Applications of Fixed Point Theorems for Coupled Systems of Fractional Integro-Differential Equations Involving Convergent Series

Mohamed Amin Abdellaoui, Zoubir Dahmani, and Nabil Bedjaoui,

Abstract—**The existence and uniqueness of solutions in an appropriate Banach space is established for a coupled system of integro-differential equations involving convergent series, with derivatives of non-integer order on the unknown functions and integrals of Riemann-Liouville in the nonlinearity as well as in the initial conditions. Some illustrative examples are also presented to demonstrate our main results.**

Index Terms—**Caputo derivative, convergent series, differential system, fixed point, existence and uniqueness, Riemann-Liouville integral.**

I. Introduction

THE differential equations of fractional order arise in many scientific disciplines, such as physics, chemistry, control theory, signal processing and biophysics. For more details, we refer the reader to [5], [9], [16], [17], [18] and the references therein. Recently, there has been a significant progress in the investigation of this theory, see [1], [2], [3], [4], [6], [9], [21], [23]. Moreover, the study of coupled systems of fractional differential equations is also of a great importance. Such systems occur in various problems of applied science. For some recent results on the fractional systems, we refer the reader to ([7], [8], [11], [14], [20], [22], [24].

Let us now present some results that have inspired our work. We begin by [10], where the authors studied the following problem:

$$\left\{\begin{array}{c} D^{\alpha}x(t)+f_1\left(t,y(t),D^{\delta}y(t)\right)=0,t\in J,\\ D^{\beta}y(t)+f_2\left(t,x(t),D^{\sigma}x(t)\right)=0,t\in J,\\ x(0)=x_0^*,y(0)=y_0^*,\\ \left|x^{'}(0)\right|+\left|x^{''}(0)\right|+\left|y^{'}(0)\right|+\left|y^{''}(0)\right|=0,\\ x^{'''}(0)=J^px(\eta),y^{'''}(0)=J^qy(\xi), \end{array}\right. \quad (1)$$

where $\alpha,\beta\in\left]3,4\right]$, $\delta\leq\alpha-1$, $\sigma\leq\beta-1$, $\eta,\xi\in[0,1]$, J^p,J^q are the Riemann-Liouville fractional integrals, D^{α}, $D^{\beta},D^{\delta},D^{\sigma}$ are the Caputo fractional derivatives, $J=[0,1]$, x_0^*, y_0^* are real constants

Then M. Li and Y. Liu [14] studied the following fractional differential problem:

$$\left\{\begin{array}{c} {}^cD_{0+}^{\alpha}u(t)=f_1\left(t,u(t),v(t),{}^cD_{0+}^{p}u(t),{}^cD_{0+}^{q}v(t)\right)=0,\\ {}^cD_{0+}^{\beta}v(t)=f_2\left(t,u(t),v(t),{}^cD_{0+}^{p}u(t),{}^cD_{0+}^{q}v(t)\right)=0,\\ u(0)+\lambda_1u^{'}(0)=0,u(1)+\lambda_2^cD_{0+}^{p}u(1)=0,\\ v(0)+\lambda_1v^{'}(0)=0,v(1)+\lambda_2^cD_{0+}^{q}v(1)=0, \end{array}\right. \quad (2)$$

where ${}^cD_{0+}^{\gamma}$ denotes the Caputo fractional derivative for $\gamma>0$, $1<\alpha,\beta\leq2$, $0<p,q\leq1,t\in J=[0,T]$ $f_1,f_2\in C\left([0,T]\times R_+^2\times R^2,R_+\right)$.

Recently, in [24], the authors discussed the following coupled system with integral boundary conditions:

$$\left\{\begin{array}{c} {}^cD_{0+}^{\alpha}u(t)=f\left(t,v(t),{}^cD_{0+}^{p}v(t)\right),\\ {}^cD_{0+}^{\beta}v(t)=g\left(t,u(t),{}^cD_{0+}^{q}u(t)\right)=0,0<t<1,\\ au^{'}(0)+u(\eta_1)=\int_0^1\phi(s,v(s))ds,\\ u(\eta_2)+bu^{'}(1)=\int_0^1\psi(s,v(s))ds,\\ cv^{'}(0)+v(\xi_1)=\int_0^1\varphi(s,u(s))ds,\\ v(\xi_2)+dv^{'}(1)=\int_0^1\rho(s,u(s))ds, \end{array}\right. \quad (3)$$

where $1<\alpha,\beta<2$, $0<p,q<1$, $\alpha-p-1\geq0$, $\beta-q-1\geq0$, $0\leq\eta_1<\eta_2\leq1$, $\phi,\psi,\varphi,\rho\in L^1[0,1]$ and $f,g\in C\left((0,1)\times R^2,R\right),{}^cD_{0+}^{\gamma}$ are in the sense of Caputo.

