup_{t\to+\infty} b(t) > 0$, we prove $\limsup_{t\to+\infty} x(t) > 0$ almost surely. If it is false, set $F = \{\limsup_{t\to+\infty} x(t) = 0\}$, and suppose that $\mathcal{P}(F) > 0$. By (4),

$$\begin{aligned}&t^{-1}\ln(x(t)/x_0) = t^{-1}\int_0^t b(s)ds - t^{-1}\int_0^t ax^\theta(s)ds\\&-0.5t^{-1}\int_0^t \sigma_3^2x^{2\theta}(s)ds + (\sigma_1B_1(t) + M_2 + M_3)/t.\end{aligned} \tag{9}$$

For arbitrary $\omega \in F$, we have $\lim_{t\to+\infty} x(t,\omega) = 0$, it then follows from the law of large numbers for local martingales that $\lim_{t\to+\infty} M_3(t)/t = 0$. When this identity and (5) are used in (9), one can observe that $\limsup_{t\to+\infty}[t^{-1}\ln x(t,\omega)] = \limsup_{t\to+\infty} t^{-1}\int_0^t b(s)ds > 0$. Hence $\mathcal{P}\{\limsup_{t\to+\infty}[t^{-1}\ln x(t)] > 0\} > 0$, which is a contradiction with (7). ∎

Remark 1. From Theorems 1, 2 and 3, one can observe that $\limsup_{t\to+\infty} t^{-1}\int_0^t b(s)ds$ is the threshold between weak persistence and extinction of the species.

In the study of population system, permanence is one of the most important topics. So let us now consider the permanence of $x(t)$.

Theorem 4. *If* $\liminf_{t\to+\infty} b(t) > 0$ *and* $0 < \theta \le 1$, *then* $x(t)$ *is stochastically permanent.*

Proof: Define $V_1(x) = 1/x^{1+\theta}$, where $x > 0$. By Itô's formula,

$$\begin{aligned}
&dV_1(x(t)) = (1+\theta)V_1(x)\Big[-r_0 + r_1C_0(t) + ax^\theta\Big]dt\\
&+0.5(1+\theta)(2+\theta)\sigma_1^2V_1(x)dt\\
&+0.5(1+\theta)(2+\theta)\sigma_2^2C_0^2(t)V_1(x)dt\\
&+0.5(1+\theta)(2+\theta)\sigma_3^2x^{\theta-1}dt\\
&-(1+\theta)\sigma_1V_1(x)dB_1(t) - (1+\theta)\sigma_2V_1(x)C_0(t)dB_2(t)\\
&-(1+\theta)\sigma_3x^{-1}dB_3(t).
\end{aligned}$$

Since $\liminf_{t\to+\infty} b(t) > 0$, we can let $0 < \kappa < 1$ be sufficiently small such that

$$\liminf_{t\to+\infty} b(t) > 0.5\kappa(\theta+1)(\sigma_1^2+\sigma_2^2). \tag{10}$$

Define $V_2(x) = (1+V_1(x))^\kappa$. By Itô's formula, for sufficiently large t,

$$\begin{aligned}
&dV_2(x) = \kappa(1+V_1(x))^{\kappa-2}\times\\
&\bigg\{(1+V_1(x))(1+\theta)\bigg[V_1(x)\Big(-r_0 + r_1C_0(t) + ax^\theta\Big)\\
&+0.5(2+\theta)\sigma_1^2V_1(x) + 0.5(2+\theta)\sigma_2^2C_0^2(t)V_1(x)\\
&+0.5(2+\theta)\sigma_3^2x^{\theta-1}\bigg] + 0.5(\kappa-1)(1+\theta)^2\\
&\times\Big\{\sigma_1^2V_1^2(x) + \sigma_2^2C_0^2(t)V_1^2(x) + \sigma_3^2x^{-2}\Big\}\bigg\}dt\\
&-\kappa(1+V_1(x))^{\kappa-1}(1+\theta)\Big[\sigma_1V_1(x)dB_1(t)\\
&+\sigma_2C_0(t)V_1(x)dB_2(t) + \sigma_3x^{-1}dB_3(t)\Big]\\
&\le \kappa(1+\theta)(1+V_1(x))^{\kappa-2}\times\\
&\bigg\{-\Big[\liminf_{t\to+\infty} b(t) - \varepsilon - 0.5\kappa(\theta+1)(\sigma_1^2+\sigma_2^2)\Big]V_1^2(x)\\
&-\Big[r_0 - r_1 - 0.5(2+\theta)(\sigma_1^2+\sigma_2^2)\Big]V_1(x)\\
&+a(V_1(x)+1)x^{-1} + (1+0.5\theta)\sigma_3^2(x^{\theta-1}+x^{-2})\bigg\}dt\\
&-\kappa(1+V_1(x))^{\kappa-1}(1+\theta)\Big[\sigma_1V_1(x)dB_1(t)\\
&+\sigma_2C_0(t)V_1(x)dB_2(t) + \sigma_3x^{-1}dB_3(t)\Big],
\end{aligned}$$

where ε is sufficiently small satisfying

$$\liminf_{t\to+\infty} b(t) - 0.5\kappa(\theta+1)(\sigma_1^2+\sigma_2^2) > \varepsilon.$$

In the proof of the last inequality, we have used the facts that $C_0(t) \le 1$ and $\kappa < 1$. Let $\eta > 0$ be sufficiently small such that

$$0 < \frac{\eta}{\kappa(1+\theta)} < \liminf_{t\to+\infty} b(t) - 0.5\kappa(\theta+1)(\sigma_1^2+\sigma_2^2) - \varepsilon.$$

