Complete Stationary Fuzzy Metric Space of The Bounded Closed Fuzzy Sets and Common Fixed Point Theorems

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***Abstract*—In this paper, we introduce a stationary M_∞-fuzzy metric on the set $\mathscr{CB}(X)$, where M_∞-fuzzy metric can be thought of as the degree of nearness between two fuzzy sets with respect to any positive real number and $\mathscr{CB}(X)$ is the class of fuzzy sets with nonempty bounded closed α-cut sets. Under the ϕ-contraction conditions, we give some common fixed point theorems for self-mappings in the space $\mathscr{CB}(X)$.**

***Index Terms*—bounded closed α-cut sets, ϕ-contraction conditions, fixed point theorems, M_∞-fuzzy metric.**

I. Introduction

As a natural generalization of the concept of set, fuzzy sets was introduced initially by Zadeh [34] in 1965. Various concepts of the fuzzy metrics on ordinary set were considered in [4], [9], [11], [15], [18]. It is well known that the Hausdorff metric is very important concept not only in general topology and analysis, but also many authors have expansively developed it in the theory of fuzzy sets and application (see [1], [13], [16], [17], [19], [33], [28]). In [31], J. Rodríguez-López and S. Romaguera introduced and discussed a suitable notion for the Hausdorff fuzzy metric of a given fuzzy metric space (in the sense of George and Veeramani) on the set of its nonempty compact subsets. It is necessary to note that such fuzzy metric space has very important application in studying fixed point theorems for contraction-type mappings [2], [12], [21], [27]. In fuzzy functional analysis, many researches have been done on the fixed point theory in the space of compact fuzzy sets equipped with the supremum metric [3], [5], [7], [8], [26], [35].

We must point out that they have given most of their attention to the class of fuzzy sets with nonempty compact α-cut sets in the metric space X, but few of their attention to the class of fuzzy sets with nonempty bounded closed α-cut sets. However, it is known that all compact sets are bounded closed sets in a general metric space and the converse is not always true. In this paper, based on the Hausdorff fuzzy metric H_M, we introduce a suitable notion for the stationary M_∞-fuzzy metric on the fuzzy sets whose α-cut are nonempty bounded closed for each $\alpha \in [0,1]$. Then, under ϕ-contraction conditions, we give some common fixed point theorems in the fuzzy metric space on fuzzy sets.

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II. Preliminaries

A fuzzy set μ on X is a mapping from X into the unit interval $I = [0,1]$. An α-cut of μ is

$$[\mu]^\alpha = \{x \in X : \mu(x) \geq \alpha\},$$

where $0 < \alpha \leq 1$, and we separately specify the support $[\mu]^0$ of μ to be the closure of the union of $[\mu]^\alpha$ for $0 < \alpha \leq 1$. Denote by $\mathscr{F}(X)$, the family of all fuzzy subsets of X. Let $\mu_1, \mu_2 \in \mathscr{F}(X)$, then μ_1 is said to be included in μ_2, denoted by $\mu_1 \subseteq \mu_2$, if and only if $\mu_1(x) \leq \mu_2(x)$ for each $x \in X$. Thus we have that $\mu_1 \subseteq \mu_2$ if and only if $[\mu_1]^\alpha \subseteq [\mu_2]^\alpha$ for all $\alpha \in I$.

Definition 2.1: [14] A triangular norm (or t-norm for short) is a binary operation $*$ on I, i.e. a function $* : I^2 \to I$, such that for all $a, b, c, d \in I$ the following four axioms are satisfied:

(1) $a * 1 = a$;
(2) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
(3) $a * b = b * a$;
(4) $a * (b * c) = (a * b) * c$;

A t-norm $*$ is said to be continuous if it is a continuous function in $[0,1]^2$; a t-norm $*$ is said to be positive if $a*b > 0$ whenever $a, b \in (0,1]$. The following are examples of t-norms: $a *_P b = a \cdot b$; $a \wedge b = \min(a,b)$, where $a \cdot b$ denotes the usual multiplication for all $a, b \in I$.

Definition 2.2: [10] A stationary fuzzy metric space is an ordered triple $(X, M, *)$ such that X is an arbitrary nonempty set, $*$ is a continuous t-norm and M is a fuzzy set of $X \times X$ satisfying the following conditions, for all $x, y, z \in X$:

(1) $M(x,y) > 0$;
(2) $M(x,y) = 1$ if and only if $x = y$;
(3) $M(x,y) = M(y,x)$;
(4) $M(x,y) \geq M(x,z) * M(z,y)$.

If $(X, M, *)$ is a stationary fuzzy metric space, it will be said that $(M, *)$ is a stationary fuzzy metric on X. Since a stationary fuzzy metric is a special fuzzy metric [11], we can prove that every stationary fuzzy metric $(M, *)$ on X generates a topology τ_M on X which has as a base the family of sets of the form

$$\{B_M(x,\varepsilon) : x \in X, 0 < \varepsilon < 1\},$$

where $B_M(x,\varepsilon) = \{y \in X : M(x,y) > 1 - \varepsilon\}$ for all $\varepsilon \in (0,1)$. A sequence $\{x_i\}_{i\in\mathbb{N}}$ in a stationary fuzzy metric space $(X, M, *)$ is said to be Cauchy if $\lim_{i,j\to\infty} M(x_i, x_j) = 1$; a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X converges to x if $\lim_{i\to\infty} M(x_i, x) = 1$ (see [10]).

Definition 2.3: [24] Let $(X, M, *)$ be a stationary fuzzy metric space and $A \subseteq X$. For a point $x \in X$, if for all $\varepsilon \in$

$(0,1)$, $B_M(x,\varepsilon)\bigcap(A-\{x\})\neq\emptyset$, then x is an accumulation point of A; the set of all accumulation points of A is called the derived set of A, denote by $d(A)$; the union of A and $d(A)$ is called the closure of A, denote by $\overline{A}$. If $d(A)\subseteq A$, then A is a closed set of X.

Definition 2.4: [24] Let $(X,M,*)$ be a stationary fuzzy metric space, $*$ is a positive continuous t-norm and $A\subset X$. If there exists $r\in(0,1)$ such that for all $x,y\in A$ we have $M(x,y)>1-r$, then we say A is a bounded subset of X; if X itself is a bounded set we will say $(X,M,*)$ is a bounded stationary fuzzy metric space.

Definition 2.5: [30] A stationary fuzzy pseudo-metric s-pace is an ordered triple $(X,M,*)$ such that X is an arbitrary nonempty set, $*$ is a continuous t-norm and M is a fuzzy set of $X\times X$ satisfying the following conditions, for all $x,y,z\in X$:

(1) $M(x,x)=1$ for all $x\in X$;
(2) $M(x,y)=M(y,x)$;
(3) $M(x,y)\geq M(x,z)*M(z,y)$.

If $(X,M,*)$ is a stationary fuzzy pseudo-metric space, it will be said that $(M,*)$ is a stationary fuzzy pseudo-metric on X. Given a stationary fuzzy metric space $(X,M,*)$, we shall denote by $\mathcal{CB}(X)$, the set of nonempty bounded closed subsets of $(X,M,*)$.

