Some Inequalities for Generalized L_p-mixed Affine Surface Areas

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***Abstract*—The concept of mixed affine surface was defined by Lutwak in 1987. Recently, Wang and Leng introduced the concept of L_p-mixed affine surface areas. More recently, L_p-mixed affine surface areas have been further generalized by Ma to the entire class of so-called ith L_p-mixed affine surface areas (also called $(i,0)$-type L_p-mixed affine surface areas). In this article, we continue studying the ith L_p-mixed affine surface areas. Combining with this new notion, a result of Lutwak and two results of Wang and Leng were extended. Furthermore, we establish a monotonic inequality related to the ith L_p-mixed affine surface areas. Finally, two open questions are raised.**

***Index Terms*—mixed affine surface area, L_p-mixed affine surface area, ith L_p-mixed affine surface area.**

I. Introduction

We work in n-dimensional real vector space $\mathbf{R}^n (n \geq 2)$, equipped with the standard Euclidean structure. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean spaces $\mathbf{R}^n$. For the set of convex bodies containing origin in their interiors and the set of convex bodies with centroid in the origin in $\mathcal{K}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$, and let $V(K)$ denotes the n-dimensional volume of a body K. For the standard unit ball B in $\mathbf{R}^n$, its volume is written by $\omega_n = V(B)$.

For $K \in \mathcal{K}^n$, K is said to have a curvature function (see [14]), $f(K,\cdot) : S^{n-1} \to \mathbf{R}$, if its surface area measure $S(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{\mathrm{d}S(K,\cdot)}{\mathrm{d}S} = f(K,\cdot).$$

Let $\mathcal{F}^n$ denote the set of all bodies in $\mathcal{K}^n$ that has a positive continuous curvature function. For $K \in \mathcal{F}^n$, the classical affine surface area, $\Omega(K)$, of K is defined by

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{\frac{n}{n+1}} \mathrm{d}S(u). \quad (1)$$

During the past three decades, the investigations of the classical affine surface area have received great attention from many articles (see papers [2], [3], [4], [5], [6], [8], [9], [10], [11], [13], [14], [24], [25], [27], [28], [31] or books [7], [26]).

The classical mixed affine surface area was given by Lutwak (see [12]). For $K, L \in \mathcal{F}^n$, $j \in \mathbf{R}$, the mixed affine surface area, $\Omega_j(K,L)$, of K and L is defined by

$$\Omega_j(K,L) = \int_{S^{n-1}} f(K,u)^{\frac{n-j}{n+1}} f(L,u)^{\frac{j}{n+1}} \mathrm{d}S(u). \quad (2)$$

From definitions (1) and (2), it is obvious that $\Omega_j(K,K) = \Omega(K)$ and $\Omega_0(K,L) = \Omega(K)$.

A convex body $K \in \mathcal{K}_o^n$ is said to have a L_p-curvature function (see [16]), $f_p(K,\cdot) : S^{n-1} \to \mathbf{R}$, if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{\mathrm{d}S_p(K,\cdot)}{\mathrm{d}S} = f_p(K,\cdot).$$

Let $\mathcal{F}_o^n$ and $\mathcal{F}_c^n$ denote the set of all bodies in $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$ respectively, and both of them have a positive continuous curvature function.

In 1996, Lutwak (see [16]) showed the L_p-affine surface area as follows: For $K \in \mathcal{F}_o^n$, the L_p-affine surface area, $\Omega_p(K)$, of K is given by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}} \mathrm{d}S(u).$$

Further, Lutwak established the well-known L_p-affine isoperimetric inequality as follows:

Theorem A. *If $K \in \mathcal{F}_o^n$ and $p \geq 1$, then*

$$\Omega_p(K) \leq n\omega_n^{\frac{2p}{n+p}} V(K)^{\frac{n-p}{n+p}}, \quad (3)$$

with equality if and only if K is an ellipsoid which centered at the origin.

Regarding the more results of L_p-affine surface area, we may see in these articles [22], [27] and [30].

Recently, Wang and Leng introduced the notion of L_p-mixed affine surface area (see [30]): For $K, L \in \mathcal{F}_o^n, p \geq 1, j \in \mathbf{R}$, the L_p-mixed affine surface area, $\Omega_{p,j}(K,L)$, of K and L is defined by

$$\begin{aligned}&\Omega_{p,j}(K,L)\\ =\ &\int_{S^{n-1}} f_p(K,u)^{\frac{n-j}{n+p}} f_p(L,u)^{\frac{j}{n+p}} \mathrm{d}S(u).\end{aligned} \quad (4)$$

Let $L = B$ in (4), then write $\Omega_{p,j}(K,B) = \Omega_{p,j}(K)$. Since $f_p(B,\cdot) = 1$, the L_p-mixed affine surface area of $K \in \mathcal{F}_o^n$ is that

$$\Omega_{p,j}(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n-j}{n+p}} \mathrm{d}S(u).$$

Associated with (4), Wang and Leng proved the following cycle inequality and Minkowski's inequality for L_p-mixed affine surface area, respectively.

Theorem B. *If $K, L \in \mathcal{F}_o^n$, $p \geq 1$, $j,k,m \in \boldsymbol{R}$ and $j < k < m$, then*

$$\Omega_{p,j}(K,L)^{m-k}\Omega_{p,m}(K,L)^{k-j} \geq \Omega_{p,k}(K,L)^{m-j}, \quad (5)$$

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with equality for $p = 1$ if and only if K and L are homothetic; for $n \neq p > 1$ if and only if K and L are dilates.