Define $V_3(x) = e^{\eta t}V_2(x)$. By Itô's formula, for sufficiently large t,

$$\begin{aligned}
&dV_3(x(t)) = \eta e^{\eta t}V_2(x)dt + e^{\eta t}dV_2(x)\\
&\le (1+\theta)\kappa e^{\eta t}(1+V_1(x))^{\kappa-2}\bigg\{\frac{\eta(1+V_1(x))^2}{\kappa(1+\theta)}\\
&-\Big[\liminf_{t\to+\infty} b(t) - \varepsilon - 0.5\kappa(\theta+1)(\sigma_1^2+\sigma_2^2)\Big]V_1^2(x)\\
&-\Big[r_0 - r_1 - 0.5(2+\theta)(\sigma_1^2+\sigma_2^2)\Big]V_1(x)\\
&+a(V_1(x)+1)x^{-1} + (1+0.5\theta)\sigma_3^2(x^{\theta-1}+x^{-2})\bigg\}dt\\
&-e^{\eta t}\kappa(1+V_1(x))^{\kappa-1}(1+\theta)\Big[\sigma_1V_1(x)dB_1(t)\\
&+\sigma_2C_0(t)V_1(x)dB_2(t) + \sigma_3x^{-1}dB_3(t)\Big]\\
&= e^{\eta t}J(x)dt - e^{\eta t}\kappa(1+V_1(x))^{\kappa-1}(1+\theta)\Big[\\
&(\sigma_1dB_1(t) + \sigma_2C_0(t)dB_2(t))V_1(x) + \sigma_3x^{-1}dB_3(t)\Big]
\end{aligned}$$

where

$$\begin{aligned}
&J(x) = (1+\theta)\kappa(1+V_1(x))^{\kappa-2}\bigg\{-\Big[\liminf_{t\to+\infty} b(t)\\
&-\varepsilon - 0.5\kappa(\theta+1)(\sigma_1^2+\sigma_2^2) - \frac{\eta}{\kappa(1+\theta)}\Big]V_1^2(x)\\
&-\Big[r_0 - r_1 - 0.5(2+\theta)(\sigma_1^2+\sigma_2^2) - \frac{2\eta}{\kappa(1+\theta)}\Big]V_1(x)\\
&+aV_1(x)x^{-1} + \frac{\eta}{\kappa(1+\theta)} + ax^{-1}\\
&+(1+0.5\theta)\sigma_3^2(x^{\theta-1}+x^{-2})\bigg\}.
\end{aligned} \tag{11}$$

Now we are in the position to prove if $0 < \theta \le 1$, then $J(x)$ is upper bounded in R_+. Let

$$K = \min\left\{1,\ \left(\frac{K_1}{4a}\right)^{-\theta},\ \left(\frac{K_1}{6\sigma_3^2}\right)^{-2\theta}\right\},$$

where

$$K_1 = \liminf_{t\to+\infty} b(t) - \varepsilon - 0.5\kappa(\theta+1)(\sigma_1^2+\sigma_2^2) - \eta/[\kappa(1+\theta)].$$

(a) If $x \ge K$, then by the definition of $V_1(x)$, $J(x)$ is upper bounded, that is to say, there is a constant $J_1 > 0$ such that $\sup_{x\ge K} J(x) < J_1$.

(b) If $x < K$, then by $0 < x < 1$ and $0 < \theta \le 1$, one can obtain that

$$x^{\theta-1} = x^{2\theta}x^{-\theta-1} \le V_1(x). \tag{12}$$

On the other hand, by $x < \left(\frac{K_1}{4a}\right)^{-\theta}$, we get

$$-0.25K_1V_1^2(x) + aV_1(x)x^{-1} < 0. \tag{13}$$

In the same way, by $x < \left(\frac{K_1}{6\sigma_3^2}\right)^{-2\theta}$ one can show that

$$-0.25K_1V_1^2(x) + (1+0.5\theta)\sigma_3^2x^{-2} < 0. \tag{14}$$

When (12),(13) and (14) are used in (11), we derive that

$$J(x) \leq (1+\theta)\kappa(1+V_1(x))^{\kappa-2}\Big\{-0.5K_1V_1^2(x) + \frac{\eta}{\kappa(1+\theta)} - \Big[r_0 - r_1 - 0.5(2+\theta)(\sigma_1^2+\sigma_2^2) - \frac{2\eta}{\kappa(1+\theta)} - (1+0.5\theta)\sigma_3^2 - a\Big]V_1(x)\Big\}.$$

Hence if $x < K$, there is a positive constant J_2 such that $\sup_{x<K} J(x) < J_2$. Therefore, $J(x)$ is upper bounded in R_+, i.e., $J_3 := \sup_{x\in R_+} J(x) < +\infty$. Consequently, for sufficiently large t,

$$dV_3(x(t)) \leq J_3e^{\eta t}dt - e^{\eta t}\kappa(1+V_1(x))^{\kappa-1}(1+\theta)\Big[\sigma_1V_1(x)dB_1(t) + \sigma_2C_0(t)V_1(x)dB_2(t) + \sigma_3x^{-1}dB_3(t)\Big].$$

That is to say

$$\mathbb{E}\Big[e^{\eta t}\Big(1+V_1(x)\Big)^{\kappa}\Big] \leq \Big(1+V_1(x_0)\Big)^{\kappa} + J_3\Big(e^{\eta t}-1\Big)/\eta.$$

Hence

$$\limsup_{t\to+\infty}\mathbb{E}\Big[V_1^{\kappa}(x(t))\Big] \leq J_3/\eta. \tag{15}$$

Therefore

$$\limsup_{t\to+\infty}\mathbb{E}\Big[x^{-\kappa-\kappa\theta}(t)\Big] \leq J_3/\eta =: J_4.$$

For arbitrary $\varepsilon > 0$, let $\beta = (\varepsilon/J_4)^{-\kappa-\kappa\theta}$. It then follows from Chebyshev's inequality that

$$\mathcal{P}\Big\{x^{-\kappa-\kappa\theta}(t) > \beta^{-\kappa-\kappa\theta}\Big\} \leq \frac{\mathbb{E}[x^{-\kappa-\kappa\theta}(t)]}{\beta^{-\kappa-\kappa\theta}}.$$

In other words, $\limsup_{t\to+\infty}\mathcal{P}\{x(t) < \beta\} \leq \beta^{\kappa+\kappa\theta}J_4 = \varepsilon$. Therefore, $\liminf_{t\to+\infty}\mathcal{P}\{x(t) \geq \beta\} \geq 1-\varepsilon$.