Let B be a nonempty subset of a stationary fuzzy metric space $(X,M,*)$. For any $x\in X$, let

$$M(x,B)=\sup_{y\in B}M(x,y)=M(B,x).$$

For the empty index set $\emptyset$, we will make the convention that for $a_x\in I$,

$$\sup_{x\in\emptyset}a_x=0 \text{ and } \inf_{x\in\emptyset}a_x=1.$$

It follows that $M(x,\emptyset)=M(\emptyset,x)=0$ (see [29]). Let $\mathscr{P}(X)$ be the powerset of X. Recall that the stationary fuzzy pseudo-metric H_M (see [25]) is defined as

$$\begin{aligned} H_M(A,B) &= \inf_{x\in A}\sup_{y\in B}M(x,y)\wedge\inf_{y\in B}\sup_{x\in A}M(y,x)\\ &=\rho(A,B)\wedge\rho(B,A), \end{aligned}$$

where $A,B\in\mathscr{P}(X)$ and

$$\rho(A,B)=\inf_{x\in A}\sup_{y\in B}M(x,y)=\inf_{x\in A}M(x,B).$$

Denote by $\mathscr{B}(X)$, $\mathscr{C}(X)$ and $\mathscr{CB}(X)$, the totality of fuzzy sets which satisfy that for each $\alpha\in I$, the α-*cut* of μ is nonempty bounded, nonempty closed and nonempty bounded closed in X, respectively. Next, we give the definition of the function M_∞ on $\mathscr{F}(X)\times\mathscr{F}(X)$, which is induced by H_M on $\mathscr{P}(X)\times\mathscr{P}(X)$.

Definition 2.6: Let $(X,M,*)$ be a stationary fuzzy metric space. We define a function M_∞ by

$$\begin{aligned} &M_\infty(\mu_1,\mu_2)\\ =\;&\inf_{\alpha\in I}H_M([\mu_1]^\alpha,[\mu_2]^\alpha)\\ =\;&\inf_{\alpha\in I}(\rho([\mu_1]^\alpha,[\mu_2]^\alpha)\wedge\rho([\mu_2]^\alpha,[\mu_1]^\alpha))\\ =\;&\left(\inf_{\alpha\in I}\rho([\mu_1]^\alpha,[\mu_2]^\alpha)\right)\wedge\left(\inf_{\alpha\in I}\rho([\mu_2]^\alpha,[\mu_1]^\alpha)\right)\\ =\;&\rho_\infty(\mu_1,\mu_2)\wedge\rho_\infty(\mu_2,\mu_1) \end{aligned}$$

where $\mu_1,\mu_2\in\mathscr{F}(X)$ and

$$\rho_\infty(\mu_1,\mu_2)=\inf_{\alpha\in I}\rho([\mu_1]^\alpha,[\mu_2]^\alpha).$$

Definition 2.7: Let $(X,M,*)$, $(Y,M,*)$ be any stationary fuzzy metric spaces. A mapping F is said to be a fuzzy map-ping if and only if F is a mapping from $\mathscr{F}(X)$ into $\mathscr{F}(Y)$, i.e. $F(\mu)\in\mathscr{F}(Y)$ for each $\mu\in\mathscr{F}(X)$. $\mu_0\in\mathscr{CB}(X)$ is said to be a fixed point of a fuzzy self-mapping F of $\mathscr{CB}(X)$ if and only if $\mu_0\subseteq F(\mu_0)$.

III. Main results

Now we will establish our main theorems.

Lemma 3.1: [26] Let A be a set in X and let $\{A_\alpha:\alpha\in I\}$ be a family of subset of A such that

(i) $A_0=A=\overline{\bigcup_{\alpha\in(0,1]}A_\alpha}$;
(ii) $\alpha\leq\beta$ implies $A_\beta\subseteq A_\alpha$;
(iii) $\alpha_1\leq\alpha_2\leq\cdots$, $\lim_{n\to\infty}\alpha_n=\alpha$ implies $A_\alpha=\bigcap_{k=1}^{\infty}A_{\alpha_k}$.

Then, there exists a function $\mu:X\to I$ defined by

$$\mu(x)=\begin{cases}\sup\{\alpha\in I:x\in A_\alpha\}, & x\in A\\ 0, & x\in X-A\end{cases}$$

has the property that $[\mu]^\alpha=A_\alpha$ for all $\alpha\in I$. Conversely, for any fuzzy set μ in X, the family $\{[\mu]^\alpha:\alpha\in I\}$ of α-cut set of μ satisfies the above conditions (i)-(iii).

Theorem 3.1: Let $(X,M,*)$ be a stationary fuzzy metric space. Then, for any $\mu_1,\mu_2,\mu_3\in\mathscr{F}(X)$,

(1) $\rho_\infty(\mu_1,\mu_2)=1$ if and only if $[\mu_1]^\alpha\subseteq\overline{[\mu_2]^\alpha}$ for all $\alpha\in I$;
(2) $\rho_\infty(\mu_1,\mu_3)\geq\rho_\infty(\mu_1,\mu_2)*\rho_\infty(\mu_2,\mu_3)$;
(3) $\rho_\infty(\mu_1,\mu_3)\geq M_\infty(\mu_1,\mu_2)*\rho_\infty(\mu_2,\mu_3)$;
(4) If $[\mu_1]^\alpha\subseteq\overline{[\mu_2]^\alpha}$ for all $\alpha\in I$, then $\rho_\infty(\mu_1,\mu_3)\geq M_\infty(\mu_2,\mu_3)$;
(5) $M_\infty(\mu_1,\mu_2)=1$ if and only if $\overline{[\mu_1]^\alpha}=\overline{[\mu_2]^\alpha}$ for all $\alpha\in I$.

Proof. For (1), since $\rho_\infty(\mu_1,\mu_2)=1$ if and only if $\rho([\mu_1]^\alpha,[\mu_2]^\alpha)=1$ for all $\alpha\in I$. By (1) of Proposition 9 in [25], for all $\alpha\in I$, we have $\rho([\mu_1]^\alpha,[\mu_2]^\alpha)=1$ if and only if $[\mu_1]^\alpha\subseteq\overline{[\mu_2]^\alpha}$. Thus, we can obtain that $\rho_\infty(\mu_1,\mu_2)=1$ if and only if $[\mu_1]^\alpha\subseteq\overline{[\mu_2]^\alpha}$ for all $\alpha\in I$.