Theorem C. *If $K, L \in \mathcal{F}_o^n$, $p \geq 1$, $j \in$ **R**, then for $j < 0$ or $j > n$,*

$$\Omega_{p,j}(K,L)^n \geq \Omega_p(K)^{n-j}\Omega_p(L)^j; \tag{6}$$

for $0 < j < n$,

$$\Omega_{p,j}(K,L)^n \leq \Omega_p(K)^{n-j}\Omega_p(L)^j, \tag{7}$$

with equality in every inequality for $p = 1$ if and only if K and L are homothetic; for $n \neq p > 1$ if and only if K and L are dilates. For $j = 0$ or $j = n$, (6) *(or* (7) *is identical.*

The main aim of this article is to define the notions of ith L_p-mixed affine surface areas and (i, j)-type L_p-mixed affine surface areas, and to extend the above inequalities to the entire family of these new notions. Here, we first give the concepts of L_p-mixed curvature function and L_p-mixed curvature image of convex body.

For $K \in \mathcal{K}_o^n$, $p \geq 1$, and $i = 0, 1, \cdots, n-1$, body K is said to have a L_p-mixed curvature function, $f_{p,i}(K, \cdot) : S^{n-1} \to$ **R**, if its L_p-mixed surface area measure $S_{p,i}(K, \cdot)$ (see [18], [19], [21], [22]) is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{\mathrm{d}S_{p,i}(K,\cdot)}{\mathrm{d}S} = f_{p,i}(K,\cdot). \tag{8}$$

If the ith mixed surface area measure $S_i(K)$ is absolutely continuous with respect to spherical Lebesgue measure S, we have

$$f_{p,i}(K,u) = h(K,u)^{1-p} f_i(K,u), \tag{9}$$

for $u \in S^{n-1}$.

Let $\mathcal{F}_{o,i}^n$ and $\mathcal{F}_{c,i}^n$ denote the set of all bodies in $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively, and both of them have a positive continuous ith curvature function $f_i(K, \cdot)$ (see [15]).

For each $K \in \mathcal{F}_{o,i}^n (i = 0, 1, \cdots, n-1)$ and real $p \geq 1$, define star body $\Lambda_{p,i}K \in \mathcal{S}_o^n$, the ith L_p-mixed curvature image of K, by (see [19], [20])

$$\rho(\Lambda_{p,i}K,\cdot)^{n+p-i} = \frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\omega_n} f_{p,i}(K,\cdot). \tag{10}$$

In particular, taking $i = 0$ in (10), we immediately get Lutwak's definition of L_p-curvature image $\Lambda_p K$ of convex body $K \in \mathcal{F}_o^n$ (see [16]).

Recently, Ma introduced the notion of ith L_p-mixed affine surface area as follows (see [20], [23]): For $p \geq 1$ and $i = 0, 1, \cdots, n-1$, the ith L_p-mixed affine surface area, $\Omega_p^{(i)}(K_1, \cdots, K_{n-i})$, of $K_1, \cdots, K_{n-i} \in \mathcal{F}_{o,i}^n$ is defined by

$$\begin{aligned}&\Omega_p^{(i)}(K_1,\cdots,K_{n-i})\\ &= \int_{S^{n-1}} [f_{p,i}(K_1,u)\cdots f_{p,i}(K_{n-i},u)]^{\frac{1}{n+p-i}}\mathrm{d}S(u).\end{aligned} \tag{11}$$

Let $K_1 = \cdots = K_{n-i-j} = K$ and $K_{n-i-j+1} = \cdots = K_{n-i} = L$ $(j = 0, \cdots, n-i)$, we denote $\Omega_{p,j}^{(i)}(K, L) := \Omega_p^{(i)}(K, \cdots, K, L, \cdots, L)$, with $n-i-j$ copies of K, and j copies of L.

If j is any real, we can define that: For $K, L \in \mathcal{F}_{o,i}^n, i = 0, \cdots, n-1, p \geq 1, j \in$ **R**, the (i, j)-type L_p-mixed affine surface area, $\Omega_{p,j}^{(i)}(K, L)$, of K and L is defined by

$$\begin{aligned}&\Omega_{p,j}^{(i)}(K,L)\\ &= \int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}} f_{p,i}(L,u)^{\frac{j}{n+p-i}}\mathrm{d}S(u).\end{aligned} \tag{12}$$

Let $L = B$ in (12), then we write $\Omega_{p,j}^{(i)}(K) := \Omega_{p,j}^{(i)}(K, B)$. For $u \in S^{n-1}, S_i(B, u) = S, h(B, u) = 1$, it follows from (8) and (9) that $f_{p,i}(B, u) = 1$. Together with (12) yields

$$\Omega_{p,j}^{(i)}(K) = \int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}}\mathrm{d}S(u), \tag{13}$$

where $\Omega_{p,j}^{(i)}(K)$ is called (i, j)-type L_p-mixed affine surface area of $K \in \mathcal{F}_{o,i}^n$. If $j = 0$, we write that

$$\begin{aligned}\Omega_p^{(i)}(K) &= \Omega_{p,0}^{(i)}(K)\\ &= \int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i}{n+p-i}}\mathrm{d}S(u),\end{aligned} \tag{14}$$

where $\Omega_p^{(i)}(K)$ is called $(i, 0)$-type L_p-mixed affine surface area (or is called ith L_p-mixed affine surface area).

In [20], Ma further gives the following an expansion of the definition of the $(i, 0)$-type L_p-mixed affine surface area: If $K \in \mathcal{K}_o^n, p \geq 1$, then the $(i, 0)$-type L_p-mixed affine surface area, $\Omega_p^{(i)}(K)$, of K is defined by

$$\begin{aligned}&n^{-\frac{p}{n-i}}\Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}}\\ &= \inf\left\{nW_{p,i}(K,Q^*)\widetilde{W}_i(Q)^{\frac{p}{n-i}} : Q \in \mathcal{S}_o^n\right\}.\end{aligned} \tag{15}$$

For $i = 0$, the definition is just the definition of L_p-affine surface area by proposed by Lutwak in [16].

The main results of this article can be stated as follows: First, we establish the extended form of Theorem A, given by Theorem 1.