To complete the proof, it suffices to show that for arbitrary given $\varepsilon > 0$, there exists a constant $M > 0$ such that $\liminf_{t\to+\infty}\mathcal{P}(x(t) \leq M) \geq 1-\varepsilon$. The proof is similar to that of [31] (Lemma 3.2). Define $V(x) = x^q$, where $x \in R_+$, $0 < q \leq 1$. By Itô's formula

$$\begin{aligned} d(e^tV(x)) &= e^tV(x)dt + e^tdV(x) \\ &= e^tx^q\Big\{1 + q\Big[r_0 - 0.5(1-q)(\sigma_1^2+\sigma_2^2C_0^2(t)) \\ &\quad - r_1C_0(t) - ax^{\theta} - 0.5(1-q)\sigma_3^2x^{2\theta}\Big]\Big\}dt \\ &\quad + e^tqx^q\Big[\sigma_1dB_1(t) + \sigma_2C_0(t)dB_2(t) + \sigma_3x^{\theta}dB_3(t)\Big] \\ &\leq e^tK_2dt \\ &\quad + e^tqx^q\Big[\sigma_1dB_1(t) + \sigma_2C_0(t)dB_2(t) + \sigma_3x^{\theta}dB_3(t)\Big], \end{aligned}$$

where K_2 is a positive constant. Hence

$$\mathbb{E}[e^tx^q(t)] - x_0^q \leq \mathbb{E}\int_0^t e^sK_2ds \leq K_2(e^t-1),$$

That is to say

$$\limsup_{t\to+\infty}\mathbb{E}[x^q(t)] \leq K_2. \tag{16}$$

Then the desired assertion follows from Chebyshev's inequality. ■

Remark 2. Liu and Wang [21] have studied model (1) and have shown that

(i) If $\limsup_{t\to+\infty} t^{-1}\int_0^t b_1(s)ds < 0$, then the species, $x(t)$, represented by model (1), goes to extinction, where $b_1(t) = r_0 - 0.5\sigma_1^2 - r_1C_0(t)$.

(ii) If $\limsup_{t\to+\infty} t^{-1}\int_0^t b_1(s)ds > 0$, then the species, $x(t)$, represented by (1) is weakly persistence a.s.;

(iii) If $\liminf_{t\to+\infty} b_2(t) > 0$, then $x(t)$ is stochastic permanent, where $b_2(t) = r_0 - \sigma_1^2 - r_1C_0(t)$.

As said above, model (1) is a special case of our model (2). Therefore our Theorems 1 and 3 extends the results (i) and (ii), respectively. On the other hand, note that $b_2(t) = b(t) + 0.5\sigma_1^2 \geq b(t)$, Thus our conditions of Theorem 4 are much weaker than that of (iii).

III. Global attractivity.

In the previous section, we have studied the persistence and extinction of the population. Now let us consider the global attractivity of the positive solution of Eq. (3). Before we state and prove our main result of this section, let us give the definition of global attractivity and recall an important lemma.

Definition 2. *Let $x(t)$, $y(t)$ be two arbitrary solutions of Eq. (3) with initial values $x_0 > 0$, $y_0 > 0$ respectively. If $\lim_{t\to+\infty}\mathbb{E}|x(t)-y(t)| = 0$, then we say model (3) is globally attractive.*

Lemma 3. *([32]) If f is a non-negative, integrable, and uniformly continuous function defined on $R_+ = [0,\infty)$, then $\lim_{t\to+\infty} f(t) = 0$.*

Theorem 5. *If $\liminf_{t\to+\infty} b(t) > 0$ and $0 < \theta \leq 1$, then model (3) is globally attractive.*

Proof: Define $V(t) = |\ln x(t) - \ln y(t)|$. It then follows from Itô's formula that

$$\begin{aligned} d^+V(t) &= \mathrm{sgn}\Big(x(t)-y(t)\Big)d(\ln x(t) - \ln y(t)) \\ &= -a\Big|x^{\theta}(t)-y^{\theta}(t)\Big|dt - \tfrac{1}{2}\sigma_3^2\Big|x^{2\theta}(t)-y^{2\theta}(t))\Big|dt \\ &\quad + \sigma_3\Big|x^{\theta}(t)-y^{\theta}(t)\Big|dB_3(t) \\ &\leq -a\Big|x^{\theta}(t)-y^{\theta}(t)\Big|dt + \sigma_2\Big|x^{\theta}(t)-y^{\theta}(t)\Big|dB_3(t). \end{aligned}$$

Integrating and then taking the expectation, we have

$$\mathbb{E}(V(t)) \leq \mathbb{E}(V(0)) - a\int_0^t \mathbb{E}\Big|x^{\theta}(s)-y^{\theta}(s)\Big|ds.$$

Consequently,

$$\mathbb{E}(V(t)) + a\int_0^t \mathbb{E}\Big|x^{\theta}(s)-y^{\theta}(s)\Big|ds \leq \mathbb{E}(V(0)) < \infty.$$

Note that $V(t) \geq 0$, hence $\mathbb{E}|x^{\theta}(t)-y^{\theta}(t)| \in L^1[0,\infty)$. By (3),

$$\mathbb{E}(x^{\theta}(t)) = x_0 + \theta\int_0^t x^{\theta}(s)\Big[r_0 - r_1C_0(s) - a - 0.5(1-\theta)(\sigma_1^2+\sigma_2^2C_0^2(s)) - 0.5(1-\theta)\sigma_3^2x^{\theta}\Big]ds,$$

Consequently, $\mathbb{E}(x^\theta(t))$ is continuously differentiable with respect to t. On the other hand, in view of (16),

$$\frac{d\mathbb{E}(x^\theta(t))}{dt} \le r_0\mathbb{E}(x^\theta(t)) \le K_3,$$

where K_3 is a positive constant. Therefore, $\mathbb{E}(x^\theta(t))$ is uniformly continuous. By Lemma 3, $\lim_{t\to+\infty}\mathbb{E}|x^\theta(t)-y^\theta(t)| = 0$. Note that model (3) is permanent, hence $\lim_{t\to+\infty}\mathbb{E}|x(t)-y(t)| = 0$. ■