For (2), for any $\alpha\in I$, by (4) of Proposition 9 in [25], we have

$$\begin{aligned} \rho([\mu_1]^\alpha,[\mu_3]^\alpha) &\geq \rho([\mu_1]^\alpha,[\mu_2]^\alpha)*\rho([\mu_2]^\alpha,[\mu_3]^\alpha)\\ &\geq \inf_{\alpha\in I}\rho([\mu_1]^\alpha,[\mu_2]^\alpha)*\inf_{\alpha\in I}\rho([\mu_2]^\alpha,[\mu_3]^\alpha)\\ &= \rho_\infty(\mu_1,\mu_2)*\rho_\infty(\mu_2,\mu_3). \end{aligned}$$

By the arbitrariness of α, we have

$$\rho_\infty(\mu_1,\mu_3)\geq\rho_\infty(\mu_1,\mu_2)*\rho_\infty(\mu_2,\mu_3).$$

For (3), since $\rho_\infty(\mu_1,\mu_2)\geq M_\infty(\mu_1,\mu_2)$, by (2) we have

$$\rho_\infty(\mu_1,\mu_3)\geq M_\infty(\mu_1,\mu_2)*\rho_\infty(\mu_2,\mu_3).$$

For (4), since $[\mu_1]^\alpha\subseteq\overline{[\mu_2]^\alpha}$ for all $\alpha\in I$, we get $\rho_\infty(\mu_1,\mu_2)=1$. By (3), we have

$$\rho_\infty(\mu_1,\mu_3)\geq M_\infty(\mu_2,\mu_3).$$

For (5), it follows from (1). □

Theorem 3.2: Let $(X, M, *)$ be a stationary fuzzy metric space, then $(\mathscr{F}(X), M_\infty, *)$ is a stationary fuzzy pseudo-metric space.

Proof. Let $\mu_1, \mu_2, \mu_3 \in \mathscr{F}(X)$, by the definition of M_∞ , (1) of Theorem 3.1 and the commutativity of $\wedge$, it is clear that $M_\infty(\mu, \mu) = 1$ and $M_\infty(\mu_1, \mu_2) = M_\infty(\mu_2, \mu_1)$.

In addition, by (2) of Theorem 3.1, we obtain

$$\begin{aligned} & \rho_\infty(\mu_1, \mu_3) \wedge \rho(\mu_3, \mu_1) \\ \geq \quad & (\rho_\infty(\mu_1, \mu_2) * \rho_\infty(\mu_2, \mu_3)) \\ & \wedge(\rho_\infty(\mu_3, \mu_2) * \rho_\infty(\mu_2, \mu_1)) \\ \geq \quad & (\rho_\infty(\mu_1, \mu_2) \wedge \rho_\infty(\mu_2, \mu_1)) \\ & *(\rho_\infty(\mu_2, \mu_3) \wedge \rho_\infty(\mu_3, \mu_2)). \end{aligned}$$

Consequently, by the definition of M_∞, we get

$$M_\infty(\mu_1, \mu_3) \geq M_\infty(\mu_1, \mu_2) * M_\infty(\mu_2, \mu_3).$$

We conclude that $(\mathscr{F}(X), M_\infty, *)$ is a stationary fuzzy pseudo-metric space. □

Let $\{\mu_n\}_{n=1}^\infty$ be a sequence in $\mathscr{F}(X)$. It follows from the definition of M_∞ that μ_n converges with respect to M_∞ if and only if $[\mu_n]^\alpha$ converges uniformly in $\alpha \in I$ with respect to H_M. In the following we always suppose the continuous t-norm $*$ is positive.

Lemma 3.2: [24] Let $(X, M, *)$ be a stationary fuzzy metric space. If $A, B \subset X$ are any two bounded subsets of X, then $A \cup B$ is a bounded subset of X.

Lemma 3.3: Let $(X, M, *)$ be a stationary fuzzy metric space. If $\mu_1, \mu_2 \in \mathscr{B}(X)$, then $\mu_1 \cup \mu_2 \in \mathscr{B}(X)$.

Proof. Let $\mu_1, \mu_2 \in \mathscr{B}(X)$. By the definition of α-cut, we have

$$[\mu_1 \cup \mu_2]^\alpha = [\mu_1]^\alpha \cup [\mu_2]^\alpha, \text{ for all } \alpha \in I.$$

Hence by Lemma 3.2, for all $\alpha \in I$, we can get $[\mu_1 \cup \mu_2]^\alpha$ is a nonempty bounded subset of X, i.e. $\mu_1 \cup \mu_2 \in \mathscr{B}(X)$. □

Theorem 3.3: Let $(X, M, *)$ be a stationary fuzzy metric space. Then $(\mathscr{CB}(X), M_\infty, *)$ is a stationary fuzzy metric space.

Proof. Let $\mu_1, \mu_2, \mu_3 \in \mathscr{CB}(X)$. By Lemma 3.3, we have $\mu_1 \cup \mu_2 \in \mathscr{B}(X)$, which means there exists $r \in (0, 1)$ such that $M(x, y) > 1 - r$, for all $x, y \in [\mu_1 \cup \mu_2]^0 = [\mu_1]^0 \cup [\mu_2]^0$. Hence, for any $\alpha \in [0, 1]$ and $x \in [\mu_1]^\alpha$, we can get that

$$M(x, [\mu_2]^\alpha) = \sup_{y \in [\mu_2]^\alpha} M(x, y) > 1 - r > 0.$$

Thus we obtain

$$\rho_\infty(\mu_1, \mu_2) = \inf_{\alpha \in [0,1]} \rho([\mu_1]^\alpha, [\mu_2]^\alpha) = \inf_{\alpha \in [0,1]} \inf_{x \in [\mu_1]^\alpha} M(x, [\mu_2]^\alpha) \geq (1 - r) > 0.$$

Similarly, we can get

$$\rho_\infty(\mu_2, \mu_1) = \inf_{\alpha \in [0,1]} \rho([\mu_2]^\alpha, [\mu_1]^\alpha) \geq (1 - r) > 0.$$

Consequently, we have $M_\infty(\mu_1, \mu_2) = \rho_\infty(\mu_1, \mu_2) \wedge \rho_\infty(\mu_2, \mu_1) \geq (1 - r) > 0$.

By the definition of M_∞, (5) of Theorem 3.1 and the commutativity of $\wedge$, it is clear that $M_\infty(\mu_1, \mu_2) = 1$ if and only if $\mu_1 = \mu_2$ and $M_\infty(\mu_1, \mu_2) = M_\infty(\mu_2, \mu_1)$. In addition, by (2) of Theorem 3.1, we obtain

$$\begin{aligned} & \rho_\infty(\mu_1, \mu_3) \wedge \rho_\infty(\mu_3, \mu_1) \\ \geq \quad & (\rho_\infty(\mu_1, \mu_2) * \rho_\infty(\mu_2, \mu_3)) \wedge (\rho_\infty(\mu_3, \mu_2) * \rho_\infty(\mu_2, \mu_1)) \\ \geq \quad & (\rho_\infty(\mu_1, \mu_2) \wedge \rho_\infty(\mu_2, \mu_1)) * (\rho_\infty(\mu_2, \mu_3) \wedge \rho_\infty(\mu_3, \mu_2)). \end{aligned}$$

Consequently, by the definition of M_∞, we get

$$M_\infty(\mu_1, \mu_3) \geq M_\infty(\mu_1, \mu_2) * M_\infty(\mu_2, \mu_3).$$

We conclude that $(\mathscr{CB}(X), M_\infty, *)$ is a stationary fuzzy metric space.□

Example 3.1: Let $(X, M, *)$ be a stationary fuzzy metric space. Denote by $a \cdot b$ the usual multiplication for all $a, b \in I$, and define M_∞^H on $\mathscr{F}(X) \times \mathscr{F}(X)$ by

$$M_\infty^H(\mu_1, \mu_2) = \inf_{\alpha \in I} \frac{1}{1 + H_M([\mu_1]^\alpha, [\mu_2]^\alpha)}$$

for all $\mu_1, \mu_2 \in \mathscr{F}(X)$. Then we can easily get that $(M_\infty^H, \cdot)$ is a stationary fuzzy metric on $\mathscr{F}(X)$.