Very recently, A. Taieb and Z. Dahmani [19] established some results on the existence and uniqueness of solutions for the problem:

$$\left\{\begin{array}{c} D^{\alpha_1}x_1(t)=\sum\limits_{i=1}^{m}f_i^1(t,x_1(t),x_2(t),...,x_n(t)),t\in J,\\ D^{\alpha_2}x_2(t)=\sum\limits_{i=1}^{m}f_i^2(t,x_1(t),x_2(t),...,x_n(t)),t\in J,\\ \vdots\\ D^{\alpha_n}x_n(t)=\sum\limits_{i=1}^{m}f_i^n(t,x_1(t),x_2(t),...,x_n(t)),t\in J,\\ k=1,x_k^{(k-1)}(0)=0,\\ k=2,x_k^{(k-2)}(0)=x_k^{(k-1)}(1)=0,\\ k=3,x_k^{(k-3)}(0)=x_k^{(k-2)}(0)=x_k^{(k-1)}(1)=0,\\ k=n,x_k^{(k-n)}(0)=...=x_k^{(k-2)}(0)=x_k^{(k-1)}(1)=0, \end{array}\right. \quad (4)$$

where $k-1<\alpha_k<k,k=1,2,...,n,J:=[0,1]$ and the derivatives D^{α_k}, are in the sense of Caputo.

In this paper, we discuss the existence and uniqueness of solutions for the following coupled system of fractional

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M. A. Abdellaoui and Z. Dahmani (corresponding author e-mail : zzdahmani@yahoo.fr) are with the Department LPAM, Faculty SEI, UMAB of Mostaganem, Algeria

N. Bedjaoui is with the LAMFA laboratory of mathematics, UPJV, Rue St Leu, Amiens 80000, France

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integro-differential equations:

$$\begin{cases} D^{\alpha}u(t) = f_1(t,u(t),v(t)) \\ +\sum_{i=1}^{+\infty}\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\varphi_i(s)\, g_i(s,u(s),v(s))\,ds, t\in J \\ D^{\beta}v(t) = f_2(t,u(t),v(t)) \\ +\sum_{i=1}^{+\infty}\int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}\phi_i(s)\, h_i(s,u(s),v(s))\,ds, t\in J \\ \sum_{k=0}^{n-2}\left(\left|u^{(k)}(0)\right|+\left|v^{(k)}(0)\right|\right)=0, \\ u^{(n-1)}(0)=\gamma I^p u(\eta), \eta\in]0,1[, \\ v^{(n-1)}(0)=\delta I^q v(\zeta), \zeta\in]0,1[, \end{cases} \quad (5)$$

where D^{α}, D^{β} denote the Caputo fractional derivatives, with $\alpha,\beta\in]n-1,n[, n\in N^*, \alpha_i\geq 1, \beta_i\geq 1, p,q\in R_+^*$ and $J=[0,1]$. The functions f_1 and f_2 as well as the general terms φ_i, ϕ_i, g_i and $h_i; (i\in N^*)$ will be specified later.

To the best of our knowledge, the case of differential systems involving series, no contribution exists. So, the main aim of this work is to fill the gap in the relevant literature.

The rest of the paper is organized as follows: In section 1, we present some preliminaries and lemmas. Section 2 is devoted to existence of solutions of problem (5). In the last section, some examples are presented to illustrate our results.

The following preliminaries will be used in the proof of our main results [12], [13], [15]

Definition 1.1: A real function $f(t)$; $t>0$ is said to be in space C_μ, $\mu\in R$, if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C(0,+\infty)$ and it is said to be in the space C_μ^n if $f^{(n)}\in C_\mu$, $n\in N$.

Definition 1.2: The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$, for a function $f\in C_\mu$, $\mu\geq -1$, is defined as:

$$I^{\alpha}f(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,d\tau, \alpha>0, t\geq 0, \\ f(t), \qquad if \alpha=0.\end{cases} \quad (6)$$

Definition 1.3: The Caputo fractional derivative of order $\alpha>0$, $n-1<\alpha\leq n, n\in N^*$ of a function $f\in C_{-1}^n$, is defined as:

$$D^{\alpha}f(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)\,d\tau, t>0, \\ f^{(n)}(t), \qquad if \alpha=n.\end{cases} \quad (7)$$

The following lemma gives some properties of Riemann-Liouville fractional integral and Caputo derivative [12], [13]

Lemma 1.1: *Let $f\in C_{-1}$. Then for all $t>0$, we have:*
$I^r I^s f(t)=I^{r+s}f(t); r>0, s>0.$
$D^s I^s f(t)=f(t); s>0.$
$D^r I^s f(t)=I^{s-r}f(t); s>r>0.$

To study the coupled system (5), we need the following two lemmas [12]:

Lemma 1.2: *Let $n-1<\alpha<n, n\in N^*$. The general solution of $D^{\alpha}x(t)=0$ is given by*

$$x(t)=c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1}, \quad (8)$$

where $c_j\in R, j=0,1,2,..,n-1$.

Lemma 1.3: *Let $n-1<\alpha<n$, $n\in N^*$. Then, for a given function $f\in C_{-1}^n$, we have*

$$I^{\alpha}D^{\alpha}f(t)=f(t)+c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1}, \quad (9)$$

for some $c_j\in R, j=0,1,2,...,n-1, n=[\alpha]+1$.

