Some Features for Blow-up Solutions of a Nonlinear Parabolic Equation

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***Abstract*—In previous studies we have shown some conjectures for behavior of blow-up solutions to a nonlinear parabolic equations. They are very important features to investigate behavior of solutions near their blow-up time. The purpose of our paper is to prove one of them that we call "weak eventual monotonicity".**

***Index Terms*—parabolic equations, blow-up solutions, Type 2, eventual monotonicity.**

I. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ of C^2-class and consider the following initial-boundary value problems.

$$\begin{cases} u_t = u^\delta(\Delta u + \lambda u) & x \in \Omega, t > 0, \\ u(x,0) = u_0(x) & x \in \overline{\Omega}, \\ u(x,t) = 0 & x \in \partial\Omega, t \geq 0, \end{cases} \tag{1}$$

where $\delta > 0$, λ is greater than the first eigenvalue, $\lambda_1(\Omega)$ of $-\Delta$ in Ω, that is, $\lambda > \lambda_1(\Omega) > 0$ and the initial functions $u_0 \in C^\infty(\Omega) \bigcap C(\overline{\Omega})$ satisfies

$$u_0 > 0 \quad \text{in } \Omega \quad \text{and} \quad u_0(x) = 0 \quad \text{for } x \in \partial\Omega. \tag{2}$$

In this paper we discuss classical solutions which are approximated by functions u_ε solving the following problems:

$$\begin{cases} (u_\varepsilon)_t = u_\varepsilon^\delta(\Delta u_\varepsilon + \lambda u_\varepsilon) & x \in \Omega, t > 0, \\ u_\varepsilon(x,0) = u_0(x) + \varepsilon & x \in \overline{\Omega}, \\ u_\varepsilon(x,t) = \varepsilon & x \in \partial\Omega, t \geq 0. \end{cases} \tag{3}$$

It has been proved that each of them blows up at a finite time, that is, for any solutions approximated by u_ε, there exists $T > 0$ such that

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_\infty = \infty,$$

where $\|u(\cdot,t)\|_\infty = \sup_{x\in\Omega} u(x,t)$. (For instance, see [2].)

Here the constant T is called "the blow-up time" of u. Moreover it has been known that they are classified into two types by their blow-up rates as follows.

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Definition 1: Let u and T be a solution of (1) and the blow-up time of u, respectively. Then u is called "Type 1" if it satisfies

$$\limsup_{t \nearrow T} (T-t)^{\frac{1}{\delta}} \|u(\cdot,t)\|_\infty < +\infty \tag{4}$$

and u is called "Type 2" if it satisfies

$$\limsup_{t \nearrow T} (T-t)^{\frac{1}{\delta}} \|u(\cdot,t)\|_\infty = +\infty. \tag{5}$$

Precisely, It has been proved that if $\delta \geq 2$ then there exists solutions of (1) which are "Type 2" and if $0 < \delta < 2$ then u is "Type 1". (See [3], [4], [5], [7], [8], [9], [10], [11] and so force.) Our purpose of this paper is to investigate asymptotic behavior of "Type 2" solutions.

In [1] we investigated asymptotic behavior of solution to (1) without any special assumptions in the case of $\delta = 2$ and it is proved that if $\delta = 2$ then $2(T-t)\int_\Omega u^{-1}u_t\,dx$ becomes positive and has an upper bound after a finite time. In particular, the positivity implies that there exists $t_0 \in (0,T)$ such that

$$\int_\Omega u(\Delta u + \lambda u)\,dx = \int_\Omega u^{-1}u_t\,dx > 0$$

for $t \in [t_0, T)$. In other words, this means that if $\delta = 2$ then $\int_\Omega \log u(x,t)\,dx$ is increasing in (t_0,T). We call this property "weak eventual monotonicity".

In addition, [1] provided some conjectures for behavior of solutions which contain "weak eventual monotonicity" for the case of $\delta > 2$. Our main purpose of this paper is to prove the following theorem and to solve it in part.

Theorem 2: Let u_0 be a initial function satisfying $(u_0)^{2-\delta_1} \in L^1(\Omega)$ for some $\delta_1 > 2$. Then there exists $\delta_0 \in (2, \delta_1]$ such that we can choose $t_\delta \in (0,T)$ for any $\delta \in (2, \delta_0)$ so that

$$\int_\Omega u(x,t)\Big(\Delta u(x,t) + \lambda u(x,t)\Big)\,dx = \int_\Omega u(x,t)^{1-\delta}u_t(x,t)\,dx > 0$$

for any $t \in (t_\delta, T)$.

In the case of $\delta > 2$ Theorem 2 implies that $\int_\Omega u(x,t)^{2-\delta}\,dx$ is decreasing in (t_δ, T), and so the proof of this theorem is more difficult than one for the case of $\delta = 2$ in [1] which used the fact that $\log u(x,t)$ blows up in a subset with positive measure as t is closed to the blow-up time T and then $\int_\Omega \log u(x,t)\,dx$ also blows up. Indeed, if

$\delta > 2$ then it is known that $u(x,t)^{2-\delta}$ decays in a subset with positive measure but behavior of $\displaystyle\int_\Omega u(x,t)^{2-\delta}\,dx$ is unknown.

In this paper we will first show some features for blow-up solutions which are required to investigate of behavior of $\displaystyle\int_\Omega u(x,t)^{2-\delta}\,dx$. Precisely, in section II we provide well-known properties in previous studies and fundamental lemmas for solutions of (1). In section III we exactly estimate behavior of solutions near boundary. In section IV we show an important property for regions in which solutions blow up with the same rate as the maximum point. The similar properties was proved in previous studies (for instance, see [11]). We provide an exact proof by estimates in this paper which is required in the proof of our main theorem. In section V we investigate behavior of $\displaystyle\int_\Omega u(x,t)^{2-\delta}\,dx$ and give the proof of Theorem 2.

II. Fundamental Lemmas

In this section we provide well-known properties in previous studies and fundamental lemmas for solutions of (1).

***Lemma** 3:* Let u be a solution u of (1). Then u satisfies

$$u_t(x,t) \geq -\frac{1}{\delta t}u(x,t) \quad \text{for } x\in\Omega \text{ and } t\in(0,T).$$

This lemma has been stated in the previous studies, for instance [1] but we give the proof for the reader's convenience.