Lemma 3.4: [25] Let $(X, M, *)$ be a stationary fuzzy metric space. Then $(\mathcal{CB}(X), H_M, *)$ is complete if and only if $(X, M, *)$ is complete.

Theorem 3.4: Let $(X, M, *)$ be a stationary fuzzy metric space. Then $(\mathscr{CB}(X), M_\infty, *)$ is complete if and only if $(X, M, *)$ is complete.

Proof. Let $\{\mu_n\}_{n=1}^\infty$ is a Cauchy sequence in $(\mathscr{CB}(X), M_\infty, *)$. For any $\alpha \in I$, $\{[\mu_n]^\alpha\}_{n=1}^\infty$ is a Cauchy sequence in $(\mathcal{CB}(X), H_M, *)$. From Lemma 3.4, it following that there exists a A_α such that

$$[\mu_n]^\alpha \xrightarrow{H_M} A_\alpha \in \mathcal{CB}(X).$$

Let $\varepsilon \in (0, 1)$. Then, since $\{\mu_n\}_{n=1}^\infty$ is a Cauchy sequence, there exists a $n(\varepsilon)$ such that $n, m \geq n(\varepsilon)$ implies $M_\infty(\mu_n, \mu_m) > 1 - \varepsilon$. Let $n(\geq n(\varepsilon))$ be fixed. Then for any $\alpha \in I$, we have

$$\begin{aligned} H_M([\mu_n]^\alpha, A_\alpha) &= \lim_{m \to \infty} H_M([\mu_n]^\alpha, [\mu_m]^\alpha) \\ &\geq \lim_{m \to \infty} \inf_{\alpha \in I} H_M([\mu_n]^\alpha, [\mu_m]^\alpha) \\ &= \lim_{m \to \infty} M_\infty(\mu_n, \mu_m) \geq 1 - \varepsilon, \end{aligned}$$

which implies $[\mu_n]^\alpha \xrightarrow{H_M} A_\alpha$ uniformly in $\alpha \in I$.

Consider the family $\{A_\alpha : \alpha \in I\}$. By (3) of Proposition 9 in [25], we have

$$H_M\left(\overline{\bigcup_{\alpha \in (0,1]} [\mu_n]^\alpha}, \overline{\bigcup_{\alpha \in (0,1]} A_\alpha}\right) = H_M\left(\bigcup_{\alpha \in (0,1]} [\mu_n]^\alpha, \bigcup_{\alpha \in (0,1]} A_\alpha\right).$$

Thus, we can obtain that

$$\begin{aligned} & H_M\left(A_0, \overline{\bigcup_{\alpha \in (0,1]} A_\alpha}\right) \\ \geq \quad & H_M\left(A_0, [\mu_n]^0\right) * H_M\left([\mu_n]^0, \overline{\bigcup_{\alpha \in (0,1]} [\mu_n]^\alpha}\right) * \\ & \inf_{\alpha \in (0,1]} H_M([\mu_n]^\alpha, A_\alpha) \end{aligned}$$

where $H_M\left([\mu_n]^0, \overline{\bigcup_{\alpha \in (0,1]} [\mu_n]^\alpha}\right) = 1$ since $[\mu_n]^0 = \overline{\bigcup_{\alpha \in (0,1]} [\mu_n]^\alpha}$. Consequently,

$$H_M\left(A_0, \overline{\bigcup_{\alpha\in(0,1]} A_\alpha}\right) = 1.$$

By (7) of Proposition 9 in [25], we have $A_0 = \overline{\bigcup_{\alpha\in(0,1]} A_\alpha}$.

Taking $\alpha_1 \leq \alpha_2$, by (4) of Proposition 9 in [25], we have

$$\rho(A_{\alpha_2}, A_{\alpha_1}) \geq \rho(A_{\alpha_2}, [\mu_n]^{\alpha_2}) * \rho([\mu_n]^{\alpha_2}, [\mu_n]^{\alpha_1}) * \rho([\mu_n]^{\alpha_1}, A_{\alpha_1}).$$

By (1) of Proposition 1 in [24], we have $\rho([\mu_n]^{\alpha_2}, [\mu_n]^{\alpha_1}) = 1$. Thus

$$\rho(A_{\alpha_2}, A_{\alpha_1}) \geq \rho(A_{\alpha_2}, [\mu_n]^{\alpha_2}) * \rho([\mu_n]^{\alpha_1}, A_{\alpha_1}) \to 1 \text{ as } n \to \infty.$$

Hence $\rho(A_{\alpha_2}, A_{\alpha_1}) = 1$, and we have $A_{\alpha_2} \subseteq A_{\alpha_1}$.

Now take $\alpha_0 \in I, \alpha_k \uparrow$, and $\lim_{k\to\infty} \alpha_k = \alpha_0$. Let $\varepsilon \in (0,1)$. There exists a $n(\varepsilon)$ such that $n \geq n(\varepsilon)$, implies $H_M([\mu_n]^\alpha, A_\alpha) > 1-\varepsilon$ for all $\alpha \in I$. Since $[\mu_n(\varepsilon)]^{\alpha_k} \xrightarrow{H_M} [\mu_n(\varepsilon)]^{\alpha_0}$, there exists a $k(n(\varepsilon))$ such that $k \geq k(n(\varepsilon))$, implies $H_M([\mu_n(\varepsilon)]^{\alpha_0}, [\mu_n(\varepsilon)]^{\alpha_k}) > 1-\varepsilon$. Consequently, we have

$$A_{\alpha_0} = \lim_{k\to\infty} A_{\alpha_k} = \bigcap_{k=1}^{\infty}\left(\overline{\bigcup_{i=k}^{\infty} A_{\alpha_i}}\right) = \bigcap_{k=1}^{\infty} A_{\alpha_k}.$$

Thus by Lemma 3.1, there exists a $\mu \in \mathscr{CB}(X)$ with $[\mu]^\alpha = A_\alpha$ for every $\alpha \in I$. It follows that $[\mu_n]^\alpha \xrightarrow{H_M} [\mu]^\alpha$ uniformly in $\alpha \in I$, thus $\mu_n \xrightarrow{M_\infty} \mu$ in $\mathscr{CB}(X)$. This completes the proof. □

Lemma 3.5: [25] Let $(X, M, *)$ be a stationary fuzzy metric space and $A, B \in \mathcal{CB}(X)$. Then

(1) for arbitrarily $\varepsilon \in (0,1)$ and any $x \in A$, there exists $y \in B$ such that $M(x,y) \geq H_M(A,B) - \varepsilon$;

(2) for any $x \in A$ and any $\beta \in [0,1)$, there exists $y \in B$ such that $M(x,y) \geq \beta H_M(A,B)$.