The following result gives an integral representation for the problem (5):

Lemma 1.4: *Suppose that $x\in C_{-1}^n, n-1<\alpha<n, \alpha_i\geq 1$. If $F, G_i, \Phi_i\in C_{-1}([0,1],R)$, such that $\sum_{i=1}^{+\infty}\|\Phi_i\|_\infty<\infty$ and G_i is uniformly bounded, then the integral solution of the problem*

$$D^{\alpha}x(t)=F(t)+\sum_{i=1}^{+\infty}\int_0^t\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\Phi_i(s)\,G_i(s)\,ds, \quad (10)$$

associated with the conditions

$$\sum_{k=0}^{n-2}\left|x^{(k)}(0)\right|=0, x^{(n-1)}(0)=\gamma I^p x(\eta) \quad (11)$$

is given by:

$$\begin{aligned} x(t) &= \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)\,ds \\ &+\sum_{i=1}^{+\infty}\int_0^t \tfrac{(t-s)^{\alpha_i+\alpha-1}}{\Gamma(\alpha_i+\alpha)}\Phi_i(s)\,G_i(s)\,ds \\ &+\tfrac{\gamma\Gamma(p+n)t^{n-1}}{\Gamma(n)(\Gamma(p+n)-\gamma\eta^{p+n-1})}\times\Big(\int_0^\eta\tfrac{(\eta-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}F(s)\,ds \\ &+\sum_{i=1}^{+\infty}\int_0^\eta\tfrac{(\eta-s)^{\alpha_i+\alpha+p-1}}{\Gamma(\alpha+\alpha_i+p)}\Phi_i(s)\,G_i(s)\,ds\Big), \end{aligned} \quad (12)$$

where $\gamma\neq\frac{\Gamma(p+n)}{\eta^{p+n-1}}$.

Proof. Applying Lemma 1.3 to (10), we can write

$$\begin{aligned} x(t) &= \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)\,ds \\ &+\sum_{i=1}^{+\infty}\int_0^t\tfrac{(t-s)^{\alpha_i+\alpha-1}}{\Gamma(\alpha_i+\alpha)}\Phi_i(s)\,G_i(s)\,ds \\ &-c_0-c_1t-...-c_{n-1}t^{n-1}. \end{aligned} \quad (13)$$

Then,

$$x^{(k)}(0)=-k!\,c_k, k=1,...,n-1. \quad (14)$$

Since $x(0)=x'(0)=...=x^{(n-2)}(0)=0$, it follows then that $c_0=c_1=c_2=...=c_{n-2}=0$.

Thanks to Lemma 1.1, it yields that

$$\begin{aligned} I^p x(t) &= \tfrac{1}{\Gamma(\alpha+p)}\int_0^t (t-s)^{\alpha+p-1}F(s)\,ds+ \\ &\sum_{i=1}^{+\infty}\int_0^t\tfrac{(t-s)^{\alpha_i+\alpha+p-1}}{\Gamma(\alpha_i+\alpha+p)}\Phi_i(s)\,G_i(s)\,ds \\ &-c_{n-1}\tfrac{\Gamma(n)t^{p+n-1}}{\Gamma(p+n)}. \end{aligned} \quad (15)$$

Using (11), we get

$$c_{n-1} = -\frac{\gamma\Gamma(p+n)}{\Gamma(n)(\Gamma(p+n)-\gamma\eta^{p+n-1})} \times \left(I^{\alpha+p}F(\eta) + \sum_{i=1}^{+\infty} I^{\alpha+\alpha_i+p}\Phi_i(\eta)G_i(\eta) \right) \quad (16)$$

Substituting $c_0, c_1, c_2, ..., c_{n-1}$ in (13), we obtain (12). Lemma 1.4 is thus proved.

II. Main Results

In this paragraph, we introduce the following assumptions:

$(H1)$: There exist non negative real numbers m_j, m'_j, n_j, n'_j $(j = 1, 2)$, such that for all $t \in [0, 1]$ and $(u_1, v_1), (u_2, v_2) \in R^2$, we have:

$$|f_1(t, u_2, v_2) - f_1(t, u_1, v_1)| \le m_1 |u_2 - u_1| + m_2 |v_2 - v_1|,$$

$$|f_2(t, u_2, v_2) - f_2(t, u_1, v_1)| \le n_1 |u_2 - u_1| + n_2 |v_2 - v_1|,$$

$$|g_i(t, u_2, v_2) - g_i(t, u_1, v_1)| \le m'_1 |u_2 - u_1| + m'_2 |v_2 - v_1|,$$

$$|g_i(t, u_2, v_2) - g_i(t, u_1, v_1)| \le m'_1 |u_2 - u_1| + m'_2 |v_2 - v_1|,$$

where , $i \in N^*$.