Theorem 1. *Suppose $K \in \mathcal{F}_{c,i}^n (i = 0, 1, \cdots, n-1)$ and $p \geq 1$, then*

$$\Omega_p^{(i)}(K) \leq n\omega_n^{\frac{2p}{n+p-i}} W_i(K)^{\frac{n-i}{n+p-i}} \widetilde{W}_i(K)^{\frac{-p}{n+p-i}}, \tag{16}$$

with equality in inequality for $i = 0$ if and only if K is an ellipsoid which centered at the origin; for $0 < i \leq n-1$ if and only if $K$is a ball which centered at the origin.

In particular, taking $i = 0$ in (16), we immediately obtain L_p-affine isoperimetric inequality (3) for $K \in \mathcal{F}_c^n$.

Next, the cycle inequality of the (i, j)-type L_p-mixed affine surface areas and the Minkowski's inequality of the ith L_p-mixed affine surface areas are given as follows:

Theorem 2. *Suppose $K, L \in \mathcal{F}_{o,i}^n, p \geq 1, i = 0, 1, \cdots, n-1$, $j, k, m \in$ **R** and $j < k < m$, then*

$$\Omega_{p,j}^{(i)}(K,L)^{m-k}\Omega_{p,m}^{(i)}(K,L)^{k-j} \geq \Omega_{p,k}^{(i)}(K,L)^{m-j}, \tag{17}$$

with equality in inequality for $n-i \neq p = 1$ and $0 \leq i < n-1$ if and only if K and L are homothetic; for $n-i \neq p > 1$ and $0 < i < n$ if and only if K and L are dilates.

Let $i = 0$ in (17) of Theorem 2, we immediately get Theorem B.

Theorem 3. *Suppose $K, L \in \mathcal{F}_{o,i}^n, p \geq 1, i = 0, 1, \cdots, n-1$, $j \in$ **R**. For $j < 0$ or $j > n-i$, then*

$$\Omega_{p,j}^{(i)}(K,L)^{n-i} \geq \Omega_p^{(i)}(K)^{n-i-j}\Omega_p^{(i)}(L)^j; \tag{18}$$

for $0<j<n-i$, then

$$\Omega_{p,j}^{(i)}(K,L)^{n-i}\le\Omega_p^{(i)}(K)^{n-i-j}\Omega_p^{(i)}(L)^j, \quad (19)$$

with equality in every inequality for $n-i\neq p=1$ and $0\le i<n-1$ if and only if K and L are homothetic; for $n-i\neq p>1$ and $0\le i<n$ if and only if K and L are dilates. For $j=0$ or $j=n-i$, (18) *(or* (19) *is identical.*

Obviously, the case $i=0$ of Theorem 3 is just Theorem C.

Further, we obtain the more general form of inequality (19):

Theorem 4. *Suppose $K_1,K_2,\cdots,K_{n-i}\in\mathcal{F}_{o,i}^n(i=0,1,\cdots,n-1)$, then for $m\le n-i$,*

$$\begin{aligned}&\Omega_p^{(i)}(K_1,K_2,\cdots,K_{n-i})^m\\ \le&\prod_{j=1}^{m}\Omega_p^{(i)}(\underbrace{K_j,\cdots,K_j}_{m},K_{m+1},\cdots,K_{n-i}),\end{aligned} \quad (20)$$

with equality in inequality if and only if $K_1,K_2,\cdots,K_{n-i}$ $(i=0,1,\cdots,n-1)$ are all dilations of each other (with the origin as the center of dilation).

Taking $m=n-i,K_1=K_2=\cdots=K_{n-i-j}=K$ and $K_{n-i-j+1}=\cdots=K_{n-i}=L(0<j<n-i)$ in (20), we immediately obtain inequality (19).

Finally, we show that a monotonic result as follows:

Theorem 5. *If $K\in\mathcal{K}_o^n$ and $i=0,1,\cdots,n-1$, then for $1\le p\le q$,*

$$\left(\frac{\Omega_q^{(i)}(K)}{n\widetilde{W}_i(K^*)}\right)^{n+q-i}\le\left(\frac{\Omega_p^{(i)}(K)}{n\widetilde{W}_i(K^*)}\right)^{n+p-i}. \quad (21)$$

Particularly, taking $i=0$ in Theorem 5, we immediately get a result of literature [16].

II. Preliminaries

If $K\in\mathcal{K}^n$, then its support function, $h_K=h(K,\cdot):\mathbf{R}^n\to(-\infty,\infty)$, is defined by (see [1])

$$h(K,x)=\max\{\langle x,y\rangle:\ y\in K\},\quad x\in\mathbf{R}^n,$$

where $\langle x,y\rangle$ denotes the standard inner product of x and y.

If K is a compact star-shaped (about the origin) in $\mathbf{R}^n$, its radial function, $\rho_K=\rho(K,\cdot):\mathbf{R}^n\setminus\{0\}\to[0,\infty)$, is defined by (see [1], [25])

$$\rho(K,x)=\max\{\lambda\ge0:\ \lambda x\in K\},x\in\mathbf{R}^n\backslash\{0\},$$

when ρ_K is positive and continuous, K is called a star body (about the origin).

Let S_o^n denote the set of star bodies (about the origin) in $\mathbf{R}^n$. Two star bodies K and L are said to be dilates each other if $\rho_K(u)/\rho_L(u)$ is independent on $u\in S^{n-1}$.

For $K\in\mathcal{K}_o^n$, the polar body, K^*, of K is defined by(see [1], [25])

$$K^*=\{x\in\mathbf{R}^n:\ \langle x,y\rangle\le1,y\in K\}.$$

Obviously, $(K^*)^*=K$. If $\phi\in\mathrm{GL}(n)$, then $(\phi K)^*=\phi^{-t}K^*$; If $\lambda>0$, then $(\lambda K)^*=\lambda^{-1}K^*$.

If $K\in\mathcal{K}_o^n$, then the support and radial functions of the polar body K^* of K are given respectively by

$$h_{K^*}(u)=\frac{1}{\rho_K(u)}\quad\text{and}\quad\rho_{K^*}(u)=\frac{1}{h_K(u)} \quad (22)$$

for all $u\in S^{n-1}$.