Theorem 3.5: Let $(X, M, *)$ is a stationary fuzzy metric space and $\mu_1, \mu_2 \in \mathscr{CB}(X)$, then

(1) for arbitrarily $\varepsilon \in (0,1)$ and any $\mu_3 \in \mathscr{CB}(X)$ satisfying $\mu_3 \subseteq \mu_1$, there exists a $\mu_4 \in \mathscr{CB}(X)$ such that $\mu_4 \subseteq \mu_2$ and $M_\infty(\mu_3, \mu_4) \geq M_\infty(\mu_1, \mu_2) - \varepsilon$;

(2) for any any $\mu_3 \in \mathscr{CB}(X)$ satisfying $\mu_3 \subseteq \mu_1$ and any $\beta \in [0,1)$, there exists a $\mu_4 \in \mathscr{CB}(X)$ such that $\mu_4 \subseteq \mu_2$ and $M_\infty(\mu_3, \mu_4) \geq \beta M_\infty(\mu_1, \mu_2)$.

Proof. we only prove (1) since it is equivalent to (2).

Since μ_1, μ_2 and μ_3 are in $\mathscr{CB}(X)$, we have $\phi \neq [\mu_3]^\alpha \subseteq [\mu_1]^\alpha$ and $[\mu_2]^\alpha \neq \phi$ for all $\alpha \in I$. Let

$$C_\alpha = \{y : \text{ there exists an } x \in [\mu_3]^\alpha \text{ such that } M(x,y) \geq M_\infty(\mu_1, \mu_2) - \varepsilon\},$$

and let

$$D_\alpha = \{z : \rho_\infty(z, [\mu_3]^\alpha) \geq M_\infty(\mu_1, \mu_2) - \varepsilon\}.$$

By the proof of Lemma 17 in [25], we can get that $\overline{C_\alpha} \subseteq D_\alpha$. Let $[\mu_4]^\alpha = D_\alpha \cap [\mu_2]^\alpha$. For any $x \in [\mu_3]^\alpha \subseteq [\mu_1]^\alpha$, by Lemma 3.5, there exists a $y \in [\mu_2]^\alpha$ such that

$$M(x,y) \geq H_M([\mu_1]^\alpha, [\mu_2]^\alpha) - \varepsilon \geq M_\infty(\mu_1, \mu_2) - \varepsilon.$$

Thus $[\mu_4]^\alpha \in \mathscr{B}(X)$, moreover $[\mu_4]^{\alpha_2} \subseteq [\mu_4]^{\alpha_1}$ if $0 \leq \alpha_1 \leq \alpha_2 \leq 1$.

From the proof of Lemma 17 in [25], we have

$$H_M([\mu_3]^\alpha, [\mu_4]^\alpha) \geq M_\infty(\mu_1, \mu_2) - \varepsilon.$$

Now take $\alpha_0 > 0, \alpha_k \uparrow$, and $\lim_{k\to\infty} \alpha_k = \alpha_0$. From the continuity of M and $\bigcap_{k=1}^{\infty} [\mu_3]^{\alpha_k} = [\mu_3]^{\alpha_0}$, we have $\bigcap_{k=1}^{\infty} D_{\alpha_k} = D_{\alpha_0}$. Then we have

$$\bigcap_{k=1}^{\infty} [\mu_4]^{\alpha_k} = \bigcap_{k=1}^{\infty} (D_{\alpha_k} \cap [\mu_2]^{\alpha_k}) = \left(\bigcap_{k=1}^{\infty} D_{\alpha_k}\right) \cap \left(\bigcap_{k=1}^{\infty} [\mu_2]^{\alpha_k}\right) = D_{\alpha_0} \cap [\mu_2]^{\alpha_0} = [\mu_4]^{\alpha_0}.$$

By Lemma 3.1, we can get that $\mu \in \mathscr{CB}(X)$ such that $\mu_4 \subseteq \mu_2$ and

$$M_\infty(\mu_3, \mu_4) \geq M_\infty(\mu_1, \mu_2) - \varepsilon.$$

This completes the proof. □

Lemma 3.6: [27] Let $\phi : [0,1] \to [0,1]$ be a nondecreasing function satisfying the following condition:

(i) ϕ is continuous from the left;

(ii) $\phi^n(h) \to 1 \ (n \to \infty) \quad for\ all\ h \in (0,1]$,

where ϕ^n denote the $n$thiterative function of ϕ.Then

(1) for each $h \in (0,1)$, such that $\phi(h) > h$;

(2) $\phi(1) = 1$.

Theorem 3.6: Let $(X, M, *)$ be a complete stationary fuzzy metric space and let $\{F_n\}_{n=1}^{\infty}$ be a sequence of fuzzy self-mappings of $\mathscr{CB}(X)$. If there exists a constant $q \in (1, +\infty)$, such that for each $\mu_1, \mu_2 \in \mathscr{CB}(X)$, and for arbitrary positive integers i and j, $i \neq j$,

$$M_\infty(F_i(\mu_1), F_j(\mu_2)) \geq q\phi(M_\infty(\mu_1, \mu_2)) \tag{3.1}$$

where ϕ satisfy the conditions of Lemma 3.6. Then there exists an $\mu^* \in \mathscr{CB}(X)$ such that $\mu^* \subseteq F_i(\mu^*)$, for all $i \in \mathbb{N}^+$.

Proof. Let $\mu_0, \mu_1 \in \mathscr{CB}(X)$ and $\mu_1 \subseteq F_1(\mu_0)$, and $\beta = \frac{1}{q} \in (0,1)$. By Theorem 3.5, there exists $\mu_2 \in \mathscr{CB}(X)$, such that $\mu_2 \subseteq F_2(\mu_1)$ and

$$M_\infty(\mu_1, \mu_2) \geq \beta M_\infty(F_1(\mu_0), F_2(\mu_1)).$$

Again by Theorem 3.5, we can find $\mu_3 \in \mathscr{CB}(X)$ such that $\mu_3 \subseteq F_3(\mu_2)$ and

$$M_\infty(\mu_2, \mu_3) \geq \beta M_\infty(F_2(\mu_1), F_3(\mu_2)).$$

By induction, we produce a sequence $\{\mu_n\}_{n=1}^{\infty}$ of points of $\mathscr{CB}(X)$ such that

$$\begin{cases} \mu_{n+1} \subseteq F_{n+1}(\mu_n) \ n = 0,1,2,\cdots \\ M_\infty(\mu_{n+1}, \mu_n) \geq \beta M_\infty(F_{n+1}(\mu_n), F_n(\mu_{n-1})). \end{cases} \tag{3.2}$$

Now we prove that $\{\mu_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathscr{CB}(X)$. In fact, for arbitrary positive integer n, by the inequality (3.1) and formula (3.2), we have

$$\begin{aligned} M_\infty(\mu_{n+1}, \mu_n) &\geq \beta M_\infty(F_{n+1}(\mu_n), F_n(\mu_{n-1})) \\ &\geq \beta q\phi(M_\infty(\mu_n, \mu_{n-1})) \\ &= \phi(M_\infty(\mu_n, \mu_{n-1})). \end{aligned} \tag{3.3}$$

Thus, from the above inequality (3.3), we easily obtain the following relations:

$$\begin{aligned} M_\infty(A_{n+1}, A_n) &\geq \phi(M_\infty(\mu_n, \mu_{n-1})) \\ &\geq \phi^2(M_\infty(\mu_n, \mu_{n-1})) \\ &\geq \cdots \geq \phi^n(M_\infty(\mu_1, \mu_0)). \end{aligned}$$