$(H2)$: (i) : Suppose that $\varphi_i, \phi_i \in C_{-1}([0, 1], R)$, and g_i and h_i are uniformly bounded i.e. there exist nonnegative real numbers L'_1, L'_2 such that for all $t \in [0, 1]$ and $(u, v) \in R^2$, we have:

$$|g_i(t, u, v)| \le L'_1, |h_i(t, u, v)| \le L'_2, i = 1, 2, 3...$$

(ii) : Assume that $\sum_{i=1}^{+\infty} \|\varphi_i\|_\infty < +\infty, \sum_{i=1}^{+\infty} \|\phi_i\|_\infty < +\infty.$

$(H3)$: The functions f_1, f_2, g_i and $h_i : [0, 1] \times R^2 \to R; i \in N^*$ are continuous and there exist nonnegative real numbers L_1, L_2 such that for all $t \in [0, 1]$ and $(u, v) \in R^2$, we have:

$$|f_1(t, u, v)| \le L_1, |f_2(t, u, v)| \le L_2.$$

We also need to introduce the quantities:

$$\omega_1 := \frac{\gamma\Gamma(p+n)}{\Gamma(n)(\Gamma(p+n) - \gamma\eta^{p+n-1})},$$

$$\omega_2 := \frac{\delta\Gamma(q+n)}{\Gamma(n)(\Gamma(q+n) - \delta\zeta^{q+n-1})},$$

$$M_1 := \frac{1}{\Gamma(\alpha+1)} + \frac{|\omega_1|}{\Gamma(\alpha+p+1)}, + \sum_{i=1}^{+\infty} \left(\frac{\|\varphi_i\|_\infty}{\Gamma(\alpha+\alpha_i+1)} + \frac{|\omega_1| \|\varphi_i\|_\infty}{\Gamma(\alpha+\alpha_i+p+1)} \right),$$

$$M_2 := \frac{1}{\Gamma(\beta+1)} + \frac{|\omega_2|}{\Gamma(\beta+q+1)} + \sum_{i=1}^{+\infty} \left(\frac{\|\phi_i\|_\infty}{\Gamma(\beta+\beta_i+1)} + \frac{|\omega_2| \|\varphi_i\|_\infty}{\Gamma(\beta+\beta_i+q+1)} \right),$$

$$\overline{L} = \max\left\{m_j, n_j, m'_j, n'_j\right\}_{j=1,2}.$$

Now, we are ready to state and prove our main results. We begin by the following theorem:

Theorem 2.1: *Suppose that* $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}, \delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}}$ *and assume that* $(H1)$ *and* $(H2)$ *hold. If*

$$\overline{L}(M_1 + M_2) < 1, \quad (17)$$

then the fractional system (5) *has exactly one solution in* $X \times X$.

Proof. We apply the Banach contraction principle. To this end, we introduce the Banach space: $X := C^n_{-1}([0, 1], R)$, equipped with the norm $||.||_X = ||.||_\infty$.
So, the product space $\left(X \times X, \|(u, v)\|_{X\times X}\right)$ is also a Banach space with norm $\|(u, v)\|_{X\times X} = \|u\|_X + \|v\|_X$.
We also define the operator $\Psi : X \times X \to X \times X$ by:

$$\Psi(u, v)(t) = \left(\Psi_1(u, v)(t), \Psi_2(u, v)(t) \right), \quad (18)$$

where,

$$\begin{aligned}\Psi_1(u, v)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, u(s), v(s)) ds \\ &+ \sum_{i=1}^{+\infty} \int_0^t \frac{(t-s)^{\alpha+\alpha_i-1}}{\Gamma(\alpha+\alpha_i)} \varphi_i(s) g_i(s, u(s), v(s)) ds \\ &+ \omega_1 t^{n-1} \int_0^\eta \frac{(\eta-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} f_1(s, u(s), v(s)) ds + \\ &+ \omega_1 t^{n-1} \sum_{i=1}^{+\infty} \int_0^\eta \frac{(\eta-s)^{\alpha+\alpha_i+p-1}}{\Gamma(\alpha+\alpha_i+p)} \varphi_i(s) g_i(s, u(s), v(s)) ds\end{aligned} \quad (19)$$

and

$$\begin{aligned}\Psi_2(u, v)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, u(s), v(s)) ds \\ &+ \sum_{i=1}^{+\infty} \int_0^t \frac{(t-s)^{\beta+\beta_i-1}}{\Gamma(\beta+\beta_i)} \phi_i(s) h_i(s, u(s), v(s)) ds \\ &+ \omega_2 t^{n-1} \int_0^\zeta \frac{(\zeta-s)^{\beta+q-1}}{\Gamma(\beta+q)} f_2(s, u(s), v(s)) ds \\ &+ \omega_2 t^{n-1} \sum_{i=1}^{+\infty} \int_0^\zeta \frac{(\zeta-s)^{\beta+\beta_i+q-1}}{\Gamma(\beta+\beta_i+q)} \phi_i(s) h_i(s, u(s), v(s)) ds.\end{aligned} \quad (20)$$

We shall prove that Ψ is contractive:
Let $(u_1, v_1), (u_2, v_2) \in X \times X$. Then, for each $t \in [0, 1]$, we have

$$\begin{aligned}&|\Psi_1(u_2, v_2)(t) - \Psi_1(u_1, v_1)(t)| \le \\ &\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + |\omega_1| \int_0^\eta \frac{(\eta-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} ds \right) \\ &\times \sup_{0\le s\le 1} |f_1(s, u_2(s), v_2(s)) - f_1(s, u_1(s), v_1(s))| \\ &+ \sup_{0\le s\le 1} |g_i(s, u_2(s), v_2(s)) - g_i(s, u_1(s), v_1(s))|) \\ &\times \sum_{i=1}^{+\infty} \left(\sup_{0\le s\le 1} |\varphi_i(s)| \int_0^t \frac{(t-s)^{\alpha+\alpha_i-1}}{\Gamma(\alpha+\alpha_i)} ds \right. \\ &\left. + |\omega_1| \int_0^\eta \frac{(\eta-s)^{\alpha+\alpha_i+p-1}}{\Gamma(\alpha+\alpha_i+p)} ds \right).\end{aligned} \quad (21)$$