Proof. Let $v(x,t) = u(x,t)^{-1}u_t(x,t)$. Then it is verified that

$$\begin{aligned} v_t(x,t) = u(x,t)^\delta \Delta v(x,t) \\ + 2u(x,t)^{\delta-1}\langle \nabla u(x,t), \nabla v(x,t)\rangle \\ + \delta v(x,t)^2, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^n$. Hence, by the maximum principle, we have

$$v(x,t) \geq -\frac{1}{\delta t}$$

for $x\in\Omega$ and $t\in(0,T)$ which completes this proof. □

***Lemma** 4:* Let u be a solution of (1). Then $\varphi(x,t) := t^{\frac{1}{\delta}}u(x,t)$ is increasing with respect to t for any $x\in\Omega$.

Proof. By Lemma 3, we have

$$\begin{aligned} \frac{\partial}{\partial t}\Big(t^{\frac{1}{\delta}}u(x,t)\Big) &= t^{\frac{1}{\delta}}\frac{\partial u}{\partial t}(x,t) + \frac{1}{\delta}t^{\frac{1}{\delta}-1}u(x,t) \\ &= t^{\frac{1}{\delta}}\left(\frac{\partial u}{\partial t}(x,t) + \frac{1}{\delta t}u(x,t)\right) \geq 0 \end{aligned}$$

for any $x\in\Omega$ and $t\in(0,T)$. □

***Lemma** 5:* Let $\delta > 2$. Suppose that $(u_0)^{2-\delta}\in L^1(\Omega)$. Then there exists $K_\delta > 0$ such that

$$\int_\Omega u(x,t)^{2-\delta}\,dx \leq K_\delta \quad \text{for any } t\in(0,T).$$

***Remark**:* K_δ depends on n, Ω, λ and u_0 in addition to δ.

Proof. Let $t_0\in(0,T)$ be fixed arbitrarily. Since there exists a constant $C_{t_0} > 0$ such that

$$\begin{aligned} &\left|\int_\Omega u(x,t)^{1-\delta}u_t(x,t)\,dx\right| \\ &= \left|\int_\Omega u(x,t)\Big(\Delta u(x,t) + \lambda u(x,t)\Big)\,dx\right| \leq C_{t_0} \end{aligned}$$

for $t\in(0,t_0]$, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u(x,t)^{2-\delta}\,dx \\ &= (2-\delta)\int_\Omega u(x,t)^{1-\delta}u_t(x,t)\,dx \\ &\leq (\delta-2)C_{t_0} \end{aligned}$$

for $t\in(0,t_0]$. This impiles that

$$\begin{aligned} &\int_\Omega u(x,t)^{2-\delta}\,dx \\ &\leq (\delta-2)C_{t_0}t_0 + \int_\Omega u_0(x)^{2-\delta}\,dx < \infty \end{aligned}$$

for $t\in(0,t_0]$.

Next, consider the case of $t\in(t_0,T)$. By Lemma 4,

$$\int_\Omega \Big(t^{\frac{1}{\delta}}u(x,t)\Big)^{2-\delta}\,dx \leq \int_\Omega \Big(t_0^{\frac{1}{\delta}}u(x,t_0)\Big)^{2-\delta}\,dx$$

for $t\in(t_0,T)$. And then

$$\begin{aligned} &\int_\Omega u(x,t)^{2-\delta}\,dx \\ &\leq \left(\frac{t}{t_0}\right)^{\frac{\delta-2}{\delta}}\int_\Omega u(x,t_0)^{2-\delta}\,dx \\ &\leq \left(\frac{T}{t_0}\right)^{\frac{\delta-2}{\delta}}\left((\delta-2)C_{t_0}t_0 + \int_\Omega u_0(x)^{2-\delta}\,dx\right) \end{aligned}$$

for $t\in(t_0,T)$. Therefore,

$$\begin{aligned} &K_\delta := \\ &\inf_{t_0\in(0,T)}\left(\frac{T}{t_0}\right)^{\frac{\delta-2}{\delta}}\left((\delta-2)C_{t_0}t_0 + \int_\Omega u_0(x)^{2-\delta}\,dx\right) \end{aligned}$$

satisfies the assertion in this lemma. □

***Lemma** 6:* Let u be a solution of (1) and

$$m_\delta := \min\left(\Big(\max_{x\in\Omega}u_0(x)\Big)^{-\delta},\ 1\right).$$

Then it holds

$$u(x,t) \leq (m_\delta - \lambda\delta t)^{-\frac{1}{\delta}}$$

if $x\in\Omega$ and $0\leq t < \dfrac{m_\delta}{\lambda\delta}$.

Proof. It is verified that

$$m(t) := (m_\delta - \lambda\delta t)^{-\frac{1}{\delta}}$$

satisfies

$$\frac{dm}{dt} = \lambda m(t)^{\delta+1} \text{ if } 0 < t < \frac{m_\delta}{\lambda\delta}$$

and

$$u(x,0) \leq m(0) \text{ if } x\in\Omega.$$

Hence, by the maximum principle, we have $u(x,t) \leq m(t)$ if $x \in \Omega$ and $0 \leq t < \frac{m_\delta}{\lambda\delta}$. □

***Lemma** 7:* If $0 < t < s < T$ then

$$\int_\Omega u(x,t)^{1-\delta} u_t(x,t)\,dx < \int_\Omega u(x,s)^{1-\delta} u_t(x,s)\,dx.$$

The main idea for the proof of this lemma is essentially included in [1]. We also give it and a self contained proof for the reader's convenience.

Proof. Let $\{u_{0,n}\}_{n\in\mathbb{N}} \in C^\infty(\Omega)$ satisfy

$$\|u_{0,n} - u_0\|_{L^\infty(\Omega)} < \frac{1}{n} \text{ and } \alpha_n\Theta \leq u_{0,n} \leq \beta_n\Theta$$

for any $n \in \mathbb{N}$ with constants $0 < \alpha_n < \beta_n$, where Θ denotes the principal eigenfunction of $-\Delta$ in Ω with $\max_{x\in\Omega}\Theta(x) = 1$. Then it has been shown in [10] that if $\delta > 1$ then for any $n \in \mathbb{N}$ there exists the unique solution $u_n \in C(\overline{\Omega} \times [0, T(u_n)) \bigcap C^\infty(\Omega \times (0, T(u_n)))$ of the problems

$$\begin{cases} (u_n)_t = u_n^\delta(\Delta u_n + \lambda u_n) & x \in \Omega, t \in (0, T(u_n)), \\ u_n(x,0) = u_{0,n}(x) & x \in \overline{\Omega}, \\ u_n(x,t) = 0 & x \in \partial\Omega, t \geq 0, \end{cases}$$

where $T(u_n)$ is the blow-up time of u_n satisfying $T(u_n) \to T$ as $n \to \infty$.

