Dynamics of a Harvesting Schoener's Competition Model with Time-varying Delays and Impulsive Effects

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Abstract—**In this paper, we consider a kind of delayed Schoener's competition model with harvesting terms and impulsive effects. By means of Mawhin's continuation theorem of coincidence degree theory, some sufficient conditions are obtained for the existence of at least four positive almost periodic solutions for the above model. Further, by using the comparison theorem and constructing a suitable Lyapunov functional, the global asymptotic stability of the model is also investigated. To the best of the author's knowledge, so far, the results of this paper are completely new. An example and numerical simulations are employed to illustrate the main results in this paper.**

Index Terms—**Multiplicity; Positive almost periodic solution; Coincidence degree; Schoener; Harvesting.**

I. Introduction

In recent years, the Schoener's competition system has been studied by many scholars. Topics such as existence, uniqueness and global attractivity of positive periodic solutions of the system were extensively investigated and many excellent results have been derived (see [1-9] and the references cited therein). In [6], Liu, Xu and Wang proposed and studied the global stability of the following Schoener's competition model with pure-delays:

$$\begin{cases} \dot{y}_1(t) = y_1(t)\Big[\frac{a_{10}(t)}{y_1(t-\tau_{10})+m_1(t)} - a_{11}(t)y_1(t-\tau_{11}) \\ \qquad -a_{12}(t)y_2(t-\tau_{12}) - c_1(t)\Big], \\ \dot{y}_2(t) = y_2(t)\Big[\frac{a_{20}(t)}{y_2(t-\tau_{20})+m_2(t)} - a_{21}(t)y_1(t-\tau_{21}) \\ \qquad -a_{22}(t)y_2(t-\tau_{22}) - c_2(t)\Big], \end{cases} \tag{1.1}$$

where $y_1(t)$, $y_2(t)$ are population densities of species y_1, y_2 at time t, respectively.

In real world, the ecological systems are usually perturbed by human exploitation activities such as planting and harvesting and so on. In order to obtain a more accurate description for such phenomenon, the impulsive differential equations play an important role. In [9], Zhang et al. studied the following almost periodic Schoener's competition model with pure-delays and impulsive effects:

$$\begin{cases} \dot{y}_1(t) &= y_1(t)\Big[\frac{a_{10}(t)}{y_1(t-\tau_{10})+m_1(t)} - a_{11}(t)y_1(t-\tau_{11}) \\ &\qquad -a_{12}(t)y_2(t-\tau_{12}) - c_1(t)\Big], \\ \dot{y}_2(t) &= y_2(t)\Big[\frac{a_{20}(t)}{y_2(t-\tau_{20})+m_2(t)} - a_{21}(t)y_1(t-\tau_{21}) \\ &\qquad -a_{22}(t)y_2(t-\tau_{22}) - c_2(t)\Big], \\ y_1(t_k) &= (1+p_{1k})y_1(t_k), \\ y_2(t_k) &= (1+p_{2k})y_2(t_k), \quad k \in \mathbb{Z}^+ := \{0,1,\ldots\}. \end{cases}$$

In [9], sufficient conditions which guarantee the permanence of the model and the existence of a unique uniformly asymptotically stable positive almost periodic solution are obtained. Topics such as existence, uniqueness and global attractivity of positive periodic solutions or almost periodic solutions of the system with impulses were extensively investigated, and many excellent results have been derived (see [9-12] and the references cited therein).

In many earlier studies, it has been shown that harvesting has a strong impact on dynamic evolution of a population, e.g., see [13-16]. So the study of the population dynamics with harvesting is becoming a very important subject in mathematical bio-economics. On the other hand, time delays represent an additional level of complexity that can be incorporated in a more detailed analysis of a particular system. Specially, in the real world, the delays in differential equations of biological phenomena are usually time-varying. Thus, it is worthwhile continuing to consider the following impulsive Schoener's competition model with harvesting terms and time-varying delays:

$$\begin{cases} \dot{y}_1(t) &= y_1(t)\Big[\frac{a_{10}(t)}{y_1(t-\tau_{10}(t))+m_1(t)} \\ &\qquad -\sum_{j=1}^{2} a_{1j}(t)y_j(t-\tau_{1j}(t)) - c_1(t)\Big] - h_1(t), \\ \dot{y}_2(t) &= y_2(t)\Big[\frac{a_{20}(t)}{y_2(t-\tau_{20}(t))+m_2(t)} \\ &\qquad -\sum_{j=1}^{2} a_{2j}(t)y_j(t-\tau_{2j}(t)) - c_2(t)\Big] - h_2(t), \\ y_1(t_k) &= (1+p_{1k})y_1(t_k), \\ y_2(t_k) &= (1+p_{2k})y_2(t_k), \quad k \in \mathbb{Z}^+, \end{cases} \tag{1.2}$$

where h_1 and h_2 represent harvesting terms, $p_{ik} > -1$, $i = 1, 2$, $k \in \mathbb{Z}^+$.

For the last few years, by utilizing Mawhin's continuation theorem of coincidence degree theory, many scholars are concerning with the existence of multiple positive periodic

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solutions for some non-linear ecosystems with harvesting terms, e.g., see [17-23]. However, in real world phenomenon, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples, see Example 1.1) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity. Unlike the periodic oscillation, owing to the complexity of the almost periodic oscillation, it is hard to study the existence of positive almost periodic solutions of non-linear ecosystems by using Mawhin's continuation theorem. Therefore, to the best of the author's knowledge, so far, there is no paper concerning with the multiplicity of positive almost periodic solutions of system (1.2) by using Mawhin's continuation theorem. Stimulated by the above reason, the main purpose of this paper is to establish sufficient conditions for the existence of multiple positive almost periodic solutions to system (1.2) by applying Mawhin's continuation theorem of coincidence degree theory.

Example 1. Let us consider the following simple single population model with harvesting term:

$$\dot{N}(t) = N(t)\Big[\frac{2+|\sin(\sqrt{2}t)|}{N(t)+1} \\ -(3+|\sin(\sqrt{3}t)|)N(t)\Big] - 0.01. \qquad (1.3)$$

In Eq. (1.3), $2+|\sin(\sqrt{2}t)|$ is $\frac{\sqrt{2}\pi}{2}$-periodic function and $3+|\sin(\sqrt{3}t)|$ is $\frac{\sqrt{3}\pi}{3}$-periodic function, which imply that Eq. (1.3) is with incommensurable periods. Then there is no a priori reason to expect the existence of positive periodic solutions of Eq. (1.3). Thus, it is significant to study the existence of positive almost periodic solutions of Eq. (1.3).

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}^+$ denote the sets of real numbers, integers and positive integers, respectively, $C(\mathbb{X},\mathbb{Y})$ and $C^1(\mathbb{X},\mathbb{Y})$ be the space of continuous functions and continuously differential functions which map $\mathbb{X}$ into $\mathbb{Y}$, respectively. Especially, $C(\mathbb{X}) := C(\mathbb{X},\mathbb{X})$, $C^1(\mathbb{X}) := C^1(\mathbb{X},\mathbb{X})$. Related to a continuous bounded function f, we use the following notations:

$$f^- = \inf_{s\in\mathbb{R}} f(s), \quad f^+ = \sup_{s\in\mathbb{R}} f(s).$$

Throughout this paper, we always make the following assumption for system (1.2):

(H_1) All the coefficients in system (1.2) are nonnegative almost periodic functions with $a_{ii}^- > 0$, $m_i^- > 0$ and $h_i^- > 0$, $i = 1,2$.

(H_2) $P_i(t) = \prod_{0<t_k<t}(1+p_{ik}), i = 1,2, k\in\mathbb{Z}$ is almost periodic function.

The organization of this Letter is as follows. In Section 2, we change system (1.2) into the corresponding non-impulsive system. In Section 3, some preparations are made. In Section 4, by using Mawhin's continuation theorem of coincidence degree theory, we establish sufficient conditions for the existence of at least four positive almost periodic solutions to system (1.2). An illustrative example and numerical simulations are given in Section 5.