IV. Generalization.

In the above sections, we have investigated some dynamics of model (3). As matter of fact, some results can be extended to the multi-dimensional cases. Consider the following n-species model:

$$\begin{aligned} dx_i(t) &= x_i(t)\Big(r_{i0} - r_{i1}C_0(t) - \sum_{j=1}^n a_{ij}x_{ij}^{\theta_{ij}}(t)\Big)dt \\ &+\sigma_{i1}x_i(t)dB_{i1}(t) + \sigma_{i2}C_0(t)x_i(t)dB_{i2}(t) \\ &+\sum_{j=1}^n \sigma_{ij3}x_i(t)x_{ij}^{\theta_{ij}}(t)dB_{ij3}(t), \quad i = 1,2,...,n. \end{aligned} \tag{17}$$

where $a_{ij} > 0$, $\theta_{ij} > 0$; $B_{i1}(t)$, $B_{i2}(t)$ and $B_{ij3}(t)$ are independent standard Brownian motions defined on $(\Omega,\mathcal{F},\mathcal{P})$, $1 \le i,j \le n$.

Theorem 6. *If* $\limsup_{t\to+\infty} t^{-1}\int_0^t b_i(s)ds < 0$, *then the species,* $x_i(t)$, *modeled by (17), goes to extinction a.s.,* $i = 1,...,n$, *where* $b_i(t) = r_{i0} - \frac{1}{2}\sigma_{i1}^2 - r_{i1}C_0(t) - \frac{1}{2}\sigma_{i2}^2C_0^2(t)$.

Proof: By Itô's formula

$$\begin{aligned} \ln[x_i(t)/x_{i0}] &= \int_0^t \Big[b_i(s) - \sum_{j=1}^n a_{ij}x_j^{\theta_{ij}}(s) \\ &-\frac{1}{2}\sum_{j=1}^n \sigma_{ij3}^2(s)x_j^{2\theta_{ij}}(s)\Big]ds \\ &+\sigma_{i1}B_{i1}(t) + M_{i2}(t) + \sum_{j=1}^n M_{ij3}(t), \end{aligned} \tag{18}$$

where $M_{i2}(t) = \int_0^t \sigma_{i2}C_0(s)dB_{i2}(s)$, $M_{ij3}(t) = \int_0^t \sigma_{ij3}x_j^{\theta_{ij}}(s)dB_{ij3}(s)$. The quadratic variation of $M_{i2}(t)$ is

$$\langle M_{i2}(t), M_{i2}(t)\rangle = \int_0^t \sigma_{i2}^2C_0^2(s)ds \le \sigma_{i2}^2 t.$$

By the strong law of large numbers for martingales,

$$\lim_{t\to+\infty} M_{i2}(t)/t = 0, \quad a.s. \tag{19}$$

The quadratic variation of $M_{ij3}(t)$ is

$$\langle M_{ij3}, M_{ij3}\rangle = \int_0^t \sigma_{ij3}^2x_j^{2\theta_{ij}}(s)ds.$$

By virtue of the exponential martingale inequality,

$$\mathcal{P}\Big\{\sup_{0\le t\le k}\Big[M_{ij3}(t) - \frac{1}{2}\langle M_{ij3}, M_{ij3}\rangle\Big] > 2\ln k\Big\} \le 1/k^2.$$

It then follows from the Borel-Cantelli lemma that, for almost all $\omega\in\Omega$, there is a stochastic integer $k_0 = k_0(\omega)$ such that for $k \ge k_0$,

$$\sup_{0\le t\le k}\Big[M_{ij3}(t) - \frac{1}{2}\langle M_{ij3}(t), M_{ij3}(t)\rangle\Big] \le 2\ln k.$$

That is to say, $M_{ij3}(t) \le 2\ln k + 0.5\int_0^t \sigma_{ij3}^2x^{2\theta_{ij}}(s)ds$ for all $0 \le t \le k,\ k \ge k_0$ almost surely. When this inequality is used in (18), we have

$$\begin{aligned} &\ln x_i(t) - \ln x_{i0} \\ &\le \int_0^t b_i(s)ds - \sum_{j=1}^n \int_0^t a_{ij}x_j^{\theta_{ij}}(s)ds \\ &+\sigma_{i1}B_{i1}(t) + M_{i2}(t) + 2n^2\ln k \\ &\le \int_0^t b_i(s)ds + \sigma_{i1}B_{i1}(t) + M_{i2}(t) + 2n^2\ln k \end{aligned} \tag{20}$$

for all $0 \le t \le k,\ k \ge k_0$ almost surely. The following proof is similar to that of Theorem 1 and hence is omitted. ■

Theorem 7. *If* $\limsup_{t\to+\infty} t^{-1}\int_0^t b_i(s)ds = 0$, *then* $x_i(t)$ *is non-persistent in the mean a.s.*

Proof: By (20), for arbitrarily given $\varepsilon > 0$, there is a positive constant T such that for $T \le k-1 \le t \le k$ and $k \ge k_0$, $\ln x_i(t) - \ln x_{i0} \le \varepsilon t - a_{ii}\int_0^t x_i^{\theta_{ii}}(s)ds$. The following proof is a slight modification of that in Theorem 2 and hence is omitted. ■

Theorem 8. *The solution of model (17) obeys*

$$\limsup_{t\to+\infty}\frac{\ln\sum_{i=1}^n x_i(t)}{\ln t} \le 1, \quad a.s. \tag{21}$$