Furthermore, it has also been known that $\{u_n\}$ have some properties that $u_n \to u$ in the topology of $C_{loc}(\overline{\Omega} \times [0,T)) \bigcap C^\infty_{loc}(\Omega \times (0,T))$ as $n \to \infty$ and there is a constant $c(n,\tau) > 0$ for any $n \in \mathbb{N}$ and any $\tau > 0$ such that

$$|\text{dist}\,(x,\Omega)^{\alpha-1+j(2-\delta)} \partial_x^\alpha \partial_t^j u_n(x,t)| \leq c(n,\tau)$$

for $j = 0,1$, $\alpha = 0,1,2$ and $(x,t) \in \Omega \times (0,\tau)$. In particular, since $|\nabla(u_n)_t| \leq \text{dist}\,(x,\partial\Omega)^{\delta-2} c(n,\tau)$ for $(x,t) \in \Omega \times (0,\tau)$, it is obtained for any $n \in \mathbb{N}$ that

$$\begin{aligned} 0 &\leq \int_\Omega u_n^{-\delta}\Big((u_n)_t\Big)^2 dx \\ &= \int_\Omega (\Delta u_n + \lambda u_n)(u_n)_t\,dx \\ &= \int_\Omega \Big(-\nabla u_n \cdot \nabla(u_n)_t + \lambda u_n (u_n)_t\Big)\,dx \\ &= \frac{1}{2}\left(\int_\Omega (-|\nabla u_n|^2 + \lambda u_n^2)\,dx\right)_t \\ &= \frac{1}{2}\left(\int_\Omega u_n^{1-\delta}(u_n)_t\,dx\right)_t. \end{aligned}$$

This implies that $\int_\Omega u_n^{1-\delta}(u_n)_t\,dx$ is increasing with respect to t and then

$$\int_\Omega u_n(x,t)^{1-\delta}(u_n)_t(x,t)\,dx < \int_\Omega u_n(x,s)^{1-\delta}(u_n)_t(x,s)\,dx$$

if $0 < t < s < T$.

In addition, since it is true for any $t \in (0,T)$ that $u_n(\cdot,t) \to u(\cdot,t)$ uniformly in $\overline{\Omega}$ and $\Delta u_n(\cdot,t) \rightharpoonup \Delta u(\cdot,t)$ in $C^*(\Omega)$ as $n \to \infty$, we have

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega u_n(x,t)^{1-\delta}(u_n)_t(x,t)\,dx \\ &= \lim_{n\to\infty} \int_\Omega u_n(x,t)\Big(\Delta u_n(x,t) + \lambda u_n(x,t)\Big)\,dx \\ &= \int_\Omega u(x,t)\Big(\Delta u(x,t) + \lambda u(x,t)\Big)\,dx \\ &= \int_\Omega u(x,t)^{1-\delta} u_t(x,t)\,dx. \end{aligned}$$

Similarly

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega u_n(x,s)^{1-\delta}(u_n)_t(x,s)\,dx \\ &= \int_\Omega u(x,s)^{1-\delta} u_t(x,s)\,dx. \end{aligned}$$

Hence, it holds

$$\int_\Omega u(x,t)^{1-\delta} u_t(x,t)\,dx < \int_\Omega u(x,s)^{1-\delta} u_t(x,s)\,dx. \ \square$$

III. Estimates near Boundary

Estimates near boundaries have been investigated in previous studies for solutions of (1). But we would like to know more exact estimates because it is very important in our purpose whether solutions can be estimated near boundaries independently of time parameter and δ.

In this section we prove the following theorem for behavior of solutions near boundaries.

***Theorem** 8:* Let $\zeta \in (0, 2^{-1})$ be arbitrarily fixed. Then there exists $\rho_\zeta > 0$ such that if $\tau \in (0,T)$, $t \in [0,\tau]$ and $\text{dist}\,(x,\partial\Omega) \leq \rho_\zeta$ then

$$u(x,t) \leq 2\zeta \max_{x\in\Omega, t\in[0,\tau]} u(x,t),$$

where Ω is a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ of C^2-class, u is a solution of (1) and $T > 0$ is the blow-up time of u.

Proof. Let $Q > 0$ and $L_\zeta > 0$ be constants satisfying

$$Q > 2\lambda \tag{6}$$

and

$$u_0(x) < \zeta \max_{x\in\Omega} u_0(x) + L_\zeta \text{dist}\,(x,\partial\Omega) \tag{7}$$

for any $x \in \Omega$.

Next, since $\partial\Omega$ is C^2-class, we can consider $R > 0$ that for any $z \in \partial\Omega$ there exists $x_z \notin \Omega$ such that a ball $B_R(x_z) := \{y\,;\,|y - x_z| < R\}$ is touching $\partial\Omega$ at $z \in \partial\Omega$ from outside of Ω.

Then we can choose constants $q > 0$ and $r > 0$ so that

$$(2R)^2(Q-\lambda) < q\Big(q - (n-2)\Big) \tag{8}$$

and

$$1 + \frac{r}{R} < \left(\frac{Q}{2\lambda}\right)^{1/q} < \min\left(2,\ \frac{\max_{x\in\Omega} u_0(x)}{L_\zeta R} + 1\right). \tag{9}$$

Remark that q and r depend only on n, Ω, λ, u_0 and ζ.

Now, for any $\tau \in (0,T)$ and $z \in \partial\Omega$, choose $x_z \notin \Omega$ so that a ball $B_R(x_z) := \{y\,;\,|y-x_z|<R\}$ is touching $\partial\Omega$ at $z \in \partial\Omega$ from outside of Ω and define

$$g_{\zeta,\tau,z,r}(x) := A_{\tau,r}\Big(R^{-q} - |x-x_z|^{-q}\Big) + \eta_{\zeta,\tau}, \quad (10)$$

where

$$\eta_{\zeta,\tau} := \zeta \max_{x\in\Omega, t\in[0,\tau]} u(x,t) \quad (11)$$

and

$$A_{\tau,r} := \frac{1}{R^{-q}-(R+r)^{-q}} \cdot \max_{x\in\Omega,t\in[0,\tau]} u(x,t). \quad (12)$$