II. Relation of the impulsive system and the corresponding non-impulsive system

Consider the following system

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Big[\frac{a_{10}(t)}{P_1(t)x_1(t-\tau_{10}(t))+m_1(t)} \\ \qquad - \sum_{j=1}^{2} A_{1j}(t)x_j(t-\tau_{1j}(t)) - c_1(t)\Big] - h_1(t), \\ \dot{x}_2(t) = x_2(t)\Big[\frac{a_{20}(t)}{P_2(t)x_2(t-\tau_{20}(t))+m_2(t)} \\ \qquad - \sum_{j=1}^{2} A_{2j}(t)x_j(t-\tau_{2j}(t)) - c_2(t)\Big] - h_2(t), \end{cases} \qquad (2.1)$$

where $A_{ij}(t) = P_i(t)a_{ij}(t) = \prod_{0<t_k<t}(1+p_{ik})a_{ij}(t)$, $i = 1,2$, $j = 1,2$.

Lemma 1. *For systems* (1.2) *and* (2.1), *one has the following results:*

(1) *if* $(x_1(t),x_2(t))^T$ *is a solution of system* (2.1), *then*

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \prod_{0<t_k<t}(1+p_{1k})x_1(t) \\ \prod_{0<t_k<t}(1+p_{2k})x_2(t) \end{pmatrix}$$

is a solution of system (1.2);

(2) *if* $(y_1(t),y_2(t))^T$ *is a solution of system* (1.2), *then*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \prod_{0<t_k<t}(1+p_{1k})^{-1}y_1(t) \\ \prod_{0<t_k<t}(1+p_{2k})^{-1}y_2(t) \end{pmatrix}$$

is a solution of system (2.1).

Proof: **(1)** Suppose that $(x_1(t),x_2(t))^T$ is a solution of system (2.1). Let $y_i(t) = \prod_{0<t_k<t}(1+p_{ik})x_i(t)$, $i = 1,2$, then for any $t \neq t_k$, by substituting $x_i(t) = \prod_{0<t_k<t}(1+p_{ik})^{-1}y_i(t)$, $i = 1,2$ into system (2.1), we can easily verify that the first two equations of system (1.2) hold.

For $t = t_k$, we have

$$\begin{aligned} y_i(t_k^+) &= \lim_{t\to t_k^+} \prod_{0<t_k<t}(1+p_{ik})x_i(t) \\ &= \prod_{0<t_j\le t_k}(1+p_{ik})x_i(t_k) \\ &= (1+p_{ik}) \prod_{0<t_j<t_k}(1+p_{ik})x_i(t_k) \\ &= (1+p_{ik})y_i(t_k), \ \ i = 1,2. \end{aligned}$$

So the last two equations of system (1.2) also hold. Thus $(y_1(t),y_2(t))^T$ is a solution of system (1.2). This proves the conclusion of **(1)**.

(2) We first show that $x_i(t)$, $i = 1,2$ is continuous. Since $x_i(t)$ is continuous on each interval $(t_k,t_{k+1}]$, it is sufficient to check the continuity of $x_i(t)$ at the impulse points t_k, $i = 1,2$. Since $x_i(t) = \prod_{0<t_k<t}(1+p_{ik})^{-1}y_i(t)$, $i = 1,2$, we have

$$\begin{aligned} x_i(t_k^+) &= \prod_{0<t_j\le t_k}(1+p_{ik})^{-1}y_i(t_k^+) \\ &= \prod_{0<t_j<t_k}(1+p_{ik})^{-1}y_i(t_k) = x_i(t_k), \ \ i = 1,2, \end{aligned}$$

$$x_i(t_k^-) = \prod_{0<t_j<t_k}(1+p_{ik})^{-1}y_i(t_k^-)$$

$$= \prod_{0<t_j<t_k} (1+p_{ik})^{-1} y_i(t_k) = x_i(t_k), \ \ i=1,2.$$

Thus $x_i(t)$, $i=1,2$ is continuous on $[0,+\infty)$. It is easy to check that $(x_1(t),x_2(t))^T$ satisfies (2.1). Therefore, it is a solution of system (2.1). This completes the proof of Lemma 2. ■

III. Some lemmas

Definition 1. ([24]) $x \in C(\mathbb{R},\mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $\|x(t+\tau)-x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$, where $\|\cdot\|$ is arbitrary norm of $\mathbb{R}^n$. τ is called to the ϵ-almost period of x, $T(x,\epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x,\epsilon)$. The collection of those functions is denoted by $AP(\mathbb{R},\mathbb{R}^n)$. Let $AP(\mathbb{R}) := AP(\mathbb{R},\mathbb{R})$.

Lemma 2. ([25]) *Assume that $x \in AP(\mathbb{R}) \cap C^1(\mathbb{R})$ with $\dot{x} \in C(\mathbb{R})$, for $\forall \epsilon > 0$, we have the following conclusions:*

(1) *there is a point $\xi_\epsilon \in [0,+\infty)$ such that $x(\xi_\epsilon) \in [x^* - \epsilon, x^*]$ and $\dot{x}(\xi_\epsilon) = 0$;*

(2) *there is a point $\eta_\epsilon \in [0,+\infty)$ such that $x(\eta_\epsilon) \in [x_*, x_* + \epsilon]$ and $\dot{x}(\eta_\epsilon) = 0$.*

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin [26].

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\dim\mathrm{Ker}L = \mathrm{codim}\mathrm{Im}L < +\infty$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I-Q)$. It follows that $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : (I-P)\mathbb{X} \to \mathrm{Im}L$ is invertible and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Lemma 3. ([26]) *Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. If all the following conditions hold*:

(a) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0,1)$;

(b) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;

(c) $\deg\{JQN, \Omega\cap\mathrm{Ker}L, 0\} \neq 0$, *where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is an isomorphism.*

Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

Under the invariant transformation $(x_1,x_2)^T = (e^u, e^v)^T$, system (2.1) reduces to

$$\begin{cases} \dot{u}(t) = \frac{a_{10}(t)}{P_1(t)e^{u(t-\tau_{10}(t))}+m_1(t)} - A_{11}(t)e^{u(t-\tau_{11}(t))} \\ \qquad -A_{12}(t)e^{v(t-\tau_{12}(t))} - c_1(t) - \frac{h_1(t)}{e^{u(t)}}, \\ \dot{v}(t) = \frac{a_{20}(t)}{P_2(t)e^{v(t-\tau_{20}(t))}+m_2(t)} - A_{21}(t)e^{u(t-\tau_{21}(t))} \\ \qquad -A_{22}(t)e^{v(t-\tau_{22}(t))} - c_2(t) - \frac{h_2(t)}{e^{v(t)}}. \end{cases} \tag{3.1}$$

For $f \in AP(\mathbb{R})$, we denote by

$$\bar{f} = m(f) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s,$$

$$\Lambda(f) = \Big\{\varpi \in \mathbb{R} : \lim_{T\to\infty}\frac{1}{T}\int_0^T f(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s \neq 0\Big\},$$

$$\mathrm{mod}(f) = \Big\{\sum_{j=1}^m n_j\varpi_j : n_j \in \mathbb{Z}, m \in \mathbb{N}, \varpi_j \in \Lambda(f)\Big\}$$

the mean value, the set of Fourier exponents and the module of f, respectively.

Set $\mathbb{X} = \mathbb{Y} = \mathbb{V}_1 \bigoplus \mathbb{V}_2$, where

$$\begin{aligned} \mathbb{V}_1 = \ & \Big\{z = (u,v)^T \in AP(\mathbb{R},\mathbb{R}^2) : \\ & \mathrm{mod}(u) \subseteq \mathrm{mod}(L_u), \\ & \mathrm{mod}(v) \subseteq \mathrm{mod}(L_v), \forall \varpi \in \Lambda(u)\cup\Lambda(v), |\varpi| \geq \theta_0\Big\}, \\ & \mathbb{V}_2 = \big\{z = (u,v)^T \equiv (k_1,k_2)^T, k_1,k_2 \in \mathbb{R}\big\}, \end{aligned}$$

where

$$\begin{aligned} L_u = L_u(t,\varphi) = & \frac{a_{10}(t)}{P_1(t)e^{\varphi_1(-\tau_{10}(0))}+m_1(t)} - A_{11}(t)e^{\varphi_1(-\tau_{11}(0))} \\ & -A_{12}(t)e^{\varphi_2(-\tau_{12}(0))} - c_1(t) - \frac{h_1(t)}{e^{\varphi_1(0)}}, \end{aligned}$$

$$\begin{aligned} L_v = L_v(t,\varphi) = & \frac{a_{20}(t)}{P_2(t)e^{\varphi_2(-\tau_{20}(0))}+m_2(t)} - A_{21}(t)e^{\varphi_1(-\tau_{21}(0))} \\ & -A_{22}(t)e^{\varphi_2(-\tau_{22}(0))} - c_2(t) - \frac{h_2(t)}{e^{\varphi_2(0)}}, \end{aligned}$$