For $K,L\in\mathcal{K}_o^n$, $p\ge1$ and $\varepsilon>0$, the Firey L_p-combination $K+_p\varepsilon\cdot L\in\mathcal{K}_o^n$ is defined by (see [15])

$$h(K+_p\varepsilon\cdot L,\cdot)^p=h(K,\cdot)^p+\varepsilon h(L,\cdot)^p,$$

where "$\cdot$" in $\varepsilon\cdot L$ denotes the Firey scalar multiplication.

For $K\in\mathcal{K}^n$ and $i=0,1,\ldots,n-1$, the quermassintegrals, $W_i(K)$, of K are given by (see [1], [25])

$$W_i(K)=\frac{1}{n}\int_{S^{n-1}}h(K,u)\mathrm{d}S_i(K,u), \quad (23)$$

where $S_i(K,\cdot)$ is ith surface area measure of K on S^{n-1}. From (23), we easily see that $W_0(K)=V(K)$.

Associated with the Firey L_p-combination, Lutwak defined L_p-mixed quermassintegrals as follows: For $K,L\in\mathcal{K}_o^n,\varepsilon>0$ and real $p\ge1$, the L_p-mixed quermassintegrals, $W_{p,i}(K,L)$ $(i=0,1,\ldots,n-1)$, of K and L are defined by (see [15])

$$\begin{aligned}&\frac{n-i}{p}W_{p,i}(K,L)\\=&\lim_{\varepsilon\to0^+}\frac{W_i(K+_p\varepsilon\cdot L)-W_i(K)}{\varepsilon}.\end{aligned} \quad (24)$$

Further, Lutwak (see [15]) showed that, for each $K\in\mathcal{K}_o^n$, $p\ge1$ and $i=0,1,\ldots,n-1$, there exists a positive Borel measure $S_{p,i}(K,\cdot)$ (called the L_p-mixed surface area measure of K) on S^{n-1}, such that L_p-mixed quermassintegrals $W_{p,i}(K,L)$ has the following integral representation:

$$W_{p,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}h_L^p(v)\mathrm{d}S_{p,i}(K,v) \quad (25)$$

for all $L\in\mathcal{K}_o^n$. It turns out that the measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_i(K,\cdot)$, and has the Radon-Nikodym derivative

$$\frac{\mathrm{d}S_{p,i}(K,\cdot)}{\mathrm{d}S_i(K,\cdot)}=h^{1-p}(K,\cdot). \quad (26)$$

If $i=0$, then $S_{p,0}(K,\cdot)$ is just the L_p-surface area measure $S_p(K,\cdot)$ of K.

From (23), (25) and (26), we know that

$$W_{p,i}(K,K)=W_i(K). \quad (27)$$

The Minkowski's inequality for L_p-mixed quermassintegrals $W_{p,i}$ can be stated as follows: For $K,L\in\mathcal{K}_o^n$, $p\ge1$ and $i=0,1,\ldots,n-1$, then (see [15])

$$W_{p,i}(K,L)^{n-i}\ge W_i(K)^{n-i-p}W_i(L)^p, \quad (28)$$

with equality for $p=1$ and $0\le i<n-1$ if and only if K and L are homothetic; for $p>1$ if and only if K and L are dilates. For $p=1$ and $i=n-1$, inequality (28) is identical.

According to (27) and (28), we easily get that (see [15])

Lemma 2.1. *Suppose $K,L\in\mathcal{K}_o^n$, $p\ge1$ and $i=0,1,\ldots,n-1$. If for any $Q\in\mathcal{K}_o^n$,*

$$W_{p,i}(K,Q)=W_{p,i}(L,Q),$$

then when $0\le i<n-1$ and $n-i\neq p=1$, K and L are homothetic; when $0\le i<n$ and $n-i\neq p>1$, $K=L$.

The following formula of the dual quermassintegrals will be needed.

For $K \in S_o^n$ and any real i, the dual quermassintegrals $\widetilde{W}_i(K)$, of K are defined by (see [1], [25])

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\mathrm{d}S(u). \tag{29}$$

Obviously, $\widetilde{W}_0(K) = V(K)$.

III. Proofs of Theorems

In this section, we will prove Theorems 1-5. First, we need the following three Lemmas for the proof of Theorem 1.

Lemma 3.1. *If $p \geq 1, K \in \mathcal{K}_o^n, L \in \mathcal{F}_{o,i}^n, 0 \leq i \leq n-1$, then*

$$\Omega_p^{(i)}(L)^{n+p-i} \leq n^{n+p-i} W_{p,i}(L,K)^{n-i}\widetilde{W}_i(K^*)^p, \tag{30}$$

with equality if and only if K^* and $\Lambda_{p,i}L$ are dilates.

Proof. Let $K \in \mathcal{K}_o^n$ and $L \in \mathcal{F}_{o,i}^n$, then from (12), (22), (8), (25), (29) and Hölder's inequality, we have

$$\begin{aligned}
&\Omega_p^{(i)}(L)^{n+p-i}\\
= &\left[\int_{S^{n-1}} f_{p,i}(L,u)^{\frac{n-i}{n+p-i}}\mathrm{d}S(u)\right]^{n+p-i}\\
= &\left[\int_{S^{n-1}} \left(h(K,u)^p f_{p,i}(L,u)\right)^{\frac{n-i}{n+p-i}}\right.\\
&\left.\times\left(h(K,u)^{i-n}\right)^{\frac{p}{n+p-i}}\mathrm{d}S(u)\right]^{n+p-i}\\
\leq &\ n^{n+p-i}\left(\frac{1}{n}\int_{S^{n-1}} h(K,u)^p f_{p,i}(L,u)\mathrm{d}S(u)\right)^{n-i}\\
&\times\left(\frac{1}{n}\int_{S^{n-1}} \rho(K^*,u)^{n-i}\mathrm{d}S(u)\right)^p\\
= &\ n^{n+p-i}\left(\frac{1}{n}\int_{S^{n-1}} h(K,u)^p \mathrm{d}S_{p,i}(L,u)\right)^{n-i}\\
&\times\left(\frac{1}{n}\int_{S^{n-1}} \rho(K^*,u)^{n-i}\mathrm{d}S(u)\right)^p\\
= &\ n^{n+p-i}W_{p,i}(L,K)^{n-i}\widetilde{W}_i(K^*)^p.
\end{aligned}$$