For all $t \in [0, 1]$, we can write

$$\begin{aligned}&|\Psi_1(u_2, v_2)(t) - \Psi_1(u_1, v_1)(t)| \le \\ &\left(\frac{1}{\Gamma(\alpha+1)} + \frac{|\omega_1|}{\Gamma(\alpha+p+1)} \right) \\ &\times \sup_{0\le s\le 1} |f_1(s, u_2(s), v_2(s)) - f_1(s, u_1(s), v_1(s))| \\ &+ \sum_{i=1}^{m} \left(\frac{\|\varphi_i\|_\infty}{\Gamma(\alpha+\alpha_i+1)} + \frac{|\omega_1| \|\varphi_i\|_\infty}{\Gamma(\alpha+\alpha_i+p+1)} \right) \\ &\times \sup_{0\le s\le 1} |g_i(s, u_2(s), v_2(s)) - g_i(s, u_1(s), v_1(s))|.\end{aligned} \quad (22)$$

Using $(H1)$, yields the following inequality

$$\begin{aligned}&|\Psi_1(u_2,v_2)(t)-\Psi_1(u_1,v_1)(t)|\leq\\&\quad\left(\tfrac{1}{\Gamma(\alpha+1)}+\tfrac{|\omega_1|}{\Gamma(\alpha+p+1)}\right)\\&\times\overline{L}\left(\sup_{0\leq t\leq 1}|u_2(t)-u_1(t)|+\sup_{0\leq t\leq 1}|v_2(t)-v_1(t)|\right)\\&\quad+\sum_{i=1}^{+\infty}\left(\tfrac{\|\varphi_i\|_\infty}{\Gamma(\alpha+\alpha_i+1)}+\tfrac{|\omega_1|\|\varphi_i\|_\infty}{\Gamma(\alpha+\alpha_i+p+1)}\right)\\&\times\overline{L}\left(\sup_{0\leq t\leq 1}|u_2(t)-u_1(t)|+\sup_{0\leq t\leq 1}|v_2(t)-v_1(t)|\right).\end{aligned}\tag{23}$$

By $(H2)$, we obtain

$$\begin{aligned}&|\Psi_1(u_2,v_2)(t)-\Psi_1(u_1,v_1)(t)|\leq\\&\quad M_1\overline{L}\left(\|u_2-u_1\|_X+\|v_2-v_1\|_X\right).\end{aligned}\tag{24}$$

So that,

$$\begin{aligned}&\|\Psi_1(u_2,v_2)-\Psi_1(u_1,v_1)\|_X\leq\\&\quad\overline{L}\,M_1\|(u_2-u_1,v_2-v_1)\|_{X\times X}.\end{aligned}\tag{25}$$

With the same arguments as before, we have

$$\begin{aligned}&\|\Psi_2(u_2,v_2)-\Psi_2(u_1,v_1)\|_X\leq\\&\quad\overline{L}M_2\|(u_2-u_1,v_2-v_1)\|_{X\times X}.\end{aligned}\tag{26}$$

Using (25) and (26), we deduce that

$$\begin{aligned}&\|\Psi(u_2,v_2)-\Psi(u_1,v_1)\|_{X\times X}\leq\\&\overline{L}\left(M_1+M_2\right)\|(u_2-u_1,v_2-v_1)\|_{X\times X}.\end{aligned}\tag{27}$$

Thanks to (17), we conclude that Ψ is a contraction mapping. Hence, by Banach fixed point theorem, there exists a unique fixed point which is a solution of (5).

The second main result is given by the following theorem:

Theorem 2.2: *Assume that $(H2)$ and $(H3)$ are satisfied. Then the system (5) has at least one solution in $X\times X$.*

Proof. We use Schaefer fixed point theorem to prove this result:

First of all, we show that the operator Ψ is completely continuous. (The continuity of Ψ on $X\times X$ is trivial and hence it is omitted.)

By the continuity of g_i and h_i imposed in $(H2)$, we can state that the series in (5) are uniformly convergent. On the other, thanks to (ii) of $(H2)$, for any $t\in[0,1]$, we can write:

$$\begin{aligned}&\left\|\sum_{i=1}^{+\infty}\int_0^t\frac{(t-s)^{\alpha+\alpha_i-1}}{\Gamma(\alpha+\alpha_i)}\varphi_i(s)\,g_i(s,u(s),v(s))\,ds\right\|_X\leq\\&\qquad\sum_{i=1}^{+\infty}\tfrac{L_1'\|\varphi_i\|_X}{\Gamma(\alpha+\alpha_i+1)}.\end{aligned}\tag{28}$$

The conditions imposed on α and α_i allow us to obtain:

$$\sum_{i=1}^{+\infty}\frac{L_1'\|\varphi_i\|_X}{\Gamma(\alpha+\alpha_i+1)}\leq L_1'\sum_{i=1}^{+\infty}\|\varphi_i\|_\infty<+\infty.\tag{29}$$