Since

$$\begin{aligned}\eta_{\zeta,\tau} &= \zeta \max_{x\in\Omega,t\in[0,\tau]} u(x,t)\\ &< \frac{R^{-q}}{R^{-q}-(R+r)^{-q}} \cdot \zeta \max_{x\in\Omega,t\in[0,\tau]} u(x,t)\\ &= \zeta A_{\tau,r}R^{-q},\end{aligned}$$

if $R \le |x-x_z| \le R+r$, then (8) and (9) mean

$$\begin{aligned}&\Delta g_{\zeta,\tau,z,r} + \lambda g_{\zeta,\tau,z,r}\\ &= -A_{\tau,r}q(q+2-n)|x-x_z|^{-q-2}\\ &\qquad + \lambda A_{\tau,r}(R^{-q}-|x-x_z|^{-q}) + \lambda\eta_{\zeta,\tau}\\ &< -A_{\tau,r}R^{-q-2}q(q+2-n)(1+rR^{-1})^{-q-2}\\ &\qquad + \lambda A_{\tau,r}R^{-q}\Big(1-(1+rR^{-1})^{-q}\Big) + \lambda\eta_{\zeta,\tau}\\ &< -A_{\tau,r}R^{-q}\cdot 2^2(Q-\lambda)(1+rR^{-1})^{-q-2}\\ &\qquad + \lambda A_{\tau,r}R^{-q}\Big(1-(1+rR^{-1})^{-q}\Big) + \lambda\eta_{\zeta,\tau}\\ &< -A_{\tau,r}R^{-q}\Big(Q(1+rR^{-1})^{-q}-\lambda\Big) + \lambda\eta_{\zeta,\tau}\\ &< -\lambda A_{\tau,r}R^{-q} + \lambda\eta_{\zeta,\tau}\\ &< -\lambda(1-\zeta)A_{\tau,r}R^{-q} < 0.\end{aligned}$$

Hence, it holds

$$\frac{\partial g_{\zeta,\tau,z,r}}{\partial t} - u(x,t)^{\delta}(\Delta g_{\zeta,\tau,z,r} + \lambda g_{\zeta,\tau,z,r}) \ge 0 \quad (13)$$

in $\Big(\Omega \cap B_{R+r}(x_z)\Big) \times (0,\tau)$.

Next, we consider u on the parabolic boundaries. It holds that

$$u(x,t) = 0 < g_{\zeta,\tau,z,r}(x) \quad (14)$$

for $x \in \partial\Omega \cap B_{R+r}(x_z)$ and $t \in [0,\tau]$. Furthermore, it is verified by (7), (9), (10), (11) and (12) that if $0 \le t \le \tau$ and $|x-x_z| = R+r$ then

$$u(x,t) \le \max_{x\in\Omega,t\in[0,\tau]} u(x,t) < g_{\zeta,\tau,z,r}(x) \quad (15)$$

and

$$\begin{aligned}u_0(x) &< \zeta \max_{x\in\Omega} u_0(x) + L_\zeta \mathrm{dist}\,(x,\partial\Omega)\\ &\le \eta_{\zeta,\tau} + L_\zeta r\\ &\le \eta_{\zeta,\tau} + \max_{x\in\Omega} u_0(x) \le g_{\zeta,\tau,z,r}(x).\end{aligned}$$

In addition, since $-s^{-q}$ is strictly concave in $s>0$,

$$\begin{aligned}u_0(x) &\le \eta_{\zeta,\tau} + L_\zeta \mathrm{dist}\,(x,\partial\Omega)\\ &\le \eta_{\zeta,\tau} + L_\zeta(|x-x_z|-R) < g_{\zeta,\tau,z,r}(x) \quad (16)\end{aligned}$$

for any $x \in \Omega \cap B_{R+r}(x_z)$. Since (14), (15) and (16) imply that

$$u(x,t) \le g_{\zeta,\tau,z,r}(x) \quad (17)$$

on the parabolic boundaries, it is proved by the maximum principle that

$$u(x,t) \le g_{\zeta,\tau,z,r}(x) \quad (18)$$

for any $x \in \Omega \cap B_{R+r}(x_z)$ and $t \in [0,\tau]$.

Finally, consider

$$\rho_\zeta := -R + R\Big[1-\zeta\Big(1-(1+rR^{-1})^{-q}\Big)\Big]^{-1/q}. \quad (19)$$

Then ρ_ζ is independent of $z \in \partial\Omega$ and it can be easily verified that

$$0 < \rho_\zeta < \zeta r < r \quad (20)$$

for any $\zeta \in (0,2^{-1})$ and $r>0$. Hence, by (18), (19) and (20), if $x \in \Omega \cap \overline{B_{R+\rho_\zeta}(x_z)}$ and $t \in [0,\tau]$ then

$$\begin{aligned}u(x,t) &\le g_{\zeta,\tau,z,r}(x)\\ &= A_{\tau,r}\Big(R^{-q}-|x-x_z|^{-q}\Big) + \eta_{\zeta,\tau}\\ &\le A_{\tau,r}\Big(R^{-q}-(R+\rho_\zeta)^{-q}\Big) + \eta_{\zeta,\tau}\\ &= A_{\tau,r}\cdot\zeta\Big(R^{-q}-(R+r)^{-q}\Big) + \eta_{\zeta,\tau}.\end{aligned}$$

Now, (11) and (12) implies

$$\begin{aligned}\eta_{\zeta,\tau} &= \zeta \max_{x\in\Omega,t\in[0,\tau]} u(x,t)\\ &= \zeta\Big(R^{-q}-(R+r)^{-q}\Big)A_{\tau,r}\end{aligned}$$

and then we have

$$\begin{aligned}u(x,t) &\le A_{\tau,r}\cdot\zeta\Big(R^{-q}-(R+r)^{-q}\Big) + \eta_{\zeta,\tau}\\ &= 2\eta_{\zeta,\tau}\\ &= 2\zeta \max_{x\in\Omega,t\in[0,\tau]} u(x,t)\end{aligned}$$

for any $x \in \Omega \cap \overline{B_{R+\rho_\zeta}(x_z)}$. This completes this proof because ρ_ζ is independent of $z \in \partial\Omega$. □

Remark: As a consequence of the dependence on R, r and q, ρ_ζ given in (19) is independent of δ and τ and depends only on n, Ω, λ, u_0 and ζ.

Furthermore, it immediately follows Theorem 8 that $U(x,\tau) := \dfrac{u(x,\tau)}{\max_{x\in\Omega,t\in[0,\tau]} u(x,t)}$ satisfies

$$U(x,\tau) \le 2\zeta \text{ if } \mathrm{dist}\,(x,\partial\Omega) \le \rho_\zeta.$$

Therefore, Theorem 8 gives estimates for U near boundaries which are independent of τ and δ.