Thus, it follows immediately that (30). According to the condition of equality in Hölder's inequality, and combined with the definition of L_p-mixed curvature image, we know that equality holds in (30) if and only if

$$\frac{h(K,u)^p f_{p,i}(L,u)}{h(K,u)^{i-n}} = c$$

for any $u \in S^{n-1}$, where c is a constant, i.e., for any $u \in S^{n-1}$, we have

$$\frac{\rho(\Lambda_{p,i}L,u)^{n+p-i}}{\rho(K^*,u)^{n+p-i}} = \frac{c\widetilde{W}_i(\Lambda_{p,i}L)}{\omega_n},$$

this shows that $\Lambda_{p,i}L$ and K^* are dilates. Therefore, the equality holds in inequality (30) if and only if $\Lambda_{p,i}L$ and K^* are dilates.

Lemma 3.2. ([17])*Suppose $K \in \mathcal{K}_o^n, i \in$ **R** and $0 \leq i < n$, then*

$$\widetilde{W}_i(K) \leq \omega_n^{\frac{i}{n}} V(K)^{\frac{n-i}{n}}, \tag{31}$$

with equality for $0 < i < n$ if and only if K is a ball which centered at the origin. If $i = 0$, then (31) *is a identical.*

We also need the following well-known Blaschke-Santaló inequality (see [14]):

Lemma 3.3. *Suppose $K \in \mathcal{K}_c^n$, then*

$$V(K)V(K^*) \leq \omega_n^2, \tag{32}$$

with equality if and only if K is an ellipsoid which centered at the origin.

Now we will complete the proof of Theorem 1. In fact, we prove the following more general conclusion:

Theorem 3.1. *Suppose $K \in \mathcal{K}_c^n$ and $L \in \mathcal{F}_{o,i}^n$, then for $p \geq 1$,*

$$n\omega_n^{\frac{2p}{n+p-i}} W_{p,i}(L,K)^{\frac{n-i}{n+p-i}} \geq \widetilde{W}_i(K)^{\frac{p}{n+p-i}}\Omega_p^{(i)}(L), \tag{33}$$

with equality in inequality for $i = 0$ if and only if K and L are ellipsoids which centered at the origin, and K and L are dilates; for $0 < i \leq n-1$ if and only if K and L are balls which centered at the origin, and K and L are dilates.

Proof. Using inequalities (30), (31) and (32), we have

$$\begin{aligned}
&\Omega_p^{(i)}(L)^{n+p-i}\widetilde{W}_i(K)^p\\
\leq &\ n^{n+p-i}W_{p,i}(L,K)^{n-i}(\widetilde{W}_i(K^*)\widetilde{W}_i(K))^p\\
\leq &\ n^{n+p-i}W_{p,i}(L,K)^{n-i}\omega_n^{\frac{2ip}{n}}\left(V(K)V(K^*)\right)^{\frac{(n-i)p}{n}}\\
\leq &\ n^{n+p-i}\omega_n^{2p}W_{p,i}(L,K)^{n-i}.
\end{aligned}$$

According to the condition of equality for inequality (30), (31) and (32), we know that equality holds for $i = 0$ if and only if K and L are ellipsoids which centered at the origin, and K and L are dilates; for $0 < i \leq n-1$ if and only if K and L are balls which centered at the origin, and K and L are dilates.

If $K \in \mathcal{F}_{c,i}^n$, taking $L = K$ in inequality (33), this yields Theorem 1 by (27).

Proof of Theorem 2. For $i = 0$, Theorem 2 is just Theorem B in [29].

For $1 < i \leq n-1$, together with (i,j)-type L_p-mixed affine surface area (12) and Hölder's inequality, we have that

$$\begin{aligned}
&\Omega_{p,j}^{(i)}(K,L)^{\frac{m-k}{m-j}}\Omega_{p,m}^{(i)}(K,L)^{\frac{k-j}{m-j}}\\
= &\left[\int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}} f_{p,i}(L,u)^{\frac{j}{n+p-i}}\mathrm{d}S(u)\right]^{\frac{m-k}{m-j}}\\
\times &\left[\int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-m}{n+p-i}} f_{p,i}(L,u)^{\frac{m}{n+p-i}}\mathrm{d}S(u)\right]^{\frac{k-j}{m-j}}\\
= &\left[\int_{S^{n-1}} \left(f_{p,i}(K,u)^{\frac{(n-i-j)(m-k)}{(n+p-i)(m-j)}}\right.\right.\\
\times &\left.\left. f_{p,i}(L,u)^{\frac{j(m-k)}{(n+p-i)(m-j)}}\right)^{\frac{m-j}{m-k}}\mathrm{d}S(u)\right]^{\frac{m-k}{m-j}}\\
\times &\left[\int_{S^{n-1}} \left(f_{p,i}(K,u)^{\frac{(n-i-m)(k-j)}{(n+p-i)(m-j)}}\right.\right.\\
\times &\left.\left. f_{p,i}(L,u)^{\frac{m(k-j)}{(n+p-i)(m-j)}}\right)^{\frac{m-j}{k-j}}\mathrm{d}S(u)\right]^{\frac{k-j}{m-j}}\\
\geq &\int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-k}{n+p-i}} f_{p,i}(L,u)^{\frac{k}{n+p-i}}\mathrm{d}S(u)\\
= &\ \Omega_{p,k}^{(i)}(K,L).
\end{aligned}$$

Therefore, inequality (17) is obtained.