Proof: Define $W(x) = \sum_{i=1}^n x_i$. By Itô's formula,

$$\begin{aligned} &e^t\ln\sum_{i=1}^n x_i(t) - \ln\sum_{i=1}^n x_i(0) \\ &= \int_0^t e^s\Big[\ln W(x(s)) + \frac{1}{W(x(s))}\sum_{i=1}^n x_i(s) \\ &\times\Big(r_{i0} - r_{i1}C_0(s) - \sum_{j=1}^n a_{ij}x_{ij}^{\theta_{ij}}(s)\Big)\Big]ds \\ &-\int_0^t \frac{e^s}{2W(x(s))}\sum_{i=1}^n\Big[\sigma_{i1}^2 + \sigma_{i2}^2C_0(s)\Big]x_i^2(s)ds \\ &-\int_0^t \frac{e^s}{2W(x(s))}\sum_{i=1}^n\sum_{j=1}^n x_i^2(s)x_j^{2\theta_{ij}}(s) \\ &+\sum_{i=1}^n N_{i1}(t) + \sum_{i=1}^n N_{i2}(t) + \sum_{i=1}^n\sum_{j=1}^n N_{ij3}(t), \end{aligned}$$

where

$$N_{i1}(t) = \int_0^t \frac{e^s}{W(x(s))}\sigma_{i1}x_i(s)dB_{i1}(s),$$

$$N_{i2}(t) = \int_0^t \frac{e^s}{W(x(s))}\sigma_{i2}x_i(s)C_0(s)dB_{i2}(s),$$

$$N_{ij3}(t) = \int_0^t e^s\frac{e^s}{W(x(s))}\sigma_{ij3}x_i(s)x_j^{\theta_{ij}}(s)dB_{ij3}(s).$$

Denote $N(t) = \sum_{i=1}^n(N_{i1}(t) + N_{i2}(t) + \sum_{j=1}^n N_{ij3}(t))$. The following proof is similar to that of Theorem 3 by using the exponential martingale inequality and the Borel-Cantelli lemma and hence is omitted. ■

Theorem 9. *If* $\min_{1\le i\le n}\{\liminf_{t\to+\infty} b_i(t)\} > 0$, *and* $0 < \theta_{ij} \le 1$ *for all* $1 \le i,j \le n$, *then model (17) is stochastically permanent, i.e., for every* $0 < \varepsilon < 1$, *there are positive constants* β *and* M *such that* $\liminf_{t\to+\infty} P\{|x(t)| \ge \beta\} \ge 1-\varepsilon, \liminf_{t\to+\infty} P\{|x(t)| \le M\} \ge 1-\varepsilon$.

Proof: (i) Define $V_1(x) = 1/\sum_{i=1}^{n} x_i, \quad x_i > 0$. By Itô's formula,

$$\begin{aligned} &dV_1(x) \\ &= \Big[-V_1^2(x)\sum_{i=1}^{n} x_i\Big(r_{i0} - r_{i1}C_0(t) - \sum_{j=1}^{n} a_{ij}x_{ij}^{\theta_{ij}}\Big) \\ &+V_1^3(x)\sum_{i=1}^{n}\sigma_{i1}^2x_i^2 + V_1^3(x)\sum_{i=1}^{n}\sigma_{i2}^2C_0^2(t)x_i^2 \\ &+V_1^3(x)\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij3}^2x_i^2x_j^{2\theta_{ij}}\Big]dt \\ &-V_1^2(x)\sum_{i=1}^{n}\Big(\sigma_{i1}dB_{i1}(t) + \sigma_{i2}C_0(t)dB_{i2}(t)\Big)x_i \\ &-V_1^2(x)\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij3}x_i(t)x_{ij}^{\theta_{ij}}dB_{ij3}(t). \end{aligned}$$

Since $\min_{1\le i\le n}\{\liminf_{t\to+\infty} b_i(t)\} > 0$, we can let $0 < \kappa < 1$ be sufficiently small such that

$$\min_{1\le i\le n}\{\liminf_{t\to+\infty} b_i(t)\} > \frac{\kappa}{2}\max_{1\le i\le n}(\sigma_{i1}^2 + \sigma_{i2}^2). \tag{22}$$

Define $V_2(x) = (1 + V_1(x))^\kappa$. By Itô's formula,

$$\begin{aligned} &dV_2(x) = \kappa(1 + V_1(x))^{\kappa-2}\Big\{ -(1 + V_1(x))V_1^2(x) \\ &\times\sum_{i=1}^{n} x_i\Big(r_{i0} - r_{i1}C_0(t) - \sum_{j=1}^{n} a_{ij}x_{ij}^{\theta_{ij}}\Big) \\ &+V_1^3(x)\sum_{i=1}^{n}\Big(\sigma_{i1}^2 + \sigma_{i2}^2C_0^2(t) + \sum_{j=1}^{n}\sigma_{ij3}^2x_j^{2\theta_j}\Big)x_i^2 \\ &+\frac{\kappa+1}{2}V_1^4(x)\sum_{i=1}^{n}\Big(\sigma_{i1}^2 + \sigma_{i2}^2C_0^2(t)\Big)x_i^2 \\ &+\frac{\kappa+1}{2}V_1^4(x)\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij3}^2x_i^2x_j^{2\theta_{ij}}\Big\}dt \\ &-\kappa(1 + V_1(x))^{\kappa-1}V_1^2(x)\sum_{i=1}^{n}\Big(\sigma_{i1}dB_{i1}(t) \\ &+\sigma_{i2}C_0(t)x_idB_{i2}(t) + \sum_{j=1}^{n}\sigma_{ij3}x_j^{\theta_{ij}}dB_{ij3}(t)\Big)x_i \\ &\le \kappa(1 + V_1(x))^{\kappa-2}\Big\{ -V_1^2(x) \\ &\times\Big(\min_{1\le i\le n}\{\liminf_{t\to+\infty} b_i(t)\} - \frac{\kappa}{2}\max_{1\le i\le n}(\sigma_{i1}^2 + \sigma_{i2}^2)\Big) \\ &+V_1(x)\Big(\max_{1\le i\le n} r_{i1} + \max_{1\le i\le n}(\sigma_{i1}^2 + \sigma_{i2}^2)\Big) \\ &+(1 + V_1(x))V_1^2(x)\max_{1\le i,j\le n} a_{ij}\sum_{i=1}^{n}\sum_{j=1}^{n} x_ix_j^{\theta_{ij}} \\ &+V_1^3(x)\max_{1\le i,j\le n}\sigma_{ij3}^2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2x_j^{2\theta_{ij}} \\ &+V_1^4(x)\max_{1\le i,j\le n}\sigma_{ij3}^2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2x_j^{2\theta_{ij}}\Big\}dt \\ &-\kappa(1 + V_1(x))^{\kappa-1}V_1^2(x)\sum_{i=1}^{n}\Big(\sigma_{i1}dB_{i1}(t) \\ &+\sigma_{i2}C_0(t)x_idB_{i2}(t) + \sum_{j=1}^{n}\sigma_{ij3}x_j^{\theta_{ij}}dB_{ij3}(t)\Big)x_i \end{aligned}$$