IV. Regional Blow-up

In this section we discuss regions in which solutions blow up with the same rate as the maximum point.

At first we show a theorem for behavior of maximum points as follows.

Theorem 9: Let $\rho_\zeta > 0$ be given in (19) for arbitrarily fixed $\zeta \in (0,2^{-1})$. Then it holds

$$\bigcup_{\tau\in[0,T)} \mathrm{MP}_\tau^u \subset \Omega_{\rho_\zeta} := \{x\in\Omega\,;\,\mathrm{dist}\,(x,\partial\Omega) > \rho_\zeta\},$$

where

$$\mathrm{MP}^u_\tau := \Big\{ y \in \Omega\,;\, u(y,\tau) = \max_{x\in\Omega, t\in[0,\tau]} u(x,t) \Big\}. \tag{21}$$

Remark: Since ρ_ζ given in (19) is independent of δ, Ω_{ρ_ζ} is also independent of δ.

Proof. In order to prove by contradiction, assume that there exists $\tau \in (0,T)$ such that $\mathrm{MP}^u_\tau \cap (\Omega_{\rho_\zeta})^c \neq \phi$. Then we would choose $y \in \mathrm{MP}^u_\tau \cap (\Omega_{\rho_\zeta})^c$, that is,

$$u(y,\tau) = \max_{x\in\Omega, t\in[0,\tau]} u(x,t) \text{ and } \mathrm{dist}\,(x, \partial\Omega) \le \rho_\zeta.$$

On the other hands, since ρ_ζ given in (19) is independent of τ, we can apply Theorem 8 and then

$$u(y,\tau) \le 2\zeta \max_{x\in\Omega, t\in[0,\tau]} u(x,t).$$

This is contradiction because of $2\zeta < 1$. □

Let n, Ω, λ and u_0 be fixed. Since solutions satisfies $\limsup_{t\nearrow T} u(x,t) = \infty$, there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that

$$t_k \nearrow T \text{ as } k \to \infty \text{ and } \mathrm{MP}^u_{t_k} \neq \phi \text{ for any } k \in \mathbb{N}, \tag{22}$$

where $\mathrm{MP}^u_{t_k}$ is defined by (21).

In addition, Theorem 9 implies that we can take a open domain S with smooth boundary satisfying

$$\mathrm{MP^u} \subset S,\ \overline{S} \subset \Omega \text{ and } \partial S \cap \overline{\mathrm{MP}^u} = \phi, \tag{23}$$

where $\mathrm{MP^u} := \bigcup_{k\in\mathbb{N}} \mathrm{MP}^u_{t_k}$.

Remark that we can choose S independently of δ because Ω_{ρ_ζ} in Theorem 9 is independent of δ. Then we provide the following theorem.

***Theorem** 10:* Let n, Ω, λ and u_0 be fixed. Consider a solution of (1), u, and a sequence $\mathcal{T} := \{t_k\}_{k\in\mathbb{N}} \subset (0,T)$ with (22). Let $y_k \in \mathrm{MP}^u_{t_k}$ and S be a open domain with smooth boundary satisfying (23).

Then there exists a positive constant $\nu_0 = \nu_0(S,\mathcal{T}) > 0$ such that for any $\delta > 0$, all of $\mathcal{T}$ and S given by solutions of (1) satisfy

$$\liminf_{k\to\infty} \int_S \frac{u(x,t_k)}{u(y_k,t_k)}\,dx \ge \nu_0.$$

Remark: In Theorem 10, ν_0 is independent of δ but may depend on n, Ω, λ and u_0 in addition to $\mathcal{T}$ and S.

***Lemma** 11:* Let $U(x,t_k) := \dfrac{u(x,t_k)}{u(y_k,t_k)}$ and consider a solution v_k of

$$\begin{cases} -\Delta v_k(x) + v_k(x) = C(S)U(x,t_k) & \text{for } x \in S \\ v_k(x) = 0 & \text{for } x \in \partial S, \end{cases}$$

where t_k, y_k and S are given in Theorem 10 and

$$C(S) := \lambda + 1 + \frac{1}{\delta t_0 \min_{x\in\overline{S}, k\in\mathbb{N}} u(x,t_k)^\delta}.$$

Then it holds

$$U(x,t_k) \le v_k(x) + \max_{x\in\partial S} U(x,t_k) \tag{24}$$

for any $x \in S$.

Proof. Since Lemma 3 implies

$$u_t = u^\delta(\Delta u + \lambda u) \ge -\frac{1}{\delta t} u \text{ in } \Omega \times (0,T),$$

we have

$$\begin{aligned} &-\Delta U(x,t_k) + U(x,t_k) \\ &\le \left(\lambda + 1 + \frac{1}{\delta t u(x,t_k)^\delta}\right) U(x,t_k) \\ &\le C(S)U(x,t_k). \end{aligned}$$

Furthermore,

$$\begin{aligned} &-\Delta\Big(v_k(x) + \max_{x\in\partial S} U(x,t_k)\Big) \\ &\qquad\qquad + \Big(v_k(x) + \max_{x\in\partial S} U(x,t_k)\Big) \\ &= C(S)U(x,t_k) + \max_{x\in\partial S} U(x,t_k) \\ &\ge C(S)U(x,t_k) \end{aligned}$$

for any $x \in S$. Hence, (24) is obtained by the comparison principle. □

Proof of Theorem 10. In order to prove this theorem by contradiction, we assume that for any $\nu > 0$ there exists $\delta > 0$ such that a solution of (1) has $\mathcal{T} = \{t_k\}_{k\in\mathbb{N}}$ and S under assumptions of this theorem which satisfy

$$\liminf_{k\to\infty} \int_S U(x,t_k)\,dx < \nu,$$

where $U(x,t_k) := \dfrac{u(x,t_k)}{u(y_k,t_k)}$. Then we would have a subsequence $\{t_{k_i}\}_{i\in\mathbb{N}}$ such that

$$\begin{aligned} \lim_{i\to\infty} \|U(\cdot,t_{k_i})\|_{L^p(S)} &\le \lim_{i\to\infty} \left(\int_S U(x,t_{k_i})^p\,dx\right)^{\frac{1}{p}} \\ &\le \lim_{i\to\infty} \left(\int_S U(x,t_{k_i})\,dx\right)^{\frac{1}{p}} \\ &< \nu^{\frac{1}{p}} \end{aligned}$$

for any $p > 1$ because of

$$U(x,t_{k_i}) = \frac{u(x,t_{k_i})}{u(y_{k_i},t_{k_i})} = \frac{u(x,t_{k_i})}{\max_{x\in\Omega, t\in[0,t_{k_i}]} u(x,t)} \le 1.$$