According to the conditions of equality in Hölder's inequalities, we know that the equality holds in inequality (19) if and only if for any $u \in S^{n-1}$,

$$\frac{f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}} f_{p,i}(L,u)^{\frac{j}{n+p-i}}}{f_{p,i}(K,u)^{\frac{n-i-m}{n+p-i}} f_{p,i}(L,u)^{\frac{m}{n+p-i}}}$$

is a constant, i.e., $f_{p,i}(K,u)/f_{p,i}(L,u)$ is a constant for any $u\in S^{n-1}$. Combining with (8), we get

$$\mathrm{d}S_{p,i}(K,u)=c\mathrm{d}S_{p,i}(L,u),\quad \text{for any } u\in S^{n-1},$$

where c is a constant. Using formula (25), then above equality can be rewritten that

$$W_{p,i}(K,Q)=cW_{p,i}(L,Q),\quad \text{for all } Q\in\mathcal{K}_o^n.$$

Thus, if $n-i\neq p$, then

$$W_{p,i}(K,Q)=W_{p,i}\big(c^{\frac{1}{n-i-p}}L,Q\big),\quad \text{for all } Q\in\mathcal{K}_o^n.$$

If $n-i\neq p>1$ and $0\leq i<n$, from the above equation and Lemma 2.1, we see that $K=c^{\frac{1}{n-i-p}}L$, i.e., the equality holds in inequality (17) if and only if K and L are dilates. If $n-i\neq p=1$ and $0\leq i<n-1$, from the above equation and Lemma 2.1, we see that K and $c^{\frac{1}{n-1-i}}L$ are homothetic. Therefore, let $c^{\frac{1}{n-1-i}}L=\lambda K+x$ ($\lambda>0, x\in\mathbf{R}^n$), i.e., $L=aK+y$ ($a>0, y\in\mathbf{R}^n$), this shows that the equality holds in inequality (17) if and only if K and L are homothetic. The proof of Theorem 2 is completed.

Proof of Theorem 3. For $j>n-i$, using (12), (13) and Hölder's inequality, we have

$$\begin{aligned}
&\Omega_{p,j}^{(i)}(K,L)^{\frac{n-i}{j}}\Omega_p^{(i)}(K)^{\frac{i+j-n}{j}}\\
=\ &\Big[\int_{S^{n-1}}f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}}f_{p,i}(L,u)^{\frac{j}{n+p-i}}\mathrm{d}S(u)\Big]^{\frac{n-i}{j}}\\
\times\ &\Big[\int_{S^{n-1}}f_{p,i}(K,u)^{\frac{n-i}{n+p-i}}\mathrm{d}S(u)\Big]^{\frac{i+j-n}{j}}\\
=\ &\Big[\int_{S^{n-1}}\Big(f_{p,i}(K,u)^{\frac{(n-i)(n-i-j)}{j(n+p-i)}}\\
\times\ &f_{p,i}(L,u)^{\frac{n-i}{n+p-i}}\Big)^{\frac{j}{n-i}}\mathrm{d}S(u)\Big]^{\frac{n-i}{j}}\\
\times\ &\Big[\int_{S^{n-1}}\Big(f_{p,i}(K,u)^{\frac{(n-i)(i+j-n)}{j(n+p-i)}}\Big)^{\frac{j}{i+j-n}}\mathrm{d}S(u)\Big]^{\frac{i+j-n}{j}}\\
\geq\ &\int_{S^{n-1}}f_{p,i}(L,u)^{\frac{n-i}{n+p-i}}\mathrm{d}S(u)\\
=\ &\Omega_p^{(i)}(L),
\end{aligned}$$

this gives inequality (18). According to the condition of equality in Hölder's inequality, we see that equality holds in inequality (18) if and only if $f_{p,i}(K,u)/f_{p,i}(L,u)$ $(p\geq 1)$ is a constant for any $u\in S^{n-1}$. Similar to the proof of Theorem 2, we see that the equality in (18) for $n-i\neq p=1$ and $0\leq i<n-1$ if and only if K and L are homothetic, for $n-i\neq p>1$ and $0\leq i<n$ if and only if K and L are dilates.

Similar to the above proof, for $j<0$ or $0<j<n-i$, we can prove inequality (18) and (19), respectively.

For $j=0$ (or $j=n-i$), we easily see that (18) (or (19)) is identical. The proof of Theorem 3 is completed.

The following extension of Hölder's inequality will be required to prove Theorem 4.

Lemma 3.4. ([17])*If $f_0,f_1,\cdots,f_m$ are (strictly) positive continuous functions defined on S^{n-1} and $\alpha_1,\cdots,\alpha_m$ are positive constants the sum of whose reciprocals is unity, then*

$$\begin{aligned}
&\int_{S^{n-1}}f_0(u)f_1(u)\cdots f_m(u)\mathrm{d}S(u)\\
\leq\ &\prod_{i=1}^m\Big[\int_{S^{n-1}}f_0(u)f_i^{\alpha_i}(u)\mathrm{d}S(u)\Big]^{\frac{1}{\alpha_i}},
\end{aligned}\tag{34}$$

with equality if and only if there exist positive constants $\lambda_1,\cdots,\lambda_m$ such that $\lambda_1f_1^{\alpha_1}(u)=\cdots=\lambda_mf_m^{\alpha_m}(u)$ for all $u\in S^{n-1}$.