$\varphi = (\varphi_1,\varphi_2)^T \in C([-\tau,0],\mathbb{R}^2)$, $\tau = \max_{i=1,2;j=0,1,2}\{\tau_{ij}^+\}$, θ_0 is a given positive constant. Define the norm

$$\|z\|_{\mathbb{X}} = \max\Big\{\sup_{s\in\mathbb{R}}|u(s)|, \sup_{s\in\mathbb{R}}|v(s)|\Big\}.$$

Similar to the proof as that in articles [25,27], it follows that

Lemma 4. *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with $\|\cdot\|_{\mathbb{X}}$.*

Lemma 5. *Let $L : \mathbb{X} \to \mathbb{Y}$, $Lz = L(u,v)^T = (\dot{u},\dot{v})^T$, then L is a Fredholm mapping of index zero.*

Lemma 6. *Define $N : \mathbb{X} \to \mathbb{Y}$, $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ by*

$$N\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{a_{10}(t)}{P_1(t)e^{u(t-\tau_{10}(t))}+m_1(t)} - A_{11}(t)e^{u(t-\tau_{11}(t))} \\ -A_{12}(t)e^{v(t-\tau_{12}(t))} - c_1(t) - \frac{h_1(t)}{e^{u(t)}} \\ \frac{a_{20}(t)}{P_2(t)e^{v(t-\tau_{20}(t))}+m_2(t)} - A_{21}(t)e^{u(t-\tau_{21}(t))} \\ -A_{22}(t)e^{v(t-\tau_{22}(t))} - c_2(t) - \frac{h_2(t)}{e^{v(t)}} \end{pmatrix},$$

$$P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m(u) \\ m(v) \end{pmatrix} = Qz, \quad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{X} = \mathbb{Y}.$$

Then N is L-compact on $\bar{\Omega}$(Ω is an open and bounded subset of $\mathbb{X}$).

IV. Main results

Let

$$\mu=\ln\frac{a_{10}^{+}}{m_1^{-}A_{11}^{-}}+\frac{a_{10}^{+}}{m_1^{-}}\tau_{11}^{+},\quad \nu=\ln\frac{a_{20}^{+}}{m_2^{-}A_{22}^{-}}+\frac{a_{20}^{+}}{m_2^{-}}\tau_{22}^{+},$$

$$\lambda_1=\frac{a_{10}^{-}}{P_1^{+}e^{\mu}+m_1^{+}}-A_{12}^{+}e^{\nu}-c_1^{+},$$

$$\lambda_2=\frac{a_{20}^{-}}{P_2^{+}e^{\nu}+m_2^{+}}-A_{21}^{+}e^{\mu}-c_2^{+}.$$

Now we give a assumption as follows:

(H_3) $\lambda_i>2\sqrt{A_{ii}^{+}h_i^{+}}$, $i=1,2$.

And define

$$l_i^{\pm}=\frac{\lambda_i\pm\sqrt{\lambda_i^2-4A_{ii}^{+}h_i^{+}}}{2A_{ii}^{+}},\quad i=1,2.$$

Theorem 1. *Assume that (H_1)-(H_3) hold, then system (1.2) admits at least four positive almost periodic solutions.*

Proof: It is easy to see that if system (3.1) has one almost periodic solution $(u,v)^T$, then $(x_1,x_2)^T=(e^u,e^v)^T$ is a positive almost periodic solution of system (2.1). First of all, we show that system (2.1) has four positive almost periodic solutions.

In order to use Lemma 3, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ as those in Lemma 4 and L,N,P,Q the same as those defined in Lemmas 5 and 6, respectively. It remains to search for an appropriate open and bounded subset $\Omega\subseteq\mathbb{X}$.

Corresponding to the operator equation $Lz=\lambda z$, $\lambda\in(0,1)$, we have

$$\begin{cases}\dot u(t)=\lambda\Big[\frac{a_{10}(t)}{P_1(t)e^{u(t-\tau_{10}(t))}+m_1(t)}-A_{11}(t)e^{u(t-\tau_{11}(t))}\\ \qquad -A_{12}(t)e^{v(t-\tau_{12}(t))}-c_1(t)-\frac{h_1(t)}{e^{u(t)}}\Big],\\ \dot v(t)=\lambda\Big[\frac{a_{20}(t)}{P_2(t)e^{v(t-\tau_{20}(t))}+m_2(t)}-A_{21}(t)e^{u(t-\tau_{21}(t))}\\ \qquad -A_{22}(t)e^{v(t-\tau_{22}(t))}-c_2(t)-\frac{h_2(t)}{e^{v(t)}}\Big].\end{cases}\tag{4.1}$$

Suppose that $z=(u,v)^T\in\mathrm{Dom}L\subseteq\mathbb{X}$ is a solution of system (4.1) for some $\lambda\in(0,1)$, where $\mathrm{Dom}L=\{z=(u,v)^T\in\mathbb{X}:u,v\in C^1(\mathbb{R}),\dot u,\dot v\in C(\mathbb{R})\}$.

By Lemma 2, for $\forall\epsilon\in(0,1)$, there are four points $\xi=\xi(\epsilon)$, $\zeta=\zeta(\epsilon)$, $\eta=\eta(\epsilon)$ and $\varsigma=\varsigma(\epsilon)\in[0,+\infty)$ such that

$$\dot u(\xi)=0,\quad u(\xi)\in[u^*-\epsilon,u^*];\qquad \dot v(\zeta)=0,\quad v(\zeta)\in[v^*-\epsilon,v^*],\tag{4.2}$$

$$\dot u(\eta)=0,\quad u(\eta)\in[u_*,u_*+\epsilon];\qquad \dot v(\varsigma)=0,\quad v(\varsigma)\in[v_*,v_*+\epsilon],\tag{4.3}$$

where $u^*=\sup_{s\in\mathbb{R}}u(s)$ and $v^*=\sup_{s\in\mathbb{R}}v(s)$, $u_*=\inf_{s\in\mathbb{R}}u(s)$ and $v_*=\inf_{s\in\mathbb{R}}v(s)$.

Further, in view of (H_3), we may assume that the above ϵ is small enough so that

$$a_{i0}^{+}>2m_i^{-}\sqrt{A_{ii}^{-}h_i^{-}e^{-\epsilon}},\quad \lambda_i>2\sqrt{A_{ii}^{+}h_i^{+}e^{\epsilon}},\quad i=1,2.$$

From system (4.1), it follows from (4.2)-(4.3) that

$$\begin{cases}0=\frac{a_{10}(\xi)}{P_1(t)e^{u(\xi-\tau_{10}(\xi))}+m_1(\xi)}-A_{11}(\xi)e^{u(\xi-\tau_{11}(\xi))}\\ \quad -A_{12}(\xi)e^{v(\xi-\tau_{12}(\xi))}-c_1(\xi)-\frac{h_1(\xi)}{e^{u(\xi)}},\\ 0=\frac{a_{20}(\zeta)}{P_2(t)e^{v(\zeta-\tau_{20}(\zeta))}+m_2(\zeta)}-A_{21}(\zeta)e^{u(\zeta-\tau_{21}(\zeta))}\\ \quad -A_{22}(\zeta)e^{v(\zeta-\tau_{22}(\zeta))}-c_2(\zeta)-\frac{h_2(\zeta)}{e^{v(\zeta)}},\end{cases}\tag{4.4}$$

$$\begin{cases}0=\frac{a_{10}(\eta)}{P_1(t)e^{u(\eta-\tau_{10}(\eta))}+m_1(\eta)}-A_{11}(\eta)e^{u(\eta-\tau_{11}(\eta))}\\ \quad -A_{12}(\eta)e^{v(\eta-\tau_{12}(\eta))}-c_1(\eta)-\frac{h_1(\eta)}{e^{u(\eta)}},\\ 0=\frac{a_{20}(\varsigma)}{P_2(t)e^{v(\varsigma-\tau_{20}(\varsigma))}+m_2(\varsigma)}-A_{21}(\varsigma)e^{u(\varsigma-\tau_{21}(\varsigma))}\\ \quad -A_{22}(\varsigma)e^{v(\varsigma-\tau_{22}(\varsigma))}-c_2(\varsigma)-\frac{h_2(\varsigma)}{e^{v(\varsigma)}}.\end{cases}\tag{4.5}$$