Let $\eta > 0$ be sufficiently small such that

$$\frac{\eta}{\kappa(1+\theta)} < \liminf_{t\to+\infty} b(t) - 0.5\kappa(\theta + 1)(\sigma_1^2 + \sigma_2^2) - \varepsilon.$$

Define

$$V_3(x) = e^{\eta t}V_2(x).$$

By Itô's formula, for sufficiently large t,

$$\begin{aligned} &dV_3(x(t)) = \eta e^{\eta t}V_2(x)dt + e^{\eta t}dV_2(x) \\ &\le \kappa e^{\eta t}(1 + V_1(x))^{\kappa-2}\Big\{ -\Big(\min_{1\le i\le n}\{\liminf_{t\to+\infty} b_i(t)\} \\ &-\frac{\eta}{\kappa} - \frac{\kappa}{2}\max_{1\le i\le n}(\sigma_{i1}^2 + \sigma_{i2}^2)\Big)V_1^2(x) \\ &+V_1(x)\Big(\frac{2\eta}{\kappa} + \max_{1\le i\le n} r_{i1} + \max_{1\le i\le n}(\sigma_{i1}^2 + \sigma_{i2}^2)\Big) \\ &+(1 + V_1(x))V_1^2(x)\max_{1\le i,j\le n} a_{ij}\sum_{i=1}^{n}\sum_{j=1}^{n} x_ix_j^{\theta_{ij}} \\ &+V_1^3(x)\max_{1\le i,j\le n}\sigma_{ij3}^2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2x_j^{2\theta_{ij}} \\ &+V_1^4(x)\max_{1\le i,j\le n}\sigma_{ij3}^2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2x_j^{2\theta_{ij}}\Big\}dt \\ &-\kappa e^{\eta t}(1 + V_1(x))^{\kappa-1}V_1^2(x) \\ &\times\sum_{i=1}^{n}\Big\{\sigma_{i1}dB_{i1}(t) + \sigma_{i2}C_0(t)dB_{i2}(t) \\ &+\sum_{j=1}^{n}\sigma_{ij3}x_j^{\theta_{ij}}(t)dB_{ij3}(t)\Big\}x_i(t) \end{aligned}$$

Denote

$$\begin{aligned} &J(x) = \kappa(1 + V_1(x))^{\kappa-2}\Big\{ -\Big(\min_{1\le i\le n}\{\liminf_{t\to+\infty} b_i(t)\} \\ &-\frac{\eta}{\kappa} - \frac{\kappa}{2}\max_{1\le i\le n}(\sigma_{i1}^2 + \sigma_{i2}^2)\Big)V_1^2(x) \\ &+V_1(x)\Big(\frac{2\eta}{\kappa} + \max_{1\le i\le n} r_{i1} + \max_{1\le i\le n}(\sigma_{i1}^2 + \sigma_{i2}^2)\Big) \\ &+(1 + V_1(x))V_1^2(x)\max_{1\le i,j\le n} a_{ij}\sum_{i=1}^{n}\sum_{j=1}^{n} x_ix_j^{\theta_{ij}} \\ &+V_1^3(x)\max_{1\le i,j\le n}\sigma_{ij3}^2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2x_j^{2\theta_{ij}} \\ &+V_1^4(x)\max_{1\le i,j\le n}\sigma_{ij3}^2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2x_j^{2\theta_{ij}}\Big\}. \end{aligned}$$

Then similar to the proof of Theorem 4 we can show that $J(x)$ is upper bounded in R_+^n, i.e., $J_3 := \sup_{x\in R_+} J(x) < +\infty$. Consequently, for sufficiently large t,

$$\begin{aligned} dV_3(x(t)) &\le J_3e^{\eta t}dt - \kappa e^{\eta t}(1 + V_1(x))^{\kappa-1}V_1^2(x) \\ &\times\sum_{i=1}^{n}\Big\{\sigma_{i1}dB_{i1}(t) + \sigma_{i2}C_0(t)dB_{i2}(t) \\ &+\sum_{j=1}^{n}\sigma_{ij3}x_j^{\theta_{ij}}(t)dB_{ij3}(t)\Big\}x_i(t). \end{aligned}$$

The following proof is similar to that of Theorem 4 and hence is omitted.

The following proof is similar to that of Theorem 4 by applying Itô's formula to $V(x) = \sum_{i=1}^{n} x^q$, where $x > 0$, $0 < q \le 1$, and hence is omitted. ■

V. Numerical simulations

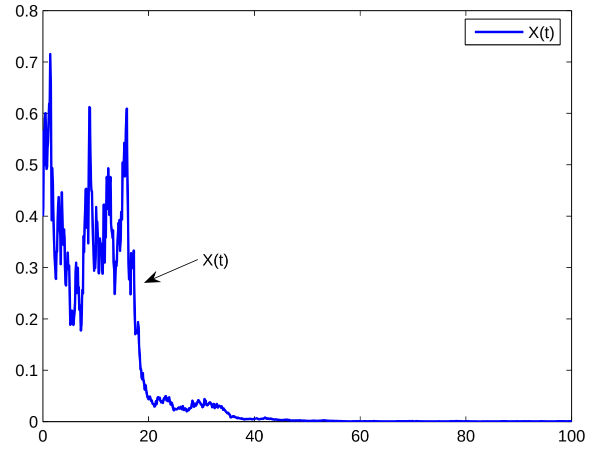
In this section, we introduce some numerical figures to illustrate our main results. To begin with, let us consider model (3). For the sake of simplicity, we choose $\theta = 1$. Then Eq. (3) becomes

$$dx(t) = x(t)[r_0 - r_1C_0(t) - ax(t)]dt + \sigma_1 x(t)dB_1(t) + \sigma_2 C_0(t)x(t)dB_2(t) + \sigma_3 x^2(t)dB_3(t). \quad (23)$$