Proof of Theorem 4. By the definition (11) of ith L_p-mixed affine surface area and Hölder's inequality (34), we have

$$\begin{aligned}
&\Omega_p^{(i)}(K_1,\cdots,K_{n-i})\\
=\ &\int_{S^{n-1}}[f_{p,i}(K_1,u)\cdots f_{p,i}(K_{n-i},u)]^{\frac{1}{n+p-i}}\mathrm{d}S(u)\\
\leq\ &\prod_{j=1}^m\Big[\int_{S^{n-1}}f_{p,i}(K_j,u)^{\frac{m}{n+p-i}}\\
&\times\Big(f_{p,i}(K_{m+1},u)\cdots f_{p,i}(K_{n-i},u)\Big)^{\frac{1}{n+p-i}}\mathrm{d}S(u)\Big]^{\frac{1}{m}}\\
=\ &\prod_{j=1}^m\Big[\Omega_p^{(i)}(\underbrace{K_j,\cdots,K_j}_{m},K_{m+1},\cdots,K_{n-i})\Big]^{\frac{1}{m}}.
\end{aligned}$$

According to the condition of equality for Hölder's inequality (34), we know that equality holds for inequality (20) if and only if $K_1,K_2,\cdots,K_{n-i}$ are all dilations of each other (with the origin as the center of dilation).

Taking $m=n-i$ in inequality (20), it follows that

Corollary 3.1. *Suppose $K_1,K_2,\cdots,K_{n-i}\in\mathcal{F}_{o,i}^n$ $(i=0,1,\cdots,n-1)$, then*

$$\begin{aligned}
&\Omega_p^{(i)}(K_1,K_2,\cdots,K_{n-i})^{n-i}\\
\leq\ &\Omega_p^{(i)}(K_1)\cdots\Omega_p^{(i)}(K_{n-i}),
\end{aligned}\tag{35}$$

with equality in inequality if and only if $K_1,K_2,\cdots,K_{n-i}$ $(i=0,1,\cdots,n-1)$ are all dilations of each other (with the origin as the center of dilation).

Proof of Theorem 5. The inequality of Theorem 5 follows immediately from the definition (15) of ith L_p-affine surface area once the following fact is established: Given $Q\in\mathcal{S}_o^n$, there exists a $\overline{Q}\in\mathcal{S}_o^n$, such that

$$\begin{aligned}
&W_{q,i}(K,\overline{Q}^*)^{n-i}\frac{\widetilde{W}_i(\overline{Q})^q}{\widetilde{W}_i(K^*)^q}\\
\leq\ &W_{p,i}(K,Q^*)^{n-i}\frac{\widetilde{W}_i(Q)^p}{\widetilde{W}_i(K^*)^p}.
\end{aligned}\tag{36}$$

To show this, define $\overline{Q}\in\mathcal{S}_o^n$ by

$$\rho_{\overline{Q}}=\big[\widetilde{W}_i(K^*)^{p-q}\widetilde{W}_i(Q)^{-p}\big]^{\frac{1}{q(n-i)}}\rho_Q^{\frac{p}{q}}\rho_{K^*}^{\frac{q-p}{q}}.\tag{37}$$

From (37) we have

$$\rho_{\overline{Q}}^{-q}h_K^{1-q}=\widetilde{W}_i(K^*)^{\frac{q-p}{n-i}}\widetilde{W}_i(Q)^{\frac{p}{n-i}}\rho_Q^{-p}h_K^{1-p},$$

the integral representation of $W_{p,i}(K,Q^*)$ shows that

$$W_{q,i}(K,\overline{Q}^*)=\widetilde{W}_i(K^*)^{\frac{q-p}{n-i}}\widetilde{W}_i(Q)^{\frac{p}{n-i}}W_{p,i}(K,Q^*).\tag{38}$$

The definition of $\overline{Q}$, together with the Hölder inequality with the formula of dual quermassintegrals, show that

$$\begin{aligned}
& \widetilde{W}_i(\overline{Q}) \\
= & \widetilde{W}_i(K^*)^{\frac{p-q}{q}} \widetilde{W}_i(Q)^{-\frac{p}{q}} \\
& \times\left[\frac{1}{n}\int_{S^{n-1}} \rho_Q(u)^{\frac{(n-i)p}{q}} \rho_{K^*}(u)^{\frac{(n-i)(q-p)}{q}} \mathrm{d}S(u)\right] \\
\leq & \widetilde{W}_i(K^*)^{\frac{p-q}{q}} \widetilde{W}_i(Q)^{-\frac{p}{q}} \left(\frac{1}{n}\int_{S^{n-1}} \rho_Q^{n-i}(u)\mathrm{d}S(u)\right)^{\frac{p}{q}} \\
& \times\left(\int_{S^{n-1}} \rho_{K^*}^{n-i}(u)\mathrm{d}S(u)\right)^{\frac{q-p}{q}} \\
= & 1.
\end{aligned}$$

Together with (38), this yields (21).

IV. Open Problem

In this section, we propose the following two open questions:

Question 4.1. *Suppose $K, L \in \mathcal{F}_{c,i}^n (i = 0, 1, \cdots, n-1), p \geq 1, j \in \boldsymbol{R}$ and $0 \leq j \leq n$. Does it follow that*

$$\Omega_{p,j}^{(i)}(K, L)\Omega_{p,j}^{(i)}(K^*, L^*) \leq (n\omega_n)^2 \ ? \tag{39}$$

with equality for $0 < j < n$ and $p = 1$ if and only if K and L are homothetic ellipsoids; for $0 < j < n$ and $n \neq p > 1$ if and only if K and L are dilate ellipsoids; for $j = 0$ (or $j = n$) if and only if K (or L) is an ellipsoid.

Obviously, the case $i = 0$ of Question 4.1 is just the result of Wang and Leng (see [30]).

Question 4.2. *Suppose $K \in \mathcal{F}_{o,i}^n (i = 0, 1, \cdots, n-1)$ and $p \geq 1$. Does it follow that*

$$\Omega_p^{(i)}(K) \leq n\omega_n^{\frac{2p}{n+p-i}} \widetilde{W}_i(K)^{\frac{n-p-i}{n+p-i}} \ ? \tag{40}$$

or

$$\Omega_p^{(i)}(K) \leq n\omega_n^{\frac{2p}{n+p-i}} W_i(K)^{\frac{n-p-i}{n+p-i}} \ ? \tag{41}$$

with equality in every inequality for $i = 0$ if and only if K is an ellipsoid which centered at the origin; for $0 < i \leq n-1$ if and only if $K$is a ball which centered at the origin.