With the same arguments, we can write:

$$\sum_{i=1}^{+\infty}\frac{L_2'\|\phi_i\|_X}{\Gamma(\beta+\beta_i+1)}\leq L_2'\sum_{i=1}^{+\infty}\|\phi_i\|_\infty<+\infty.\tag{30}$$

Step 1: Let us take $r>C>0$, and define $B_r:=\left\{(u,v)\in X\times X,\|(u,v)\|_{X\times X}\leq r\right\};C:=L(M_1+M_2),L:=\max\left\{L_j,L_j'\right\}_{j=1,2}$. Then for $(u,v)\in B_r$ and $t\in[0,1]$, we have:

$$\begin{aligned}&|\Psi_1(u,v)(t)|\\&\leq\left(\tfrac{t^\alpha}{\Gamma(\alpha+1)}+\tfrac{\eta^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\sup_{0\leq t\leq 1}|f_1(t,u(t),v(t))|\\&+\sum_{i=1}^{+\infty}\left(\tfrac{\|\varphi_i\|_\infty t^{\alpha+\alpha_i}}{\Gamma(\alpha+\alpha_i+1)}+\tfrac{\|\varphi_i\|_\infty\eta^{\alpha+\alpha_i+p}}{\Gamma(\alpha+\alpha_i+p+1)}\right)\\&\qquad\times\sup_{0\leq t\leq 1}|g_i(t,u(t),v(t))|.\end{aligned}\tag{31}$$

By (29) and $(H3)$, yields

$$\|\Psi_1(u,v)\|_X\leq LM_1<+\infty.\tag{32}$$

Similarly, for Ψ_2, we have

$$\|\Psi_2(u,v)\|_X\leq LM_2<+\infty.\tag{33}$$

Thanks to (32) and (33), we can write

$$\|\Psi(u,v)\|_{X\times X}\leq C\leq r.\tag{34}$$

Hence, $\Psi(B_r)\subset B_r$.

Step 2: Let $t_1,t_2\in[0,1],t_1<t_2$ and $(u,v)\in B_r$. We have:

$$\begin{aligned}&|\Psi_1(u,v)(t_2)-\Psi_1(u,v)(t_1)|\leq\\&\quad|\int_0^{t_2}\tfrac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,u(s),v(s))\,ds\\&\quad-\int_0^{t_1}\tfrac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,u(s),v(s))\,ds|\\&+|\sum_{i=1}^{+\infty}\int_0^{t_2}\tfrac{(t_2-s)^{\alpha+\alpha_i-1}}{\Gamma(\alpha+\alpha_i)}\varphi_i(s)\,g_i(s,u(s),v(s))\,ds\\&-\sum_{i=1}^{+\infty}\int_0^{t_1}\tfrac{(t_1-s)^{\alpha+\alpha_i-1}}{\Gamma(\alpha+\alpha_i)}\varphi_i(s)\,g_i(s,u(s),v(s))\,ds|\\&+|\omega_1|\left(t_2^{n-1}-t_1^{n-1}\right)|\\&\quad\int_0^{\eta}\tfrac{(\eta-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}f_1(s,u(s),v(s))\,ds|\\&+|\omega_1|\left(t_2^{n-1}-t_1^{n-1}\right)\\&\times\sum_{i=1}^{+\infty}|\int_0^{\eta}\tfrac{(\eta-s)^{\alpha+\alpha_i+p-1}}{\Gamma(\alpha+\alpha_i+p)}\varphi_i(s)\,g_i(s,u(s),v(s))\,ds|.\end{aligned}\tag{35}$$

Using $(H2)$ and $(H3)$, we can write

$$\begin{aligned}&|\Psi_1(u,v)(t_2)-\Psi_1(u,v)(t_1)|\leq\tfrac{L_1\left(t_2^\alpha-t_1^\alpha+(t_2-t_1)^\alpha\right)}{\Gamma(\alpha+1)}\\&+\sum_{i=1}^{+\infty}\tfrac{L_1'\|\varphi_i\|_\infty\left(t_2^{\alpha+\alpha_i}-t_1^{\alpha+\alpha_i}+(t_2-t_1)^{\alpha+\alpha_i}\right)}{\Gamma(\alpha+\alpha_i+1)}\\&+\tfrac{|\omega_1|L_1\left(t_2^{n-1}-t_1^{n-1}\right)}{\Gamma(\alpha+p+1)}+|\omega_1|\sum_{i=1}^{+\infty}\tfrac{L_1'\|\varphi_i\|_\infty\left(t_2^{n-1}-t_1^{n-1}\right)}{\Gamma(\alpha+\alpha_i+p+1)}.\end{aligned}\tag{36}$$