Now, it is well-known that there exists a positive constant $C > 0$ such that v_k given in Lemma 11 satisfies

$$\|v_k\|_{W^{2,p}(S)} \le C \cdot C(S) \|U(\cdot,t_k)\|_{L^p(S)} \tag{25}$$

for any $p > 1$. Hence, (25) leads to

$$\lim_{i\to\infty} \|v_{k_i}\|_{W^{2,p}(S)} < C \cdot C(S)\nu^{\frac{1}{p}}$$

for $p > 1$. In particular, when $p > \max\left(\dfrac{n}{2},\ 1\right)$, it is shown by the embedding theorem that

$$\lim_{i\to\infty} \max_{x\in S} |v_{k_i}(x)| < C_0 \nu^{\frac{1}{p}} \text{ for some constants } C_0 > 0$$

which implies that there exists $K_\nu \in \mathbb{N}$ such that

$$\max_{x\in S} |v_{k_i}(x)| < 2C_0\nu^{\frac{1}{p}} \text{ if } k_i \geq K_\nu.$$

Furthermore, it is verified by Lemma 11 that if $x \in S$ and $k_i \geq K_\nu$ then

$$U(x,t_{k_i}) \leq v_{k_i}(x) + \max_{x\in\partial S} U(x,t_{k_i}) < 2C_0\nu^{\frac{1}{p}} + \max_{x\in\partial S} U(x,t_{k_i}).$$

On the other hands, by $\partial S \cap \overline{\mathrm{MP}^u} = \phi$, there exists $\varepsilon_0 >$ such that

$$\max_{x\in\partial S} u(x,t_k) < u(y_k,t_k) - \varepsilon_0 \quad \text{for any } k \in \mathbb{N}.$$

Hence, if $2C_0\nu^{\frac{1}{p}} < \varepsilon_0$ then

$$U(y_{k_i},t_{k_i}) < \varepsilon_0 + \max_{x\in\partial S} U(x,t_{k_i}) < U(y_{k_i},t_{k_i})$$

for $k_i \geq K_\nu$ which is contradiction. This proof is complete. □

The following theorem follows Theorem 10 and is an important property for regions in which solutions blow up with the same rate as the maximum points.

***Theorem** 12:* Let u be a solution of (1). For any $\mu \in (0,1)$ there exists a sequence $\{t_i\}_{i\in\mathbb{N}}$ with (22) such that

$$S_\mu := \bigcap_{i\in\mathbb{N}} \left\{x \in \Omega\,;\, u(x,t_i) \geq \mu \max_{x\in\Omega, t\in[0,t_i]} u(x,t)\right\}$$

has positive Lebesgue measure.

***Remark**:* The similar properties were proved in previous studies (for instance, see [11]). We provide an exact proof by estimates in this paper.

Proof. Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence with (22) and $y_k \in \mathrm{MP}^u_{t_k}$. Since

$$\int_\Omega |U(x,t_k)|^2\,dx = \int_\Omega \left|\frac{u(x,t_k)}{u(y_k,t_k)}\right|^2 dx \leq \mathcal{L}(\Omega) < \infty,$$

we can choose a subsequence $\{t_{k_i}\}_{i\in\mathbb{N}} \subset \{t_k\}_{k\in\mathbb{N}}$ and $U_\infty \in L^2(\Omega)$ so that

$$U(x,t_{k_i}) \to U_\infty(x) \text{ almost all of } x \in \Omega \text{ as } i \to \infty,$$

where $\mathcal{L}(\cdot)$ is the Lebesgue measure. Applying the Egoroff's Theorem, it may hold that for any $\varepsilon > 0$ there exists $E \subset \Omega$ with $\mathcal{L}(E) < \varepsilon$ such that

$$U(x,t_{k_i}) \to U_\infty(x) \text{ uniformly in } \Omega\backslash E \text{ as } i \to \infty.$$

And then there exists $I_\varepsilon \in \mathbb{N}$ such that

$$\max_{x\in\Omega\backslash E} |U(x,t_{k_i}) - U(x,t_{k_j})| < \varepsilon$$

for any $j, i \geq I_\varepsilon$. This implies if we choose $\varepsilon > 0$ so that $\varepsilon < 1-\mu$ and $\left\{x \in \Omega\backslash E\,;\, U(x,t_{k_i}) \geq \mu+\varepsilon\right\} \neq \phi$ for some $i \geq I_\varepsilon$ then $\left\{x \in \Omega\backslash E\,;\, U(x,t_{k_i}) \geq \mu\right\} \neq \phi$ for any $i \geq I_\varepsilon$ and it holds

$$\begin{aligned}\bigcup_{i\geq I_\varepsilon} &\left\{x \in \Omega\backslash E\,;\, U(x,t_{k_i}) \geq \mu+\varepsilon\right\} \\ &\subset \bigcap_{i\geq I_\varepsilon} \left\{x \in \Omega\backslash E\,;\, U(x,t_{k_i}) \geq \mu\right\} \\ &\subset \bigcap_{i\geq I_\varepsilon} \left\{x \in \Omega\,;\, U(x,t_{k_i}) \geq \mu\right\}.\end{aligned} \tag{26}$$

Let $S_{\mu,k} := \left\{x \in \Omega\,;\, U(x,t_k) \geq \mu\right\}$ and consider a open domain $S \subset \bigcup_{i\geq I_\varepsilon} S_{\mu+\varepsilon,k_i}$ with smooth boundary satisfying $\partial S \cap \left(\bigcup_{i\geq I_\varepsilon} \mathrm{MP}^u_{t_{k_i}}\right) = \phi$ and $S \supset \{y_{k_i}\}_{i\geq I_\varepsilon}$. Then, by Theorem 10 and (26), there exists J_ε $(\geq I_\varepsilon)$ such that if $j \geq J_\varepsilon$ then

$$\begin{aligned}\nu_0 - \varepsilon \leq \int_S U(x,t_{k_j})\,dx &\leq \mathcal{L}(S) \\ &\leq \mathcal{L}\left(\bigcup_{i\geq I_\varepsilon} S_{\mu+\varepsilon,k_i}\right) \\ &\leq \mathcal{L}\left(\bigcup_{i\geq I_\varepsilon} S_{\mu+\varepsilon,k_i}\backslash E\right) + \varepsilon \\ &\leq \mathcal{L}\left(\bigcap_{i\geq I_\varepsilon} S_{\mu,k_i}\right) + \varepsilon,\end{aligned}$$

where ν_0 is given in Theorem 10. Therefore, this theorem is proved by letting $\varepsilon < \min\left(\frac{\nu_0}{3},\ 1-\mu\right)$ and $\{t_i\}_{i\in\mathbb{N}} := \{t_{k_i}\}_{i\geq I_\varepsilon}$. □

Furthermore, we have the following corollary which is an essential property in the next section.