In view of (4.5), we have from (4.3) that

$$\begin{aligned}A_{11}^{-}e^{u_*}+\frac{h_1^{-}}{e^{u_*+\epsilon}}&\le A_{11}(\eta)e^{u(\eta-\tau_{11}(\eta))}+\frac{h_1(\eta)}{e^{u(\eta)}}\\ &\le\frac{a_{10}(\eta)}{P_1(\eta)e^{u(\eta-\tau_{10}(\eta))}+m_1(\eta)}\\ &\le\frac{a_{10}^{+}}{m_1^{-}}.\end{aligned}$$

That is,

$$A_{11}^{-}e^{2u_*}-\frac{a_{10}^{+}}{m_1^{-}}e^{u_*}+h_1^{-}e^{-\epsilon}\le 0.\tag{4.6}$$

Letting $\epsilon\to0$ in (4.6) leads to

$$A_{11}^{-}e^{2u_*}-\frac{a_{10}^{+}}{m_1^{-}}e^{u_*}+h_1^{-}\le 0,$$

which implies that

$$\ln\rho^{-}\le u_*\le\ln\rho^{+},\tag{4.7}$$

where

$$\rho^{\pm}=\frac{\frac{a_{10}^{+}}{m_1^{-}}\pm\sqrt{\left[\frac{a_{10}^{+}}{m_1^{-}}\right]^2-4A_{11}^{-}h_1^{-}}}{2A_{11}^{-}}.$$

Similarly,

$$\ln\varrho^{-}\le v_*\le\ln\varrho^{+},\tag{4.8}$$

where

$$\varrho^{\pm}=\frac{\frac{a_{20}^{+}}{m_2^{-}}\pm\sqrt{\left[\frac{a_{20}^{+}}{m_2^{-}}\right]^2-4A_{22}^{-}h_2^{-}}}{2A_{22}^{-}}.$$

By (4.4), we have

$$\begin{aligned}A_{11}^{-}e^{u(\xi-\tau_{11}(\xi))}&\le A_{11}(\xi)e^{u(\xi-\tau_{11}(\xi))}\\ &\le\frac{a_{10}(\xi)}{P_1(\xi)e^{u(\xi-\tau_{10}(\xi))}+m_1(\xi)}\le\frac{a_{10}^{+}}{m_1^{-}},\end{aligned}$$

which implies that

$$u(\xi-\tau_{11}(\xi))<\ln\frac{a_{10}^{+}}{m_1^{-}A_{11}^{-}}.\tag{4.9}$$

Since

$$\begin{aligned}\int_{\xi-\tau_{11}(\xi)}^{\xi}\dot u(s)\,\mathrm{d}s=\int_{\xi-\tau_{11}(\xi)}^{\xi}\Big[&\frac{a_{10}(s)}{P_1(s)e^{u(s-\tau_{10}(s))}+m_1(s)}\\ &-A_{11}(s)e^{u(s-\tau_{11}(s))}\\ &-A_{12}(s)e^{v(s-\tau_{12}(s))}\end{aligned}$$

$$-c_1(s) - \frac{h_1(s)}{e^{u(s)}}\Big] \mathrm{d}s$$

$$\leq \int_{\xi-\tau_{11}(\xi)}^{\xi} \frac{a_{10}(s)}{P_1(s)e^{u(s-\tau_{10}(s))} + m_1(s)} \mathrm{d}s$$

$$\leq \frac{a_{10}^+}{m_1^-}\tau_{11}^+. \quad (4.10)$$

From (4.9)-(4.10), it follows that

$$u(\xi) = u(\xi - \tau_{11}(\xi)) + \int_{\xi-\tau_{11}(\xi)}^{\xi} \dot{u}(s)\,\mathrm{d}s \leq \ln \frac{a_{10}^+}{m_1^- A_{11}^-} + \frac{a_{10}^+}{m_1^-}\tau_{11}^+ := \mu,$$

which yields from (4.2) that

$$u^* \leq \mu + \epsilon.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$u^* \leq \mu. \quad (4.11)$$

Similarly,

$$v^* \leq \ln \frac{a_{20}^+}{m_2^- A_{22}^-} + \frac{a_{20}^+}{m_2^-}\tau_{22}^+ := \nu. \quad (4.12)$$

In view of (4.4), we obtain

$$\begin{aligned}\frac{a_{10}^-}{P_1^+e^{\mu} + m_1^+} &\leq \frac{a_{10}(\xi)}{P_1(\xi)e^{u(\xi-\tau_{10}(\xi))} + m_1(\xi)} \\ &= A_{11}(\xi)e^{u(\xi-\tau_{11}(\xi))} + A_{12}(\xi)e^{v(\xi-\tau_{12}(\xi))} \\ &\quad + c_1(\xi) + \frac{h_1(\xi)}{e^{u(\xi)}} \\ &\leq A_{11}^+e^{u^*} + A_{12}^+e^{\nu} + c_1^+ + \frac{h_1^+}{e^{u^*-\epsilon}},\end{aligned}$$

that is,

$$A_{11}^+e^{2u^*} - \left[\frac{a_{10}^-}{P_1^+e^{\mu} + m_1^+} - A_{12}^+e^{\nu} - c_1^+\right]e^{u^*} + h_1^+e^{\epsilon} \geq 0, \quad (4.13)$$

which implies that

$$u^* \geq \ln l_1^+(\epsilon) \quad \text{or} \quad u^* \leq \ln l_1^-(\epsilon), \quad (4.14)$$

where

$$l_1^{\pm}(\epsilon) = \frac{\lambda_1 \pm \sqrt{\lambda_1^2 - 4A_{11}^+h_1^+e^{\epsilon}}}{2A_{11}^+}.$$

Letting $\epsilon \to 0$ in (4.14) leads to

$$u^* \geq \ln l_1^+ \quad \text{or} \quad u^* \leq \ln l_1^-. \quad (4.15)$$

Similarly,

$$v^* \geq \ln l_2^+ \quad \text{or} \quad v^* \leq \ln l_2^-. \quad (4.16)$$

Obviously, $\ln\rho^{\pm}$, $\ln\varrho^{\pm}$, $\ln l^{\pm}$, $\ln k^{\pm}$, μ and ν are independent of λ. Let $\varepsilon_i = \frac{\ln l_i^+ - \ln l_i^-}{4}$ $(i = 1, 2)$ and

$$\Omega_1 = \left\{ (u,v)^T \,\middle|\, \begin{array}{l} \ln\rho^- - 1 < u_* \leq u^* < \ln l_1^- + \varepsilon_1, \\ \ln\varrho^- - 1 < v_* \leq v^* < \ln l_2^- + \varepsilon_2 \end{array} \right\},$$

$$\Omega_2 = \left\{ (u,v)^T \,\middle|\, \begin{array}{l} u_* \in (\ln\rho^- - 1, \ln\rho^+ + 1), \\ u^* \in (\ln l_1^+ - \varepsilon_1, \mu + 1), \\ \ln\varrho^- - 1 < v_* \leq v^* < \ln l_2^- + \varepsilon_2 \end{array} \right\},$$

$$\Omega_3 = \left\{ (u,v)^T \,\middle|\, \begin{array}{l} \ln\rho^- - 1 < u_* \leq u^* < \ln l_1^- + \varepsilon_1, \\ v_* \in (\ln\varrho^- - 1, \ln\varrho^+ + 1), \\ v^* \in (\ln l_2^+ - \varepsilon_2, \nu + 1) \end{array} \right\},$$

$$\Omega_4 = \left\{ (u,v)^T \,\middle|\, \begin{array}{l} u_* \in (\ln\rho^- - 1, \ln\rho^+ + 1), \\ u^* \in (\ln l_1^+ - \varepsilon_1, \mu + 1), \\ v_* \in (\ln\varrho^- - 1, \ln\varrho^+ + 1), \\ v^* \in (\ln l_2^+ - \varepsilon_2, \nu + 1) \end{array} \right\}.$$

Then Ω_1, Ω_2, Ω_3 and Ω_4 are bounded open subsets of $\mathbb{X}$, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, $i, j = 1, 2, 3, 4$. Therefore, Ω_1, Ω_2, Ω_3 and Ω_4 satisfy condition (a) of Lemma 3.