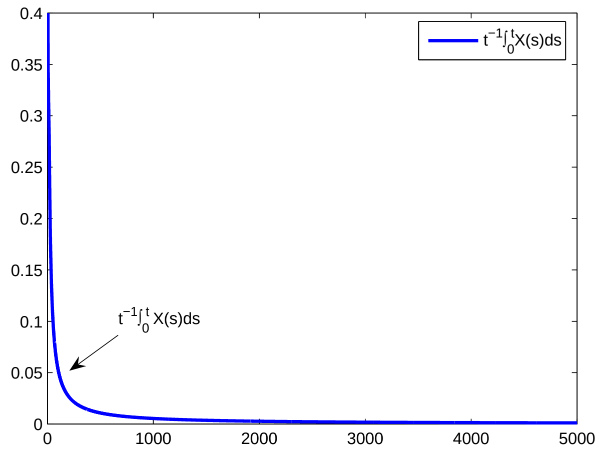
By virtue of the Milstein methods given in [33] (see also [34]), consider the discretization equation of Eq. (23):

$$\begin{aligned} x_{k+1} &= x_k + x_k\left[r_0 - r_1C_0(k\Delta t) - ax_k\right]\Delta t \\ &+\sigma_1 x_k\sqrt{\Delta t}\xi_k + \sigma_2 C_0(k\Delta t)x_k\sqrt{\Delta t}\gamma_k + \sigma_3 x_k^2\sqrt{\Delta t}\eta_k \\ &+0.5\sigma_1 x_k(\xi_k^2 - 1)\sqrt{\Delta t} + 0.5\sigma_2 C_0(k\Delta t)x_k(\gamma_k^2 - 1)\sqrt{\Delta t} \\ &+0.5\sigma_3 x_k^2(\eta_k^2 - 1)\sqrt{\Delta t}, \end{aligned}$$

where ξ_k, γ_k and η_k, $k = 1, 2, ..., n$, are Gaussian random variables.



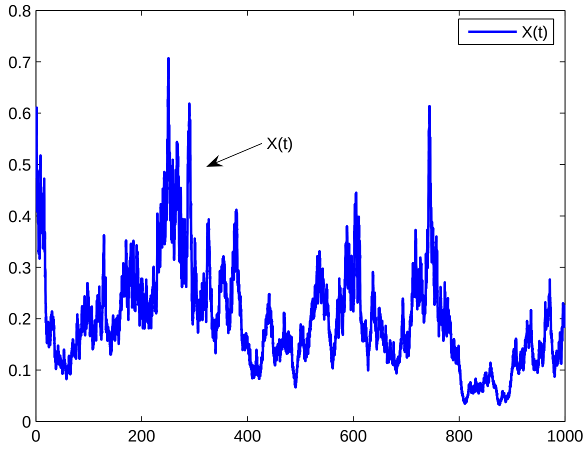
(a)



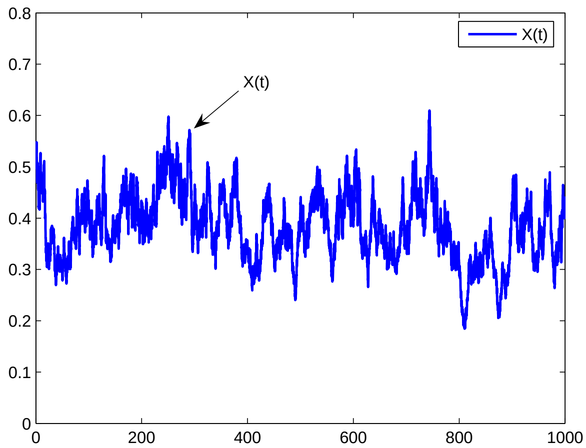
(b)

Fig. 1: Solution of Eq.(23). (a) shows that the population goes to extinction ($\sigma_1 = 0.65$); (b) shows that the population is non-persistent in the mean ($\sigma_1 = \sqrt{0.4}$); (c) shows that the population is weakly persistent ($\sigma_1 = 0.6$); (d) indicates that the population is stochastically permanent ($\sigma_1 = 0.5$).

In Fig.1, we choose $r_0 = 0.32$, $r_1 = 0.5$, $C_0(t) = 0.2 + 0.05\sin t$, $a = 0.1$, $\sigma_2 = \sigma_3 = 1$. The only difference between conditions of Fig.1(a), Fig.1(b), Fig.1(c)



(c)



(d)

Fig. 1: (Continued)

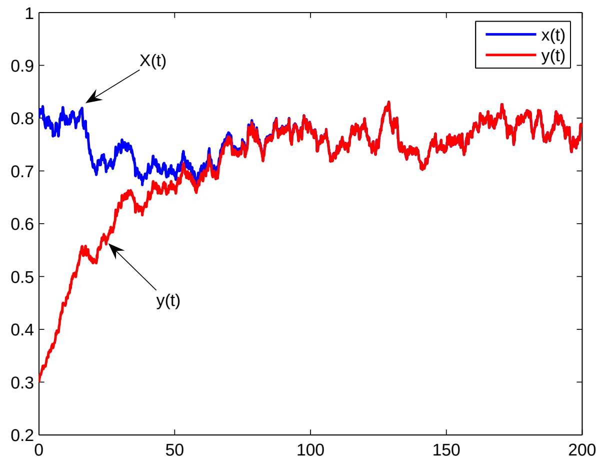
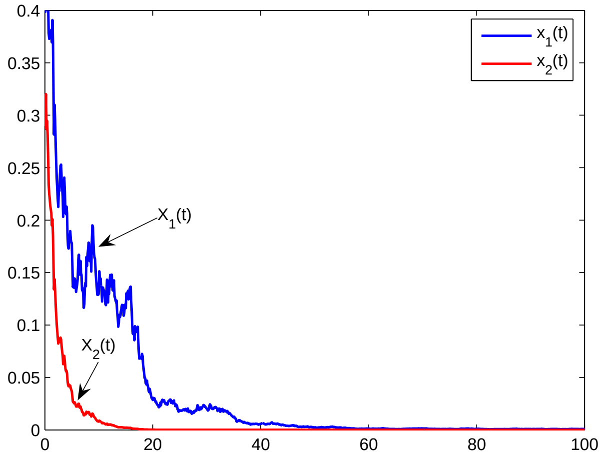
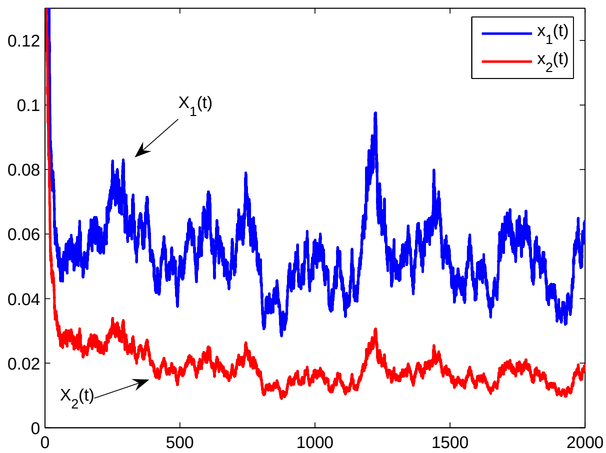