***Corollary** 13:* Let n, Ω, λ, u_0 and $\mu \in (0,1)$ be fixed. Then for any $\delta > 0$, all of sequences and S_μ given in Theorem 12 by solutions of (1) satisfy

$$\mathcal{L}(S_\mu) > \nu_1 := \frac{\nu_0}{3} > 0,$$

where ν_0 is given in Theorem 10 and independent of δ.

***Remark**:* ν_1 is independent of δ but may depend on given sequences in addition to n, Ω, λ, u_0 and μ.

Proof. At the last part in the proof of Theorem 12, if $\varepsilon < \min(\nu_1,\ 1-\mu)$ then

$$\nu_1 = \frac{\nu_0}{3} < \nu_0 - 2\varepsilon \leq \mathcal{L}\left(\bigcap_{i\geq I_\varepsilon} S_{\mu,k_i}\right) = \mathcal{L}(S_\mu). \ \square$$

V. The Proof of Main Theorem

In this section we will prove Theorem 2. The first step is to show the following lemma.

***Lemma** 14:* Let $\delta > 2$ and $(u_0)^{2-\delta} \in L^1(\Omega)$. Consider $\mathcal{T} = \{t_i\}_{i\in\mathbb{N}}$ and S_μ given in Theorem 12 for a constant $\mu \in (0,1)$ and a solution u. Then there exists τ_δ and N_δ such that

$$\limsup_{i\to\infty} \int_\Omega u(x,t_i)^{2-\delta}\,dx - \int_\Omega u(x,\tau_\delta)^{2-\delta}\,dx < \left(1 - \frac{2}{\delta}\right) N_\delta - \nu_1,$$

where ν_1 is given in Corollary 13.

Proof. Since Lemma 4 implies that if $t_i > t$ then

$$\left(t_i^{\frac{1}{\delta}} u(x,t_i)\right)^{2-\delta} \leq \left(t^{\frac{1}{\delta}} u(x,t)\right)^{2-\delta},$$

we have

$$\begin{aligned}
&\int_\Omega u(x,t_i)^{2-\delta}\,dx - \int_\Omega u(x,t)^{2-\delta}\,dx\\
&= \int_{S_\mu} u(x,t_i)^{2-\delta}\,dx + \int_{\Omega\setminus S_\mu} u(x,t_i)^{2-\delta}\,dx\\
&\qquad - \int_\Omega u(x,t)^{2-\delta}\,dx\\
&\le \int_{S_\mu} u(x,t_i)^{2-\delta}\,dx + \left(\frac{t_i}{t}\right)^{\frac{\delta-2}{\delta}} \int_{\Omega\setminus S_\mu} u(x,t)^{2-\delta}\,dx\\
&\qquad - \int_\Omega u(x,t)^{2-\delta}\,dx
\end{aligned}$$

if $t_i > t$. Furthermore,

$$\begin{aligned}
&\int_{S_\mu} u(x,t_i)^{2-\delta}\,dx\\
&\le \mu^{2-\delta}\mathcal{L}(S_\mu)\Big(\max_{x\in\Omega,t\in[0,t_i]} u(x,t)\Big)^{2-\delta} \to 0
\end{aligned}$$

as $i \to \infty$. Hence, it holds

$$\begin{aligned}
&\limsup_{i\to\infty} \int_\Omega u(x,t_i)^{2-\delta}\,dx - \int_\Omega u(x,t)^{2-\delta}\,dx\\
&\le \left(\frac{T}{t}\right)^{\frac{\delta-2}{\delta}} \int_{\Omega\setminus S_\mu} u(x,t)^{2-\delta}\,dx - \int_\Omega u(x,t)^{2-\delta}\,dx\\
&= \left[\left(\frac{T}{t}\right)^{\frac{\delta-2}{\delta}} - 1\right] \int_\Omega u(x,t)^{2-\delta}\,dx\\
&\qquad - \left(\frac{T}{t}\right)^{\frac{\delta-2}{\delta}} \int_{S_\mu} u(x,t)^{2-\delta}\,dx\\
&< \left[\left(\frac{T}{t}\right)^{\frac{\delta-2}{\delta}} - 1\right] K_\delta - \mathcal{L}(S_\mu)\cdot\Big(\max_{x\in\Omega} u(x,t)\Big)^{2-\delta}
\end{aligned}$$

for any $t \in (0,T)$, where T is the blow-up time of u and K_δ is given in Lemma 5.

Now, defined τ_δ by

$$\tau_\delta := \frac{2m_\delta}{\lambda\delta + \sqrt{(\lambda\delta)^2 + 4\mathcal{L}(S_\mu)\cdot\lambda\delta m_\delta (TK_\delta)^{-1}}},$$

where m_δ is given in Lemma 6. And choose a constant M_δ so that

$$\mathcal{L}(S_\mu) > \nu_1 = \frac{(M_\delta)^{-1}\lambda\delta T K_\delta m_\delta}{\Big(m_\delta - (M_\delta)^{-1}\Big)^2},$$

where ν_1 is given in Corollary 13. Then it is verified that M_δ satisfies

$$(m_\delta - \lambda\delta\tau_\delta)^{-1} < M_\delta$$

and then it follows Lemma 6 that

$$\max_{x\in\Omega} u(x,\tau_\delta) \le (m_\delta - \lambda\delta\tau_\delta)^{-\frac{1}{\delta}} < (M_\delta)^{\frac{1}{\delta}}.$$