Now we show that condition (b) of Lemma 3 holds, i.e., we prove that $QNz \neq 0$ for all $z = (u,v)^T \in \partial\Omega_i \cap \mathrm{Ker}L = \partial\Omega_i \cap \mathbb{R}^2$, $i = 1, 2, 3, 4$. If it is not true, then there exists at least one constant vector $z_0 = (u_0, v_0)^T \in \partial\Omega_i$ $(i = 1, 2, 3, 4)$ such that

$$\begin{cases} 0 = m\big[\frac{a_{10}}{P_1e^{u_0} + m_1}\big] - \bar{A}_{11}e^{u_0} - \bar{A}_{12}e^{v_0} - \bar{c}_1 - \frac{\bar{h}_1}{e^{u_0}}, \\ 0 = m\big[\frac{a_{20}}{P_2e^{v_0} + m_2}\big] - \bar{A}_{21}e^{u_0} - \bar{A}_{22}e^{v_0} - \bar{c}_2 - \frac{\bar{h}_2}{e^{v_0}}. \end{cases}$$

Similar to the above argument, it follows that

$$\ln\rho^- < u_0 < \ln l_1^-, \qquad \ln\varrho^- < v_0 < \ln l_2^-$$

or

$$\ln l_1^+ < u_0 < \ln\rho^+, \qquad \ln\varrho^- < v_0 < \ln l_2^-$$

or

$$\ln\rho^- < u_0 < \ln l_1^-, \qquad \ln l_2^+ < v_0 < \ln\varrho^+$$

or

$$\ln l_1^+ < u_0 < \ln\rho^+, \qquad \ln l_2^+ < v_0 < \ln\varrho^+.$$

Then $z_0 \in \Omega_1 \cap \mathbb{R}^2$ or $z_0 \in \Omega_2 \cap \mathbb{R}^2$ or $z_0 \in \Omega_3 \cap \mathbb{R}^2$ or $z_0 \in \Omega_4 \cap \mathbb{R}^2$. This contradicts the fact that $z_0 \in \partial\Omega_i$ $(i = 1, 2, 3, 4)$. This proves that condition (b) of Lemma 3 holds.

Finally, we will show that condition (c) of Lemma 3 is satisfied. Let us consider the homotopy

$$H(\iota, z) = \iota QNz + (1 - \iota)\Phi z, \ \ (\iota, z) \in [0, 1] \times \mathbb{R}^2,$$

where

$$\Phi z = \Phi \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\bar{a}_{10}}{P_1^+e^{\mu} + m_1^+} - \bar{A}_{11}e^u - \frac{\bar{h}_1}{e^u} \\ \frac{\bar{a}_{20}}{P_2^+e^{\nu} + m_2^+} - \bar{A}_{22}e^v - \frac{\bar{h}_2}{e^v} \end{pmatrix}.$$

From the above discussion it is easy to verify that $H(\iota, z) \neq 0$ on $\partial\Omega_i \cap \mathrm{Ker}L$, $\forall \iota \in [0, 1]$, $i = 1, 2, 3, 4$. Further, $\Phi z = 0$ has four distinct solutions:

$$(u_1^*, v_1^*)^T = (\ln u^-, \ln v^-)^T, \quad (u_2^*, v_2^*)^T = (\ln u^-, \ln v^+)^T,$$
$$(u_3^*, v_3^*)^T = (\ln u^+, \ln v^-)^T, \quad (u_4^*, v_4^*)^T = (\ln u^+, \ln v^+)^T,$$

where

$$u^{\pm} = \frac{\frac{\bar{a}_{10}}{P_1^+e^{\mu} + m_1^+} \pm \sqrt{(\frac{\bar{a}_{10}}{P_1^+e^{\mu} + m_1^+})^2 - 4\bar{A}_{11}\bar{h}_1}}{2\bar{A}_{11}},$$

$$v^{\pm} = \frac{\frac{\bar{a}_{20}}{P_2^+e^{\nu} + m_2^+} \pm \sqrt{(\frac{\bar{a}_{20}}{P_2^+e^{\nu} + m_2^+})^2 - 4\bar{A}_{22}\bar{h}_2}}{2\bar{A}_{22}}.$$

It is easy to verify that

$$\ln\rho^- < \ln u^- < \ln l_1^- < \ln l_1^+ < \ln u^+ < \mu,$$

$$\ln\varrho^- < \ln v^- < \ln l_2^- < \ln l_2^+ < \ln v^+ < \nu.$$

Therefore

$$(u_1^*, v_1^*)^T \in \Omega_1, \quad (u_2^*, v_2^*)^T \in \Omega_2,$$
$$(u_3^*, v_3^*)^T \in \Omega_3, \quad (u_4^*, v_4^*)^T \in \Omega_4.$$

A direct computation yields

$$\begin{aligned}
&\deg\big(\Phi, \Omega_i \cap \mathrm{Ker}L, 0\big)\\
&= \mathrm{sign}\begin{vmatrix} -\bar{A}_{11}e^{u_i^*} + \frac{\bar{h}_1}{e^{u_i^*}} & 0 \\ 0 & -\bar{A}_{22}e^{v_i^*} + \frac{\bar{h}_2}{e^{v_i^*}} \end{vmatrix}\\
&= \mathrm{sign}\bigg[\bigg(-\bar{A}_{11}e^{u_i^*} + \frac{\bar{h}_1}{e^{u_i^*}}\bigg)\bigg(-\bar{A}_{22}e^{v_i^*} + \frac{\bar{h}_2}{e^{v_i^*}}\bigg)\bigg]\\
&= \mathrm{sign}\bigg[\bigg(\frac{\bar{a}_{10}}{P_1^+e^{\mu} + m_1^+} - 2\bar{A}_{11}e^{u_i^*}\bigg)\\
&\quad \times\bigg(\frac{\bar{a}_{20}}{P_2^+e^{\nu} + m_2^+} - 2\bar{A}_{22}e^{v_i^*}\bigg)\bigg],
\end{aligned}$$

where $i = 1, 2, 3, 4$. Thus

$$\begin{aligned}
&\deg\big(\Phi, \Omega_1 \cap \mathrm{Ker}L, 0\big)\\
&= \mathrm{sign}\bigg[\bigg(\frac{\bar{a}_{10}}{P_1^+e^{\mu} + m_1^+} - 2\bar{A}_{11}u^-\bigg)\\
&\quad \times\bigg(\frac{\bar{a}_{20}}{P_2^+e^{\nu} + m_2^+} - 2\bar{A}_{22}v^-\bigg)\bigg]\\
&= 1,
\end{aligned}$$

$$\begin{aligned}
&\deg\big(\Phi, \Omega_2 \cap \mathrm{Ker}L, 0\big)\\
&= \mathrm{sign}\bigg[\bigg(\frac{\bar{a}_{10}}{P_1^+e^{\mu} + m_1^+} - 2\bar{A}_{11}u^-\bigg)\\
&\quad \times\bigg(\frac{\bar{a}_{20}}{P_2^+e^{\nu} + m_2^+} - 2\bar{A}_{22}v^+\bigg)\bigg]\\
&= -1,
\end{aligned}$$

$$\begin{aligned}
&\deg\big(\Phi, \Omega_3 \cap \mathrm{Ker}L, 0\big)\\
&= \mathrm{sign}\bigg[\bigg(\frac{\bar{a}_{10}}{P_1^+e^{\mu} + m_1^+} - 2\bar{A}_{11}u^+\bigg)\\
&\quad \times\bigg(\frac{\bar{a}_{20}}{P_2^+e^{\nu} + m_2^+} - 2\bar{A}_{22}v^-\bigg)\bigg]\\
&= -1,
\end{aligned}$$

$$\begin{aligned}
&\deg\big(\Phi, \Omega_4 \cap \mathrm{Ker}L, 0\big)\\
&= \mathrm{sign}\bigg[\bigg(\frac{\bar{a}_{10}}{P_1^+e^{\mu} + m_1^+} - 2\bar{A}_{11}u^+\bigg)\\
&\quad \times\bigg(\frac{\bar{a}_{20}}{P_2^+e^{\nu} + m_2^+} - 2\bar{A}_{22}v^+\bigg)\bigg]\\
&= 1.
\end{aligned}$$

By the invariance property of homotopy, we have

$$\begin{aligned}
\deg\big(JQN, \Omega_i \cap \mathrm{Ker}L, 0\big) &= \deg\big(QN, \Omega_i \cap \mathrm{Ker}L, 0\big)\\
&= \deg\big(\Phi, \Omega_i \cap \mathrm{Ker}L, 0\big)\\
&\neq 0, \quad i = 1, 2, 3, 4,
\end{aligned}$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$. Obviously, all the conditions of Lemma 3 are satisfied. Therefore, system (3.1) has at least four almost periodic solutions, that is, system (2.1) has at least four positive almost periodic solutions.