Fig. 2: Plot of two solution trajectories for Eq.(23) with two sets of initial conditions $x_0 = 0.8$ and $y_0 = 0.6$. This figure shows that model (23) is globally attractive.



(a)



(b)

Fig. 3: Solution of Eq.(17). (a) shows that all the populations go to extinction ($r_{10} = 0.08$ and $r_{20} = 0.12$); (b) shows that Eq.(17) is stochastic permanent ($r_{10} = 0.11$ and $r_{20} = 0.153$).

and Fig.1(d) is that the value of σ_1 is different. In Fig.1(a), we let $\sigma_1 = 0.65$. Hence

$$\limsup_{t\to+\infty} t^{-1}\int_0^t b(s)ds = -0.01 < 0.$$

In view of Theorem 1, $x(t)$ goes to extinction. Fig.1(a) confirms this. In Fig.1(b), we let $\sigma_1 = \sqrt{0.4}$. Therefore, $\liminf_{t\to+\infty} b(t) = 0$. It then follows from Theorem 2 that $x(t)$ is non-persistent in the mean. See Fig.1(b). In Fig.1(c), we choose $\sigma_1 = 0.6$. Thus $\limsup_{t\to+\infty} t^{-1}\int_0^t b(s)ds = 0.04 > 0$. According to Theorem 3, $x(t)$ is weakly persistent, Fig.1(c) confirms this. In Fig.1(d), we let $\sigma_1 = 0.5$. Therefore, $\liminf_{t\to+\infty} b(t) = 0.025$. It then follows from Theorem 4 that $x(t)$ is stochastic permanent. See Fig.1(d).

In Fig.2, the parameters are same with that in Fig.1(d). Then by Theorem 5, model (3) is globally attractive. Fig.2 confirms this.

Now let us turn to model (17). For the sake of simplicity, we choose $n = 2$ and $\theta_{ij} = 1$. In Fig.3, we choose $r_{11} = r_{21} = 0.5$, $C_0(t) = 0.2$, $a_{11} = 0.4$, $a_{12} = 0.2$, $a_{21} = 0.3$, $a_{22} = 0.4$, $\sigma_{11} = 0.4$, $\sigma_{21} = 0.5$, $\sigma_{113} = \sigma_{123} = 0.8$, $\sigma_{21} = 0.5$, $\sigma_{22} = 0.5$, $\sigma_{213} = \sigma_{223} = 0.9$. The only difference between conditions of Fig.3(a) and Fig.3(b) is that the values of r_{10} and r_{20} are different. In Fig.3(a), we choose $r_{10} = 0.08$ and $r_{20} = 0.12$. Hence

$$\limsup_{t\to+\infty} t^{-1}\int_0^t b_1(s)ds = -0.005 < 0,$$

$$\limsup_{t\to+\infty} t^{-1}\int_0^t b_2(s)ds = -0.01 < 0.$$

By Theorem 6, both x_1 and x_2 go to extinction. See Fig.3(a). In Fig.3(b), we let $r_{10} = 0.11$ and $r_{20} = 0.153$. Therefore,

$$\liminf_{t\to+\infty} b_1(t) = 0.025,\quad \liminf_{t\to+\infty} b_2(t) = 0.023.$$

By Theorem 9, model (17) is stochastic permanent. See Fig.3(b).

VI. Conclusion

In this paper, under the assumptions that all the coefficients are affected by white noise, we have proposed and investigated a stochastic single-species Gilpin-Ayala population model in a polluted environment. We have established the sufficient conditions for extinction, non-persistence in the mean, weak persistence and stochastic permanence of the population. The critical value between weak persistence and extinction have been obtained. We have also demonstrated that the solution of the model is globally attractive. Some recent results have been extended and improved.

Our results indicate that a different type of environmental noise has a different effect on the persistence and extinction of the species (see Remark 1). By the definition of $b(t)$, the white noise $\sigma_1\dot{B}_1(t)$ is unfavorable for the persistence of the population, the white noise $\sigma_2\dot{B}_2(t)$ has no impact on the persistence or extinction of the population, the white noise $\sigma_3\dot{B}_3(t)$ is also unfavorable for the persistence of the population.

Our Theorems 1-4 have some important and interesting biological meanings. From Theorems 1 and 3 one can observe that persistence and extinction of the population $x(t)$ depend only on the growth rate r_0, the power of the white noises σ_1^2 and σ_3^2, the dose-response parameter of the population to the organismal toxicant concentration r_1, the concentration of toxicant in the organism $C_0(t)$, but are independent of initial population size x_0, the parameters θ and a, as well as the power of the white noise σ_2^2. So in order to conserve a species, one has the following ways.

(i) To reduce the values of σ_1^2 and σ_3^2.
(ii) To reduce the concentration of toxicant in the organism (i.e., to reduce the pollutant output $u(t)$).

However, one could not conserve a population by influencing σ_2^2 and θ.

Some interesting topics deserve further investigation. In Theorem 4 and Theorem 5, our conditions have some limitations on θ. It is of interest to consider whether these limitations can be dropped. It is also interesting to investigate other multi-species population models (see e.g. [35], [36], [37], [38]).

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