Furthermore, since

$$x^\alpha - 1 \le \alpha(x-1) \text{ if } 0<\alpha<1 \text{ and } x>0$$

and

$$a^x \ge 1 + (\log a)x \text{ if } a>0 \text{ and } x\in\mathbb{R},$$

we have

$$\begin{aligned}
&\limsup_{i\to\infty} \int_\Omega u(x,t_i)^{2-\delta}\,dx - \int_\Omega u(x,\tau_\delta)^{2-\delta}\,dx\\
&< \left[\left(\frac{T}{\tau_\delta}\right)^{\frac{\delta-2}{\delta}} - 1\right] K_\delta - \mathcal{L}(S_\mu)\cdot\Big(\max_{x\in\Omega} u(x,\tau_\delta)\Big)^{2-\delta}\\
&\le \frac{\delta-2}{\delta}\left(\frac{T}{\tau_\delta} - 1\right) K_\delta\\
&\qquad - \mathcal{L}(S_\mu)\left[1 + (2-\delta)\log\max_{x\in\Omega} u(x,\tau_\delta)\right]\\
&< \left(1 - \frac{2}{\delta}\right) N_\delta - \nu_1,
\end{aligned}$$

where

$$N_\delta := \left(\frac{T}{\tau_\delta} - 1\right) K_\delta + \mathcal{L}(S_\mu)\log M_\delta \qquad (27)$$

and ν_1 is given in Corollary 13. Hence, this proof is complete. □

Remark: τ_δ and N_δ in Lemma 14 depend on n, Ω, λ, u_0, μ and δ. In addition, if u_0 can be fixed for any δ then $\limsup_{\delta\nearrow\infty} N_\delta = \infty$ because of $\mathcal{L}(S_\mu) > \nu_1$ and

$$\limsup_{\delta\nearrow\infty} M_\delta > \limsup_{\delta\nearrow\infty} (m_\delta - \lambda\delta\tau_\delta)^{-1} = \infty.$$

Then we can prove the essential property for behavior of $\int_\Omega u(x,t)^{2-\delta}\,dx$ as follows.

***Lemma** 15:* Let u_0 be a initial function satisfying $(u_0)^{2-\delta_1} \in L^1(\Omega)$ for some $\delta_1 > 2$. Then there exists $\delta_0 \in (2,\delta_1]$ such that we can choose σ_δ and τ_δ for any $\delta \in (2,\delta_0)$ so that $\sigma_\delta > \tau_\delta$ and

$$\int_\Omega u(x,\sigma_\delta)^{2-\delta}\,dx < \int_\Omega u(x,\tau_\delta)^{2-\delta}.$$

Proof. It is easily verified that $(u_0)^{2-\delta} \in L^1(\Omega)$ for any $\delta \in (2,\delta_1)$.

Let τ_δ and N_δ be given in Lemma 14.

In the case that δ_1 satisfies $\sup_{\delta\in(2,\delta_1)} N_\delta \le \nu_1$, if $2<\delta<\delta_1$ then Lemma 14 leads to

$$\begin{aligned}
&\limsup_{i\to\infty} \int_\Omega u(x,t_i)^{2-\delta}\,dx\\
&< \int_\Omega u(x,\tau_\delta)^{2-\delta}\,dx + \left(1 - \frac{2}{\delta}\right) N_\delta - \nu_1 \qquad (28)\\
&\le \int_\Omega u(x,\tau_\delta)^{2-\delta}\,dx - \frac{2}{\delta} N_\delta.
\end{aligned}$$

Otherwise, in the case that δ_1 satisfies $\sup_{\delta\in(2,\delta_1)} N_\delta > \nu_1$, define $\delta_2 > 2$ by

$$\delta_2 := \min\left(\delta_1,\ 2\left(1 - \frac{\nu_1}{\sup_{\delta\in(2,\delta_1)} N_\delta}\right)^{-1}\right).$$

Since

$$\left(1 - \frac{2}{\delta_2}\right) \sup_{\delta\in(2,\delta_1)} N_\delta \le \nu_1,$$

Lemma 14 also leads to

$$\begin{aligned}&\limsup_{i\to\infty}\int_\Omega u(x,t_i)^{2-\delta}\,dx\\&<\int_\Omega u(x,\tau_\delta)^{2-\delta}\,dx+\left(1-\frac{2}{\delta}\right)N_\delta-\nu_1\\&<\int_\Omega u(x,\tau_\delta)^{2-\delta}\,dx-\left(\frac{2}{\delta}-\frac{2}{\delta_2}\right)N_\delta\end{aligned}\qquad(29)$$

for any $\delta\in(2,\delta_2)$.

Therefore, it is proved that if $(u_0)^{2-\delta_1}\in L^1(\Omega)$ for some $\delta_1>2$ then

$$\delta_0:=\begin{cases}\delta_1 & \text{if } \sup_{\delta\in(2,\delta_1)} N_\delta\le\nu_1,\\ \delta_2 & \text{if } \sup_{\delta\in(2,\delta_1)} N_\delta>\nu_1\end{cases}$$

satisfies $\delta_0>2$ and it follows (28) and (29) that for any $\delta\in(2,\delta_0)$ there exists $\sigma_\delta\in\{t_i\}_{i\in\mathbb{N}}$ such that $\sigma_\delta>\tau_\delta$ and

$$\int_\Omega u(x,\sigma_\delta)^{2-\delta}\,dx<\int_\Omega u(x,\tau_\delta)^{2-\delta}$$

which completes this proof. □

Theorem 2 follows Lemma 15.

Proof of Theorem 2. Let δ_0 be given in Lemma 15. Then for any $\delta\in(2,\delta_0)$ there exists $t_\delta\in(\tau_\delta,\sigma_\delta)$ such that

$$\frac{d}{dt}\int_\Omega u(x,t)^{2-\delta}\,dx\bigg|_{t=t_\delta}<0,$$

where τ_δ and σ_δ is given in Lemma 15. This implies that

$$\begin{aligned}&\int_\Omega u(x,t)^{1-\delta}u_t(x,t)\,dx\bigg|_{t=t_\delta}\\&=-\frac{1}{\delta-2}\,\frac{d}{dt}\int_\Omega u(x,t)^{2-\delta}\,dx\bigg|_{t=t_\delta}>0,\end{aligned}$$

Therefore, it follows Lemma 7 that

$$\int_\Omega u(x,t)^{1-\delta}u_t(x,t)\,dx>0 \text{ for any } t\in(t_\delta,T).$$

Therefore, the proof of Theorem 2 is complete. □

***Remark**:* [1] provided a conjecture that “weak eventual mononicity” holds for any $\delta>2$. On the other hands, δ_0 given in theorem 2 is upper bounded even if $\delta_1\to\infty$. This might be the outcome from the technique of our proof, and so the case for large enough δ is open.

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