Next, we prove that system (1.2) has at least four positive almost periodic solutions. From Lemma 1, we know that $(y_1(t), y_2(t))^T = \big(\prod_{0<t_k<t}(1 + p_{1k})x_1(t), \prod_{0<t_k<t}(1 + p_{2k})x_2(t)\big)^T$ is a solution of system (1.2). Since condition (H_2) holds, similar to the proofs of Lemma 31 and Theorem 79 in [10], we can prove that $(y_1(t), y_2(t))^T$ is almost periodic. This completes the proof. ■

Corollary 1. *Assume that (H_1)-(H_3) hold. Suppose further that a_{ij}, τ_{ij}, m_i, c_i and h_i of system (1.2) are continuous nonnegative periodic functions with periods α_{ij}, β_i, σ_i, δ_i and χ_i, respectively, $i = 1, 2$, $j = 0, 1, 2$, then system (1.2) has at least four positive almost periodic solutions.*

In Corollary 1, let $\alpha_{ij} = \beta_i = \sigma_i = \delta_i = \chi_i = \omega$, $i = 1, 2$, $j = 0, 1, 2$, then we obtain that

Corollary 2. *Assume that (H_1)-(H_3) hold. Suppose further that a_{ij}, τ_{ij}, m_i, c_i and h_i of system (1.2) are continuous nonnegative ω-periodic functions, $i = 1, 2$, $j = 0, 1, 2$, then system (1.2) has at least four positive ω-periodic solutions.*

Remark 1. By Corollary 1, it is easy to obtain the existence of at least four positive almost periodic solutions of Eq. (1.3) in Example 1, although the positive periodic solution of Eq. (1.3) is nonexistent.

V. Permanence

In this section, we establish a permanence result for system (1.2).

Lemma 7. (*See,* [19, *Lemma* 2.2]). *Assume that for $y(t) > 0$, $t \geq 0$, it holds that*

$$\dot{y}(t) \leq y(t)\Big(a - by(t - \tau(t))\Big),$$

with initial conditions, $y(s) = \phi(s) \geq 0$ for $s \in [-\tau, 0]$, where a, b are positive constants. Then there exists a positive constant y^* such that

$$\lim_{t\to+\infty} \sup y(t) \leq y^* := \frac{ae^{a\tau^+}}{b}.$$

Let

$$p_i^+ = \sup_{t\in\mathbb{R}^+}\bigg(\frac{a_{i0}(t)}{m_i(t)} - c_i(t)\bigg),\ i = 1, 2.$$

Lemma 8. *Assume that (H_1)-(H_2) hold. Suppose further that*

(H_3) $p_i^+ > 0$, $i = 1, 2$.

Then any positive solution $(x_1(t), x_2(t))^T$ of system (2.1) satisfies

$$\limsup_{t\to\infty} x_i(t) \leq x_i^* = \frac{p_i^+ \exp\{p_i^+\tau_{ii}^+\}}{A_{ii}^-}, \quad i = 1, 2.$$

Proof: Let $(x_1(t), x_2(t))^T$ be any solution of system (2.1). From system (2.1), we have that

$$\begin{cases} \dot{x}_1(t) \leq x_1(t)\Big(p_1^+ - A_{11}^- x_1(t - \tau_{11}(t))\Big),\\ \dot{x}_2(t) \leq x_2(t)\Big(p_2^+ - A_{22}^- x_2(t - \tau_{22}(t))\Big). \end{cases}$$

Obviously, $p_i^+ \geq r_i^- > 0$, $i = 1, 2$. By Lemma 7, we obtain

$$\lim_{t\to+\infty} \sup x_i(t) \leq \frac{p_i^+ \exp\{p_i^+ \tau_{ii}^+\}}{A_{ii}^-} := x_i^*, \quad i = 1, 2.$$

This completes the proof. ■

Let

$$r_1(t) = \frac{a_{10}(t)}{x_1^* P_1(t) + m_1(t)} - x_2^* A_{12}(t) - c_1(t),$$

$$r_2(t) = \frac{a_{20}(t)}{x_2^* P_2(t) + m_2(t)} - x_1^* A_{21}(t) - c_2(t).$$

Lemma 9. *Assume that (H_1)-(H_3) hold. Suppose further that*

(H_4) $r_i^- > 2|A_{ii}^+|x_i^*$, $A_{ii}^+ < 0$, $i = 1, 2$;
(H_5) $\tau_{ii} \in C^1(\mathbf{R})$ *and* $\sup_{s\in\mathbf{R}}\{\tau'_{11}(s), \tau'_{22}(s)\} < 1$, $i = 1, 2$.

Then any positive solution $(x_1(t), x_2(t))^T$ of system (2.1) *satisfies*

$$\lim \inf_{t\to\infty} x_i(t) \geq x_{i*} = \frac{p_i^+ \exp\{p_i^+ \tau_{ii}^+\}}{A_{ii}^-}, \quad i = 1, 2.$$

Proof: Let $(x_1(t), x_2(t))^T$ be any solution of system (2.1). By Theorem 2 and (H_4), there exist a small enough positive constant ϵ and a larger enough positive constant $T = T(\epsilon) > 0$ such that

$$x_i(t) \leq x_i^* + \epsilon \quad \text{for } t \geq T, \ i = 1, 2,$$

$$r_1^-(\epsilon) = \inf_{t\in\mathbb{R}^+} \bigg(\frac{a_{10}(t)}{(x_1^* + \epsilon)P_1(t) + m_1(t)} - (x_2^* + \epsilon)A_{12}(t) - c_1(t) \bigg) > 2|A_{11}^+|(x_1^* + \epsilon),$$

$$r_2^-(\epsilon) = \inf_{t\in\mathbb{R}^+} \bigg(\frac{a_{20}(t)}{(x_2^* + \epsilon)P_2(t) + m_2(t)} - (x_1^* + \epsilon)A_{21}(t) - c_2(t) \bigg) > 2|A_{22}^+|(x_2^* + \epsilon).$$

For $t \geq T + \tau$, in view of system (2.1), we have that

$$\begin{cases} \dot{x}_1(t) \geq x_1(t)\Big(r_1^-(\epsilon) - A_{11}^+ x_1(t - \tau_{11}(t))\Big) - h_1^+, \\ \dot{x}_2(t) \geq x_2(t)\Big(r_2^-(\epsilon) - A_{22}^+ x_2(t - \tau_{22}(t))\Big) - h_2^+. \end{cases}$$

Consider the auxiliary system

$$\begin{cases} \dot{u}_1(t) = u_1(t)\Big(r_1^-(\epsilon) - A_{11}^+ u_1(t - \tau_{11}(t))\Big) - h_1^+, \\ \dot{u}_2(t) = u_2(t)\Big(r_2^-(\epsilon) - A_{22}^+ u_2(t - \tau_{22}(t))\Big) - h_2^+. \end{cases} \tag{5.1}$$

Obviously, by Theorem 2, we have

$$u_i(t) \leq x_i^* + \epsilon \quad \text{for } t \geq T, \ i = 1, 2,$$

We claim that system (5.1) has a unique globally asymptotically stable positive equilibrium point $(x_{1*}(\epsilon), x_{2*}(\epsilon))^T = \left[\frac{r_1^-(\epsilon) + \sqrt{r_1^{-2}(\epsilon) - 4A_{11}^+ h_1^+}}{2A_{11}^+}, \frac{r_2^-(\epsilon) + \sqrt{r_2^{-2}(\epsilon) - 4A_{22}^+ h_2^+}}{2A_{22}^+}\right]^T$. In fact, suppose that $u = (u_1, u_2)^T$ and $u = (u_1^*, u_2^*)^T$ are any two positive solutions of system (5.1). Define

$$V_i(t) = |u_i^*(t) - u_i(t)| - \int_{t-\tau_{ii}(t)}^{t} \frac{|A_{ii}^+|(x_i^* + \epsilon)}{1 - \tau'_{ii}(\varphi(t))} |u_i^*(s) - u_i(s)| \, \mathrm{d}s,$$