Furthermore, for arbitrary positive integers n and p, we get that

$$\begin{aligned} & M_\infty(\mu_{n+p}, \mu_n) \\ \geq & \left(\phi^{n+p-1} * \phi^{n+p-2} * \cdots * \phi^n\right)(M_\infty(\mu_1, \mu_0)). \end{aligned}$$

Since for arbitrary $h \in (0,1)$, $\phi^n(h) \to 1$ $(n \to \infty)$, and by continuity of $*$, we have

$$M_\infty(\mu_{n+p}, \mu_n) \to 1 \quad (n \to \infty),$$

i.e. $\{\mu_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{CB}(X)$. By Theorem 3.3, $\mathscr{CB}(X)$ is complete since X is complete. Consequently, there exists $\mu^* \in \mathscr{CB}(X)$ such that $\mu_n \to \mu^*$ $(n \to \infty)$, i.e. $\lim_{n\to\infty} M_\infty(\mu_n, \mu^*) = 1$.

Next, we show that $\mu^* \subseteq F_i(\mu^*)$, i.e. $\rho_\infty(\mu^*, F_i(\mu^*)) = 1$, for all $i \in \mathbb{N}^+$. In fact, for arbitrary positive integers i and j, $i \neq j$, by (3) of Theorem 3.1, we have

$$\begin{aligned} & \rho_\infty(\mu^*, F_i(\mu^*)) \\ \geq & M_\infty(\mu^*, \mu_j) * \rho_\infty(\mu_j, F_i(\mu^*)) \\ \geq & M_\infty(\mu^*, \mu_j) * M_\infty(F_j(\mu_{j-1}), F_i(\mu^*)). \end{aligned}$$

Moreover, we have

$$M_\infty(F_j(\mu_{j-1}), F_i(\mu^*)) \geq q\phi(M_\infty(\mu_{j-1}, \mu^*)) > \phi(M_\infty(\mu_{j-1}, \mu^*)),$$

Consequently, we get

$$\rho_\infty(\mu^*, F_i(\mu^*)) > M_\infty(\mu^*, \mu_j) * \phi(M_\infty(\mu_{j-1}, \mu^*)).$$

Since ϕ is continuous from the left and $*$ is a continuous positive t-norm. Hence, we can obtain

$$\rho_\infty(\mu^*, F_i(\mu^*)) \geq \lim_{j\to\infty} M_\infty(\mu^*, \mu_j) * \phi\left(\lim_{j\to\infty} M_\infty(\mu_{j-1}, \mu^*)\right) = 1 * \phi(1) = 1,$$

i.e. $\rho_\infty(\mu^*, F_i(\mu^*)) = 1$. By (1) of Theorem 3.1, we obtain $\mu^* \subseteq F_i(\mu^*)$, for all $i \in \mathbb{N}^+$. □

Corollary 3.1: Let $(X, M, *)$ be a complete stationary fuzzy metric space and let F be a fuzzy self-mappings of $\mathscr{CB}(X)$. If there exists a constant $q \in (1, +\infty)$, such that for each $\mu_1, \mu_2 \in \mathscr{CB}(X)$,

$$M_\infty(F(\mu_1), F(\mu_2)) \geq q\phi(M_\infty(\mu_1, \mu_2))$$

where ϕ satisfy the conditions of Lemma 3.6. Then there exists an $\mu^* \in \mathscr{CB}(X)$ such that $\mu^* \subseteq F(\mu^*)$.

Proof. In fact, we can define a sequence of fuzzy self-mappings of $\mathscr{CB}(X)$ as $F_i = F$, for $i = 1, 2, \cdots$. Thus, this result is a special case of Theorem 3.6. □

Theorem 3.7: Let $(X, M, *)$ be a complete stationary fuzzy metric space and let $\{F_n\}_{n=1}^\infty$ be a sequence of fuzzy self-mappings of $\mathscr{CB}(X)$. If there exists a constant $q \in (1, +\infty)$, such that for each $\mu_1, \mu_2 \in \mathscr{CB}(X)$, and for arbitrary positive integers i and j, $i \neq j$,

$$\begin{aligned} & M_\infty(F_i(\mu_1), F_j(\mu_2)) \\ \geq & q\phi(\min\{\rho_\infty(\mu_1, F_i(\mu_1)), \rho_\infty(\mu_2, F_j(\mu_2))\}) \quad (3.4) \end{aligned}$$

where ϕ satisfy the conditions of Lemma 3.6. Then there exists an $\mu^* \in \mathscr{CB}(X)$ such that $\mu^* \subseteq F_i(\mu^*)$, for all $i \in \mathbb{N}^+$.

Proof. Let $\mu_0, \mu_1 \in \mathscr{CB}(X)$ and $\mu_1 \subseteq F_1(\mu_0)$, and $\beta = \frac{1}{q} \in (0,1)$. By Theorem 3.5, there exists $\mu_2 \in \mathscr{CB}(X)$, such that $\mu_2 \subseteq F_2(\mu_1)$ and

$$M_\infty(\mu_1, \mu_2) \geq \beta M_\infty(F_1(\mu_0), F_2(\mu_1)).$$

Again by Theorem 3.5, we can find $\mu_3 \in \mathscr{CB}(X)$ such that $\mu_3 \subseteq F_3(\mu_2)$ and

$$M_\infty(\mu_2, \mu_3) \geq \beta M_\infty(F_2(\mu_1), F_3(\mu_2)).$$

By induction, we produce a sequence $\{\mu_n\}_{n=1}^\infty$ of points of $\mathscr{CB}(X)$ such that

$$\begin{cases} \mu_{n+1} \subseteq F_{n+1}(\mu_n) \ n = 0, 1, 2, \cdots \\ M_\infty(\mu_{n+1}, \mu_n) \geq \beta M_\infty(F_{n+1}(\mu_n), F_n(\mu_{n-1})). \end{cases} \quad (3.5)$$

Now we prove that $\{\mu_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{CB}(X)$. In fact, for arbitrary positive integer n, by the inequality (3.4) and formula (3.5), we have

$$\begin{aligned} & M_\infty(\mu_n, \mu_{n+1}) \\ \geq & \beta M_\infty(F_n(\mu_{n-1}), F_{n+1}(\mu_n)) \\ \geq & \beta q\phi(\min\{\rho_\infty(\mu_{n-1}, F_n(\mu_{n-1})), \rho_\infty(\mu_n, F_{n+1}(\mu_n))\}) \\ = & \phi(\min\{\rho_\infty(\mu_{n-1}, F_n(\mu_{n-1})), \rho_\infty(\mu_n, F_{n+1}(\mu_n))\}) \\ \geq & \phi(\min\{M_\infty(\mu_{n-1}, \mu_n), M_\infty(\mu_n, \mu_{n+1})\}) \end{aligned}$$

where $\mu_n \subseteq F_n(\mu_{n-1})$, which implies that $\rho_\infty(\mu_n, F_n(\mu_{n-1})) = 1$.