With the same arguments as before, we have

$$\begin{aligned}&|\Psi_2(u,v)(t_2)-\Psi_2(u,v)(t_1)|\leq\tfrac{L_2\left(t_2^\beta-t_1^\beta+(t_2-t_1)^\beta\right)}{\Gamma(\beta+1)}\\&+\sum_{i=1}^{+\infty}\tfrac{L_2\|\phi_i\|_\infty\left(t_2^{\beta+\beta_i}-t_1^{\beta+\beta_i}+(t_2-t_1)^{\beta+\beta_i}\right)}{\Gamma(\beta+\beta_i+1)}\\&+\tfrac{|\omega_2|L_2\left(t_2^{n-1}-t_1^{n-1}\right)}{\Gamma(\beta+p+1)}+|\omega_2|\sum_{i=1}^{+\infty}\tfrac{L_2'\|\phi_i\|_\infty\left(t_2^{n-1}-t_1^{n-1}\right)}{\Gamma(\beta+\beta_i+p+1)}.\end{aligned}\tag{37}$$

Thanks to (36) and (37), we get

$$\begin{aligned}
&|\Psi(u,v)(t_2)-\Psi(u,v)(t_1)| \leq \\
&\frac{L_1(t_2^{\alpha}-t_1^{\alpha}+(t_2-t_1)^{\alpha})}{\Gamma(\alpha+1)}+\frac{|\omega_1|L_1\left(t_2^{n-1}-t_1^{n-1}\right)}{\Gamma(\alpha+p+1)} \\
&+\sum_{i=1}^{+\infty}\frac{L_1'\|\varphi_i\|_\infty\left(t_2^{\alpha+\alpha_i}-t_1^{\alpha+\alpha_i}+(t_2-t_1)^{\alpha+\alpha_i}\right)}{\Gamma(\alpha+\alpha_i+1)} \\
&+|\omega_1|\sum_{i=1}^{+\infty}\frac{L_1'\|\varphi_i\|_\infty\left(t_2^{n-1}-t_1^{n-1}\right)}{\Gamma(\alpha+\alpha_i+p+1)} \\
&+\frac{L_2\left(t_2^{\beta}-t_1^{\beta}+(t_2-t_1)^{\beta}\right)}{\Gamma(\beta+1)}+\frac{|\omega_2|L_2\left(t_2^{n-1}-t_1^{n-1}\right)}{\Gamma(\beta+q+1)} \\
&+\sum_{i=1}^{+\infty}\frac{L_2'\|\varphi_i\|_\infty\left(t_2^{\beta+\beta_i}-t_1^{\beta+\beta_i}+(t_2-t_1)^{\beta+\beta_i}\right)}{\Gamma(\beta+\beta_i+1)} \\
&+|\omega_2|\sum_{i=1}^{+\infty}\frac{L_2'\|\phi_i\|_\infty\left(t_2^{n-1}-t_1^{n-1}\right)}{\Gamma(\beta+\beta_i+q+1)}.
\end{aligned} \tag{38}$$

As $t_2 \to t_1$, the right-hand side of (38) tends to zero. Then, as a consequence of Steps $1,2$ and by Arzela-Ascoli theorem, we conclude that Ψ is completely continuous. Next, we show that the set

$$\Omega := \{(u,v)\in X\times X/\,(u,v)=\lambda\Psi(u,v),\, 0<\lambda<1\} \tag{39}$$

is bounded:

Let $(u,v)\in\Omega$, then $(u,v)=\lambda\Psi(u,v)$, for some $0<\lambda<1$. Hence, for $t\in[0,1]$, we have:

$$u(t)=\lambda\Psi_1(u,v)(t),\, v(t)=\lambda\Psi_2(u,v)(t). \tag{40}$$

Thus,

$$\|(u,v)\|_{X\times X}=\lambda\|\Psi(u,v)\|_{X\times X} \tag{41}$$

By (34), we obtain

$$\|(u,v)\|_{X\times X}\leq\lambda C. \tag{42}$$

Consequently, Ω is bounded. As a conclusion of Schaefer fixed point theorem, we deduce that Ψ has at least one fixed point, which is a solution of (5).

Corollary 2.1: *Under the assumptions of Theorem 2.1, if* $\alpha_i=\beta_i=1$, then the fractional system (5) has exactly one solution in $X\times X$.

Corollary 2.2: *Under the assumptions of Theorem 2.2, if* $\alpha_i=\beta_i=1$, then the system (5) has at least one solution in $X\times X$.

III. Illustrative examples

Example 3.1: We begin by the following system

$$\begin{cases}
D^{\frac{1}{2}}u(t)=\frac{\sin(u(t)+v(t))}{64(t+1)}+\sqrt{2} \\
+\sum_{k=1}^{+\infty}\int_0^t\frac{\exp(-ks)}{k^2}\left(\frac{\sin(u(s)+v(s))}{66(k!)(s^2+1)}\right)ds,\, t\in[0,1], \\
D^{\frac{1}{2}}v(t)=\frac{\sin u(t)+\sin v(t)}{64(t^2+1)}+\sqrt{3} \\
+\sum_{k=1}^{+\infty}\int_0^t\frac{\exp\left(-ks^2\right)}{k^2}\left(\frac{\sin u(s)+\sin v(s)}{66(k!)\sqrt{k}(s\exp(ks^2)+1)}\right)ds,\, t\in[0,1], \\
u(0)=2\sqrt{\pi}I^2u\left(\frac{1}{4}\right),\, v(0)=2\sqrt{\pi}I^2u\left(\frac{1}{4}\right),
\end{cases} \tag{43}$$