$\forall t \in \mathbb{R}$, $i = 1, 2$. Calculating the upper right derivative of $V_i(t)$ along the solution of system (5.1), we have

$$D^+ V_i(t) \geq [r_i^-(\epsilon) - 2|A_{ii}^+|(x_i^* + \epsilon)]|u_i^*(t) - u_i(t)|, \tag{5.2}$$

$i = 1, 2$. Therefore, V_i is non-increasing. Integrating (5.2) from 0 to t leads to

$$V(0) + [r_i^-(\epsilon) - 2|A_{ii}^+|(x_i^* + \epsilon)] \int_0^t |u_i(s) - u_i^*(s)| \, \mathrm{d}s \leq V(t) < +\infty, \quad \forall t \geq 0,$$

that is,

$$\int_0^{+\infty} |u_i(s) - u_i^*(s)| \, \mathrm{d}s < +\infty,$$

which implies that

$$\lim_{s\to+\infty} |u_i(s) - u_i^*(s)| = 0, \quad i = 1, 2.$$

Thus, system (5.1) is globally asymptotically stable and the positive equilibrium point $(x_{1*}(\epsilon), x_{2*}(\epsilon))^T$ is globally asymptotically stable.

By the comparison theorem, $x_i(t) \geq u_i(t)$, where $u_i(t)$ is the solution of system (5.1) with $u_i(0) = x_i(0^+)$, $i = 1, 2$. And system (5.1) has a unique globally asymptotically stable positive almost periodic solution $(x_{1*}(\epsilon), x_{2*}(\epsilon))^T$. Then for any constant $\epsilon_1 > 0$, there exists $T_1 > 0$ such that $x_i(t) \geq u_i(t) > x_{i*}(\epsilon) - \epsilon_1$ for $t > T_1$, $i = 1, 2$. Letting $\epsilon, \epsilon_1 \to 0$ leads to

$$\lim \sup_{t\to\infty} x_i(t) \geq x_{i*}, \quad i = 1, 2.$$

This completes the proof. ■

By Lemmas 8-9, we have

Theorem 2. *Assume that (H_1)-(H_5) hold, then system* (1.2) *is permanent.*

VI. Global asymptotic stability

Theorem 3. *Suppose that*

(H_6) $\tau_{11} = \tau_{22} = h_1 = h_2 \equiv 0$, $\tau_{10}, \tau_{20}, \tau_{12}, \tau_{21} \in C^1(\mathbb{R})$ *and* $\sup_{s\in\mathbf{R}}\{\dot{\tau}_{10}(s), \dot{\tau}_{20}(s), \dot{\tau}_{12}(s), \dot{\tau}_{21}(s)\} < 1$.
(H_7) *there exist two positive constants λ_1 and λ_2 such that*

$$\Theta_1 = \lambda_1 A_{11}^- - \lambda_1 \frac{a_{10}^+ P_1^+}{m_1^{-2}(1 - \dot{\tau_{10}}^+)} - \lambda_2 \frac{A_{21}^+}{1 - \dot{\tau_{21}}^+} > 0,$$

$$\Theta_2 = \lambda_2 A_{22}^- - \lambda_2 \frac{a_{20}^+ P_2^+}{m_2^{-2}(1 - \dot{\tau_{20}}^+)} - \lambda_1 \frac{A_{12}^+}{1 - \dot{\tau_{12}}^+} > 0.$$

Then system (1.2) *is globally asymptotically stable.*

Proof: Assume that $(x_1, x_2)^T$ and $(\bar{x}_1, \bar{x}_2)^T$ are any two solutions of system (2.1).

Let $(u_1, u_2)^T = (\ln x_1, \ln x_2)^T$ and $(\bar{u}_1, \bar{u}_2)^T =$

$(\ln \bar{x}_1, \ln \bar{x}_2)^T$, then system (2.1) is transformed into

$$\begin{cases} \dot{u}_1(t) = \frac{a_{10}(t)}{P_1(t)x_1(t-\tau_{10}(t))+m_1(t)} \\ \qquad - \sum_{j=1}^{2} A_{1j}(t)x_j(t-\tau_{1j}(t)) - c_1(t), \\ \dot{u}_2(t) = \frac{a_{20}(t)}{P_2(t)x_2(t-\tau_{20}(t))+m_2(t)} \\ \qquad - \sum_{j=1}^{2} A_{2j}(t)x_j(t-\tau_{2j}(t)) - c_2(t), \\ \dot{\bar{u}}_1(t) = \frac{a_{10}(t)}{P_1(t)\bar{x}_1(t-\tau_{10}(t))+m_1(t)} \\ \qquad - \sum_{j=1}^{2} A_{1j}(t)\bar{x}_j(t-\tau_{1j}(t)) - c_1(t), \\ \dot{\bar{u}}_2(t) = \frac{a_{20}(t)}{P_2(t)\bar{x}_2(t-\tau_{20}(t))+m_2(t)} \\ \qquad - \sum_{j=1}^{2} A_{2j}(t)\bar{x}_j(t-\tau_{2j}(t)) - c_2(t), \end{cases} \tag{6.1}$$

where $A_{ij}(t) = P_i(t)a_{ij}(t) = \prod_{0<t_k<t}(1+p_{ik})a_{ij}(t)$, $i = 1, 2$, $j = 1, 2$. Define

$$V(t) = V_0(t) + V_1(t) + V_2(t) + V_3(t) + V_4(t),$$

where

$$V_0(t) = \lambda_1 |u_1(t) - \bar{u}_1(t)| + \lambda_2 |u_2(t) - \bar{u}_2(t)|,$$

$$V_1(t) = \lambda_1 \int_{t-\tau_{10}(t)}^{t} \frac{a_{10}^+ P_1^+}{m_1^{-2}(1-\dot{\tau_{10}}^+)} |x_1(s) - \bar{x}_1(s)| \,\mathrm{d}s,$$

$$V_2(t) = \lambda_2 \int_{t-\tau_{20}(t)}^{t} \frac{a_{20}^+ P_2^+}{m_2^{-2}(1-\dot{\tau_{20}}^+)} |x_2(s) - \bar{x}_2(s)| \,\mathrm{d}s,$$

$$V_3(t) = \lambda_1 \int_{t-\tau_{12}(t)}^{t} \frac{A_{12}^+}{1-\dot{\tau_{12}}^+} |x_2(s) - \bar{x}_2(s)| \,\mathrm{d}s,$$

$$V_4(t) = \lambda_2 \int_{t-\tau_{21}(t)}^{t} \frac{A_{21}^+}{1-\dot{\tau_{21}}^+} |x_1(s) - \bar{x}_1(s)| \,\mathrm{d}s.$$

By calculating the upper right derivative of V_1 along system (6.1), it follows that

$$\begin{aligned} & D^+V_0(t) \\ = {} & \lambda_1 \mathrm{sgn}[u_1(t) - \bar{u}_1(t)][u_1'(t) - \bar{u}_1'(t)] \\ & +\lambda_2 \mathrm{sgn}[u_2(t) - \bar{u}_2(t)][u_2'(t) - \bar{u}_2'(t)] \\ \le {} & -\lambda_1 A_{11}(t)|x_1(t) - \bar{x}_1(t)| \\ & -\lambda_2 A_{22}(t)|x_2(t) - \bar{x}_2(t)| \\ & +\lambda_1 \frac{a_{10}^+ P_1^+}{m_1^{-2}} |x_1(t-\tau_{10}(t)) - \bar{x}_1(t-\tau_{10}(t))| \\ & +\lambda_2 \frac{a_{20}^+ P_2^+}{m_2^{-2}} |x_2(t-\tau_{20}(t)) - \bar{x}_2(t-\tau_{20}(t))| \\ & +\lambda_1 A_{12}^+ |x_2(t-\tau_{12}(t)) - \bar{x}_2(t-\tau_{12}(t))| \\ & +\lambda_2 A_{21}^+ |x_1(t-\tau_{21}(t)) - \bar{x}_1(t-\tau_{21}(t))|. \end{aligned} \tag{6.2}$$