Obviously, the case $i = 0$ of Question 4.2 is just Lutwak's result Theorem A (see [16]).

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References

[1] R. J. Gardner, *Geometric Tomography*, Cambridge: Cambridge University, Press, 1995.

[2] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche and Isoperimetrie*, Berlin Göttingen Heidelberg: Springer, 1957.

[3] K. Leichtweiβ, "Zur affinoberfläche konvexer körper," *Manuscripta Mathematica*, vol. 56, no. 4, pp. 429-464, Dec. 1986.

[4] K. Leichtweiβ, "Über einige eigenschaften der affinoberfälche beliebiger konvexer körper," *Results in Mathematics*, vol. 13, no. 3-4, pp. 255-282, May. 1988.

[5] K. Leichtweiβ, "Bemerkungen zur definition einer erweiterten affinoberfläche von E.Lutwak", *Manuscripta Mathematica*, vol. 65, no. 2, pp. 181-197, Jun. 1989.

[6] K. Leichtweiβ, "On the history of the affine surface area for convex bodies," *Results in Mathematics*, vol. 20, no. 3-4, pp. 650-656, Nov. 1991.

[7] K. Leichtweiβ, *Affine Geometry of Convex Bodies*, Wiley-VCH, Heidelberg, 1998.

[8] G. S. Leng, "Affine surface areas of curvature images for convex bodies," *Acta Mathematica Sinica, English Series*, vol. 45, no. 4, pp. 797-802, May. 2002.

[9] M. Ludwig and M. Reitzner, "A characterization of affine surface area," *Advances in Mathematics*, vol. 147, no. 1, pp. 138-172, Oct. 1999.

[10] E. Lutwak, "On the Blaschke-Santaló inequality," *Annals of the New York Academy of Sciences*, vol. 440, no. 1, pp. 106-112, May. 1985.

[11] E. Lutwak, "On some affine isoperimetric inequalities," *Journal of Differential Geometry*, vol. 23, no. 1, pp. 1-13, Jan. 1986.

[12] E. Lutwak, "Mixed affine surface area," *Journal of Mathematical Analysis and Applications*, vol. 125, no. 2, pp. 351-360, Aug. 1987.

[13] E. Lutwak, "Centroid bodies and dual mixed volumes," *Proceedings of the London Mathematical*, vol. 60, no. 2, pp. 365-391, Mar. 1990.

[14] E. Lutwak, "Extended affine surface area," *Advances in Mathematics*, vol. 85, no. 1, pp. 39-68, Jan. 1991.

[15] E. Lutwak, "The Brunn-Minkowski-Firey theory I: mixed volumes and the minkowski problem," *Journal of Differential Geometry*, vol. 38, no. 1, pp. 131-150, Jul. 1993.

[16] E. Lutwak, "The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas," *Advances in Mathematics*, vol. 118, no. 2, pp. 244-294, Mar. 1996.

[17] E. Lutwak, "Dual mixed volumes," *Pacific Journal of Mathematics*, vol. 58, no. 2, pp. 531-538, Jun. 1975.

[18] L. J. Liu, W. Wang and B. W. He, "Fourier transform and L_p-mixed projection bodies," *Bulletin of the Korean Mathematical*, vol. 47, no. 5, pp. 1011-1023, Sept. 2010.

[19] F. H. Lu and W. D. Wang, "Inequalities for L_p-mixed curvature images," *Acta Mathematica Scientia*, vol. 30, no. 4, pp. 1044-102, Jul. 2010.

[20] T. Y. Ma, "Some inequalities related to (i, j)-type L_p-mixed affine surface area and L_p-mixed curvature image," *Journal of Inequalities and Applications*, vol. 2013, no. 1,pp. 1-16, Nov. 2013.

[21] T. Y. Ma and C. Y. Liu, "The generalized Busemann-Petty problem for dual L_p-mixed centroid bodies," *Journal of Southwest University, Natural Science Edition*, vol. 34, no. 4, pp. 105-112, Apr. 2012 (in Chinese).

[22] T. Y. Ma and C. Y. Liu, "The generalized Shephard problem for L_p-mixed projection bodies and Minkowski-Funk transforms", *Journal of Shandong University, Natural Science Edition*, vol. 47, no. 10, pp. 21-30, Oct. 2012 (in Chinese).

[23] T. Y. Ma, "The generalized L_p-Winternitz problem," *Journal of Mathematica Inequalities*, vol. 9, no. 2, pp. 597-614, Jun. 2015.

[24] M. Meyer and E. Werner, "On the p-affine surface area," *Advances in Mathematics*, vol. 152, no. 2, pp. 288-313, Jun. 2000.

[25] C. M. Petty, "Geominimal surface area," *Geometriae Dedicata*, vol. 3, no. 1, pp. 77-97, May. 1974.

[26] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Second edition, Cambridge: Cambridge University, Press, 2014.

[27] C. Schütt, "On the affine surface area," *Proceedings of the American Mathematical Society*, vol. 118, no. 4, pp. 1213-1218, Aug. 1993.

[28] C. Schütt and E. Werner, "Surface bodies and p-affine surface area," *Advances in Mathematics*, vol. 187, no. 1, pp. 98-145, Sept. 2004.

[29] E. Werner, "Illumination bodies and affine surface area," *Studia Mathematica*, vol. 110, no. 3, pp. 257-269, Jul. 1994.

[30] W. D. Wang and G. S. Leng, "L_p-mixed affine surface area," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 1, pp. 341-354, Nov. 2007.

[31] W. D. Wang and G. S. Leng, "Some affine isoperimetric inequalities associated with L_p-affine surface area," *Houston journal of mathematics*, vol. 34, no. 2, pp. 443-453, Mar. 2008.