where, $\alpha=\beta=\frac{1}{2}, \alpha_k=\beta_k=1\,(k=1,2,3,...)\,,\eta=\zeta=\frac{1}{4}, p=q=2, \gamma=\delta=2\sqrt{\pi}$ and $f_1(t,u,v)=\frac{\sin(u+v)}{64(t+1)}+\sqrt{2}$, $f_2(t,u,v)=\frac{\sin u+\sin v}{64(t^2+1)}+\sqrt{3}$, $g_k(t,u,v)=\frac{\sin u+\sin v}{66(k!)(t^2+1)}$, $h_k(t,u,v)=\frac{\sin u+\sin v}{66(k!)\sqrt{k}(t\exp(kt^2)+1)}$, $\varphi_k(t)=\frac{\exp(-kt)}{k^2}$ and $\phi_k(t)=\frac{\exp\left(-ks^2\right)}{k^2}$. For all $(u_1,v_1),(u_2,v_2)\in R^2, t\in[0,1]$, we have

$$\begin{aligned}
|f_1(t,u_2,v_2)-f_1(t,u_1,v_1)| &\leq \tfrac{1}{64}(|u_2-u_1|+|v_2-v_1|), \\
|f_2(t,u_2,v_2)-f_2(t,u_1,v_1)| &\leq \tfrac{1}{64}(|u_2-u_1|+|v_2-v_1|), \\
|g_k(t,u_2,v_2)-g_1(t,u_1,v_1)| &\leq \tfrac{1}{66}(|u_2-u_1|+|v_2-v_1|), \\
|h_k(t,u_2,v_2)-h_1(t,u_1,v_1)| &\leq \tfrac{1}{66}(|u_2-u_1|+|v_2-v_1|).
\end{aligned}$$

So,

$$M_1=M_2=8.23,$$

where $\sum_{k=1}^{+\infty}\left\|\frac{\exp(-kt)}{k^2}\right\|_\infty=\sum_{k=1}^{+\infty}\left\|\frac{\exp\left(-kt^2\right)}{k^2}\right\|_\infty=\sum_{k=1}^{+\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ and $\overline{L}=\max\left(m_j,m_j'\right)_{j=1,2}=\frac{1}{64}$,
Hence,

$$\overline{L}(M_1+M_2)=0.25<1.$$

The conditions of the Theorem 2.1 hold. Therefore, the problem (43) has a unique solution.

To illustrate the second main result, we give the following example:

Example 3.2: Let us take

$$\begin{cases}
D^{\frac{17}{4}}u(t)=\frac{\cos(u(t)+v(t))}{\pi+t^2}+ \\
\sum_{k=1}^{+\infty}\int_0^t\frac{(t-s)^3}{\Gamma(4)}\frac{\exp(-ks)}{k^4}\frac{e^{-ks}\cos(u(s)\times v(s))}{k^2+2}ds,\, t\in[0,1], \\
h \\
D^{\frac{21}{5}}v(t)=\frac{\sin(u(t)+v(t))}{5+t^2} \\
+\sum_{k=1}^{+\infty}\int_0^t\frac{(t-s)^3}{\Gamma(4)}\frac{\exp\left(-s^2\right)}{k^2\sqrt{k}}\frac{e^{-s}\sin(u(s)\times v(s))}{k\sqrt{k}}ds,\, t\in[0,1], \\
\sum_{j=0}^{3}\left|u^{(j)}(0)\right|+\left|v^{(j)}(0)\right|=0, \\
u^{(4)}(0)=2I^{2.1}u\left(\frac{1}{3}\right),\, v^{(4)}(0)=\sqrt{5}I^{2.3}v\left(\frac{1}{5}\right).
\end{cases} \tag{44}$$

For this example, we have $\alpha=\frac{17}{4}, \beta=\frac{21}{5}, \alpha_k=\beta_k=4; (k=1,2,...)$. For $t\in[0,1]$, we have $\varphi_k(t)=\frac{\exp(-kt)}{k^4}$, $\phi_k(t)=\frac{\exp\left(-s^2\right)}{k^2\sqrt{k}}$ and for each $(u,v)\in R^2$

$$\begin{aligned}
f_1(t,u,v)&=\frac{\cos(u+v)}{\pi+t^2}, \\
f_2(t,u,v)&=\frac{\sin(u+v)}{5+t^2}, \\
g_k(t,u,v)&=\frac{e^{-kt}\cos(u\times v)}{k^2+2}, \\
h_k(t,u,v)&=\frac{e^{-t}\sin(u\times v)}{k\sqrt{k}}.
\end{aligned}$$

For all $k=1,2,...$ and $t\in[0,1]$, we have

$$\begin{aligned}
\sum_{k=1}^{+\infty}\|\varphi_k(t)\|_\infty &= \sum_{k=1}^{+\infty}\frac{1}{k^4}<\infty, \\
\sum_{k=1}^{+\infty}\|\phi_k(t)\|_\infty &= \sum_{k=1}^{+\infty}\frac{1}{k^2\sqrt{k}}<\infty.
\end{aligned}$$

Therefore, by Theorem 2.2, the problem (44) has at least one solution.

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