If $M_\infty(\mu_{n-1}, \mu_n) \wedge M_\infty(\mu_n, \mu_{n+1}) = M_\infty(\mu_n, \mu_{n+1})$, then

$$M_\infty(\mu_n, \mu_{n+1}) \geq \phi(M_\infty(\mu_n, \mu_{n+1})). \quad (3.6)$$

From $\mu_{n+1} \subseteq F_{n+1}(\mu_n)$, it follows that $M_\infty(\mu_n, \mu_{n+1}) \in (0, 1]$. Hence, there are two cases:

Case 1: If $M_\infty(\mu_n, \mu_{n+1}) = 1$.

By (2) of Lemma 3.6, we can get

$$M_\infty(\mu_n, \mu_{n+1}) = M_\infty(\mu_{n-1}, \mu_n) = 1,$$

i.e. $M_\infty(\mu_n, \mu_{n+1}) \geq \phi(M_\infty(\mu_{n-1}, \mu_n))$.

Case 2: If $M_\infty(\mu_n, \mu_{n+1}) \in (0, 1)$.

By (1) of Lemma 3.6, we can get

$$M_\infty(\mu_n, \mu_{n+1}) < \phi(M_\infty(\mu_n, \mu_{n+1})). \quad (3.7)$$

Obviously, (3.6)and (3.7) are contradictory. Hence, we have

$$M_\infty(\mu_{n-1}, \mu_n) \wedge M_\infty(\mu_n, \mu_{n+1}) = M_\infty(\mu_{n-1}, \mu_n),$$

i.e. $M_\infty(\mu_n, \mu_{n+1}) \geq \phi(M_\infty(\mu_{n-1}, \mu_n))$.

Consequently, we easily obtain the following inequalities

$$M_\infty(\mu_{n+1}, \mu_n) \geq \phi(M_\infty(\mu_n, \mu_{n-1})) \geq \phi^2(M_\infty(\mu_{n-1}, \mu_{n-2})) \geq \cdots \geq \phi^n(M_\infty(\mu_1, \mu_0)).$$

Thus, for arbitrary positive integer p, we have

$$\begin{aligned} & M_\infty(\mu_{n+p}, \mu_n) \\ \geq & \left(\phi^{n+p-1} * \phi^{n+p-2} * \cdots * \phi^n\right)(M_\infty(\mu_1, \mu_0)). \end{aligned}$$

Since $\phi^n(h) \to 1$ $(n \to \infty)$, for all $h \in (0, 1]$, we get

$$\lim_{n\to\infty} M_\infty(\mu_{n+p}, \mu_n) \geq \lim_{n\to\infty} \left(\phi^{n+p-1} * \phi^{n+p-2} * \cdots * \phi^n\right)(M_\infty(\mu_1, \mu_0)) = 1,$$

which implies that $\lim_{n\to\infty} M_\infty(\mu_{n+p}, \mu_n) = 1$. Hence, $\{\mu_n\}_{n=1}^\infty$ is a Cauchy sequence. In addition, since $(X, M, *)$ is a complete stationary fuzzy metric space, by Theorem 3.3, we get $(\mathscr{CB}(X), M_\infty, *)$ is complete. Thus there exists an $\mu^* \in \mathscr{CB}(X)$ such that $\mu_n \to \mu^*$ as $n \to \infty$, i.e. $\lim_{n\to\infty} M_\infty(\mu_n, \mu^*) = 1$.

Next, we show that $\mu^* \subseteq F_i(\mu^*)$, i.e. $\rho_\infty(\mu^*, F_i(\mu^*)) = 1$, for all $i \in \mathbb{N}^+$. In fact, for arbitrary positive integers i and j, $i \neq j$, by (3) of Theorem 3.1 we have

$$\begin{aligned} & \rho_\infty(\mu^*, F_i(\mu^*)) \\ \geq\ & M_\infty(\mu^*, \mu_j) * \rho_\infty(\mu_j, F_i(\mu^*)) \\ =\ & M_\infty(\mu^*, \mu_j) * \rho_\infty(F_j(\mu_{j-1}), F_i(\mu^*)) \\ \geq\ & M_\infty(\mu^*, \mu_j) * M_\infty(F_j(\mu_{j-1}), F_i(\mu^*)). \end{aligned}$$

Moreover, we have

$$\begin{aligned} & M_\infty(F_j(\mu_{j-1}), F_i(\mu^*)) \\ \geq\ & q\phi(\min\{\rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), \rho_\infty(\mu^*, F_i(\mu^*))\}) \\ >\ & \phi(\min\{M_\infty(\mu_{j-1}, \mu_j), \rho_\infty(\mu^*, F_i(\mu^*))\}). \end{aligned}$$

Since ϕ is continuous from the left and $*$ is a continuous positive t-norm. Hence, we can obtain

$$\begin{aligned} & \lim_{j\to\infty} M_\infty(F_j(\mu_{j-1}), F_i(\mu^*)) \\ \geq\ & \lim_{j\to\infty} \phi(\min\{M_\infty(\mu_{j-1}, \mu_j), \rho_\infty(\mu^*, F_i(\mu^*))\}) \\ =\ & \phi(\min\{1, \rho_\infty(\mu^*, F_i(\mu^*))\}) \\ =\ & \phi(\rho_\infty(\mu^*, F_i(\mu^*))). \end{aligned}$$

Consequently, we conclude that

$$\begin{aligned} & \rho_\infty(\mu^*, F_i(\mu^*)) \\ \geq\ & \lim_{j\to\infty} M_\infty(\mu^*, \mu_j) * \lim_{j\to\infty} M_\infty(F_j(\mu_{j-1}), F_i(\mu^*)) \\ \geq\ & 1 * \phi(\rho_\infty(\mu^*, F_i(\mu^*))) \\ =\ & \phi(\rho_\infty(\mu^*, F_i(\mu^*))), \end{aligned}$$

i.e. $\rho_\infty(\mu^*, F_i(\mu^*)) = 1$. By (1) of Theorem 3.1, we obtain $\mu^* \subseteq F_i(\mu^*)$, for all $i \in \mathbb{N}^+$. □

Corollary 3.2: Let $(X, M, *)$ be a complete stationary fuzzy metric space and let F be a fuzzy self-mappings of $\mathscr{CB}(X)$. If there exists a constant $q \in (1, +\infty)$, such that for each $\mu_1, \mu_2 \in \mathscr{CB}(X)$,

$$M_\infty(F(\mu_1), F(\mu_2)) \geq q\phi(\min\{\rho_\infty(\mu_1, F(\mu_1)), \rho_\infty(\mu_2, F(\mu_2))\})$$

where ϕ satisfy the conditions of Lemma 3.6. Then there exists an $\mu^* \in \mathscr{CB}(X)$ such that $\mu^* \subseteq F(\mu^*)$.

Proof. In fact, we can define a sequence of fuzzy self-mappings of $\mathscr{CB}(X)$ as $F_i = F$, for $i = 1, 2, \cdots$. Thus, this result is a special case of Theorem 3.7. □

IV. CONCLUSIONS

In this paper, we have been established the completeness of $\mathscr{CB}(X)$ with respect to the completeness of the stationary fuzzy metric space X. We also present some common fixed point theorems for the self-mapping of stationary fuzzy metric space $\mathscr{CB}(X)$ under some ϕ-contraction conditions.