Further, by calculating the upper right derivative of V_1, V_2 and V_3 along system (6.1), it follows that

$$\begin{aligned} D^+V_1(t) \le {} & \lambda_1 \frac{a_{10}^+ P_1^+}{m_1^{-2}(1-\dot{\tau_{10}}^+)} |x_1(t) - \bar{x}_1(t)| \\ & -\lambda_1 \frac{a_{10}^+ P_1^+}{m_1^{-2}} |x_1(t-\tau_{10}(t)) - \bar{x}_1(t-\tau_{10}(t))|, \end{aligned}$$

$$\begin{aligned} D^+V_2(t) \le {} & \lambda_2 \frac{a_{20}^+ P_2^+}{m_2^{-2}(1-\dot{\tau_{20}}^+)} |x_2(t) - \bar{x}_2(t)| \\ & -\lambda_2 \frac{a_{20}^+ P_1^+}{m_2^{-2}} |x_2(t-\tau_{20}(t)) - \bar{x}_2(t-\tau_{20}(t))|, \end{aligned}$$

$$\begin{aligned} D^+V_3(t) \le {} & \lambda_1 \frac{A_{12}^+}{1-\dot{\tau_{12}}^+} |x_2(t) - \bar{x}_2(t)| \\ & -\lambda_1 A_{12}^+ |x_2(t-\tau_{12}(t)) - \bar{x}_2(t-\tau_{12}(t))|, \end{aligned}$$

$$\begin{aligned} D^+V_4(t) \le {} & \lambda_2 \frac{A_{21}^+}{1-\dot{\tau_{21}}^+} |x_1(t) - \bar{x}_1(t)| \\ & -\lambda_2 A_{21}^+ |x_1(t-\tau_{21}(t)) - \bar{x}_1(t-\tau_{21}(t))|. \end{aligned}$$

Together with (6.2), it follows that

$$\begin{aligned} D^+V(t) \le {} & -\Big\{\lambda_1 A_{11}^- - \lambda_1 \frac{a_{10}^+ P_1^+}{m_1^{-2}(1-\dot{\tau_{10}}^+)} \\ & -\lambda_2 \frac{A_{21}^+}{1-\dot{\tau_{21}}^+}\Big\} |x_1(t) - \bar{x}_1(t)| \\ & -\Big\{\lambda_2 A_{22}^- - \lambda_2 \frac{a_{20}^+ P_2^+}{m_2^{-2}(1-\dot{\tau_{20}}^+)} \\ & -\lambda_1 \frac{A_{12}^+}{1-\dot{\tau_{12}}^+}\Big\} |x_2(t) - \bar{x}_2(t)| \\ \le {} & -\Gamma_1 |x_1(t) - \bar{x}_1(t)| - \Gamma_2 |x_2(t) - \bar{x}_2(t)|. \end{aligned}$$

Therefore, V is non-increasing. Integrating of the last inequality from 0 to t leads to

$$\begin{aligned} & V(t) + \Gamma_1 \int_0^t |x_1(s) - \bar{x}_1(s)| \,\mathrm{d}s \\ & +\Gamma_2 \int_0^t |x_2(s) - \bar{x}_2(s)| \,\mathrm{d}s \\ \le {} & V(0) < +\infty, \ \ \forall t \ge 0, \end{aligned}$$

that is,

$$\int_0^{+\infty} |y_1(s) - \bar{y}_1(s)| \,\mathrm{d}s < +\infty,$$

$$\int_0^{+\infty} |y_2(s) - \bar{y}_2(s)| \,\mathrm{d}s < +\infty,$$

which imply that

$$\lim_{s\to+\infty} |y_1(s) - \bar{y}_1(s)| = \lim_{s\to+\infty} |y_2(s) - \bar{y}_2(s)| = 0.$$

This completes the proof. ■

VII. An example and numerical simulations

Example 2. Consider the following delayed Schoener's competition model with harvesting terms:

$$\begin{cases} \dot{y}_1(t) &= y_1(t)\bigg(\frac{3+\sin(\sqrt{3}t)}{y_1(t-\sin^2(\sqrt{2}t))+1} \\ & \quad -[3+\sin^2(\sqrt{3}t)]y_1(t) \\ & \quad -0.1y_2(t-1)-0.2\bigg)-0.01, \\ \dot{y}_2(t) &= y_2(t)\bigg(\frac{3+\cos(\sqrt{2}t)}{y_2(t)+1}-0.2y_1(t-\cos^2(t)) \\ & \quad -[3+\cos^2(\sqrt{2}t)]y_2(t)-0.1\bigg)-0.01 \\ y_1(t_k^+) &= (1+p_k)y_1(t_k), \\ y_2(t_k^+) &= (1+p_k)y_2(t_k), \ \{t_k\}\subset\{2k: k\in\mathbb{Z}\}, \end{cases} \tag{7.1}$$

where $\prod_{0<t_k<t}(1+p_k)\in[1,1.1]$ is almost periodic, $t\in\mathbb{R}^+$. Then system (7.1) is permanent and has at least four positive almost periodic solutions.

Proof: Corresponding to system (1.2), we have $a_{i0}^-=2$, $a_{i0}^+=3$, $a_{ii}^-=3$, $a_{ii}^+=4$, $m_i=1$, $a_{12}=0.1$, $a_{21}=0.2$, $c_1=0.2$, $c_2=0.1$, $h_1=h_2=0.01$, $i=1,2$. Obviously, (H_1) in Theorem 3.1 holds. By a easy calculation, we obtain that

$$\mu=\nu\approx 0, \quad \lambda_1=\lambda_2\approx 0.68.$$

So (H_3) in Theorem 1 holds. Clearly, (H_4)-(H_5) in Theorem 2 also hold. Therefore, all the conditions of Theorems 1-2 are satisfied. By Theorems 1-2, system (7.1) is permanent and admits at least four positive almost periodic solutions (see Figures 1-2). This completes the proof. ■

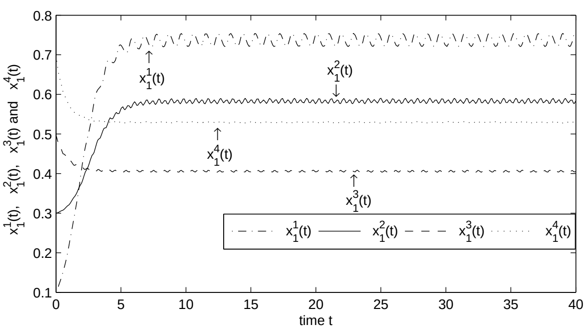

Fig. 1 Four positive almost periodic oscillations of state variable x_1 of system (5.1)

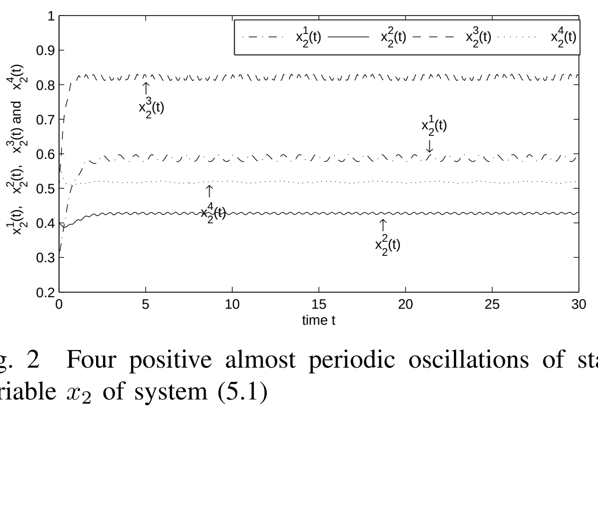

Fig. 2 Four positive almost periodic oscillations of state variable x_2 of system (5.1)

VIII. Conclusion

In this paper we have obtained the existence of at least four positive almost periodic solution for a harvesting Schoener's competition model with time-varying delays and impulsive effects. The approach is based on the continuation theorem of coincidence degree theory. Lemma 1 in Section 2 and Lemma 2 in Section 3 are critical to study the existence of positive almost periodic solution of the biological model. It is important to notice that the approach used in this paper can be extended to other types of biological models [28-29]. Future work will include models based on impulsive differential equations and biological dynamic systems on time scales.

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