Several possible applications of our results may be suggested. We briefly mention some of them. Fuzzy fixed point theory can be used in existence and continuity theorems for dynamical systems with some vague parameters [6], [20], [23], [32]. More specifically in the field of qualitative behavior, these may be used demonstrating the existence of solutions of the fuzzy differential equation [12] and fuzzy integral equation, etc. In addition, this work offers a new tool for the description and analysis of fuzzy metric space. So we hope our results would provide a mathematical background to ongoing work in the problems of those related fields.

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REFERENCES

[1] A. Abd-Elaal, H. Hefny and A. Abd-Elwahab, "Forecasting of egypt wheat imports using multivariate fuzzy time series model based on fuzzy clustering," IAENG International Journal of Computer Science, vol. 40, no.4, pp. 230-237, 2013

[2] A. Azam and I. Beg, "Common fuzzy fixed points for fuzzy mappings", Fixed Point Theory and Applications, vol. 2013, 11 pages, 2013

[3] H. Adibi, Y.J. Cho, D. O'Regan and R. Saadati, "Common fixed point theorems in $\mathscr{L}$-fuzzy metric spaces", Appl. Math. Comput, vol. 182, pp. 820-828, 2006

[4] S.S. Chang, Y.J. Cho, B.S. Lee, J.S. Jung and S.M. Kang, "Coincidence point and minimization theorems in fuzzy metric spaces", Fuzzy Sets and Systems, vol. 88, pp. 119-128, 1997

[5] Y.J. Cho and N. Petrot, "Existence theorems for fixed fuzzy points with closed α-cut sets in complete metric spaces", Commun. Korean Math. Soc., vol. 26, pp. 115-124, 2011

[6] W. Fei, "Existence and uniqueness of solution for fuzzy random differential equations with non-Lipschitz coefficients", Information Science, vol. 77, pp. 4329-4337, 2007

[7] B.M. Ghil and Y.K. Kim, "Continuity of functions defined on the space of fuzzy sets", Information Sciences, vol. 157, pp. 155-165, 2003

[8] V. Gregori and A. Sapena, "On fixed point theorem in fuzzy metric spaces", Fuzzy Sets and Systems, vol. 125, pp. 245-252, 2002

[9] V. Gregori and S. Romaguera, "On completion of fuzzy metric spaces", Fuzzy Sets and Systems, vol. 130, pp. 399-404, 2002

[10] V. Gregori and S. Romaguera, "Characterizing completable fuzzy metric spaces", Fuzzy Sets and Systems, vol. 144, pp. 411-420, 2004

[11] A. George and P. Veeramani, "On some results in fuzzy metric spaces", *Fuzzy Sets and Systems*, vol. 64, pp. 395-399, 1994

[12] O. Kaleva, "Fuzzy differential equations", Fuzzy Sets and Systems, vol. 24, pp. 301-317, 1987

[13] K. Khiabani and S. Aghabozorgi, "Adaptive time-variant model optimization for fuzzy-time-series forecasting", IAENG International Journal of Computer Science, vol. 42, no.2, pp. 107-116, 2015

[14] E.P. Klement, R. Mesiar and E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000

[15] I. Kramosil and J. Michalek, "Fuzzy metric and statistical metric spaces", Kybernetika, vol. 11, pp. 326-334, 1975

[16] H. Liu, "Morita theory for the category of fuzzy s-acts", IAENG International Journal of Computer Science, vol. 42, no.1, pp. 54-58, 2015

[17] G. Lu and R. Liu, "Discrimination for fuzzy sets related to NTV metric", IAENG International Journal of Applied Mathematics, vol. 43, no.4, pp. 204-208, 2013

[18] G. Matheron, Random Sets and Integral Geometry, Wiley, New York, 1975

[19] H. Miyajima, N. Shigei and H. Miyajima, "Approximation capabilities of interpretable fuzzy inference systems", IAENG International Journal of Computer Science, vol.42, no.2, pp. 117-124, 2015

[20] M. T. Mizukoshi, L. C. Barros, Y. Chalco-Cano, H. Roman-Flores and R. C. Bassanezi, "Fuzzy differential equations and the extension principle", Information Science, vol. 177, pp. 3627-3635, 2007

[21] S.B. Nadler Jr., "Multi-valued contraction mappings", Pacific J. Math., vol. 30, pp. 475-487, 1969

[22] M.L. Puri and D.A. Ralescu, "Fuzzy random variables", Journal of Mathematical Analysis and Applications, vol. 114, pp. 409-422, 1986

[23] G. Papaschinopoulos, G. Stefanidou and P. Efraimidis, "Existence, uniqueness and asymptotic behavior of the solutions of a fuzzy differential equation with piecewise constant argument", Information Sciences, vol. 177, pp. 3855-3870, 2007

[24] D. Qiu, C. Lu, S. Deng and L. Wang, "On the hyperspace of bounded closed sets under a generalized Hausdorff stationary fuzzy metric", Kybernetika, vol. 50, pp. 758-773, 2014

[25] D. Qiu, C. Lu, W. Zhang and Y. Mo, "On common fixed point theorems in the stationary fuzzy metric space of the bounded closed sets", Abstract and Applied Analysis Vol. 2013 Article ID 951259, 11 pages, 2013

[26] D. Qiu and L. Shu, "Supremum metric on the space of fuzzy sets and common fixed point theorems for fuzzy mappings", Information Sciences, vol. 178, pp. 3595-3604, 2008

[27] D. Qiu, L. Shu and J. Guan, "Common fixed point theorems for fuzzy mappings under Φ-contraction condition", Chaos, Solitons and Fractals, vol. 41, pp. 360-367, 2009

[28] D. Qiu and W. Zhang, "The strongest t-norm for fuzzy metric spaces", Kybernetika, vol. 49, pp.141-148, 2013

[29] D. Qiu, W. Zhang and C. Li, “On decomposable measures constructed by using stationary fuzzy pseudo-ultrametrics”, International Journal of General Systems, vol. 42, pp. 395-404, 2013
[30] D. Qiu, W. Zhang and C. Li, “Extension of a class of decomposable measures using fuzzy pseudometrics”, Fuzzy Sets and Systems, vol. 222, pp. 33-44, 2013
[31] J. Rodríguez-López and S. Romaguera, “The Hausdorff fuzzy metric on compact sets”, Fuzzy Sets and Systems , vol. 147, pp. 273-283, 2004
[32] P. Veeramani, “Best approximation in fuzzy metric space”, J. Fuzzy Math. , vol. 9, pp. 75-80, 2001.
[33] H. Yano, K. Matsui and M. Furuhashi, “Interactive decision making for multiobjective fuzzy random linear programming problems based on coefficients of variation”, IAENG International Journal of Applied Mathematics, vol. 44, no.3, pp. 137-143, 2014
[34] L.A. Zadeh, “Fuzzy sets”, Inform. Control., vol. 8, pp. 338-353, 1965
[35] W. Zhang, D. Qiu, Z. Li and G. Xiong, “Common fixed point theorems in a new fuzzy metric space”, Journal of Applied Mathematics, Vol. 2012, Article ID 890678, 18 pages, 2012