Confidence Interval Estimations of the Parameter for One Parameter Exponential Distribution

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***Abstract*—The objective of this paper was to propose TestSTAT confidence interval estimation for a one parameter exponential distribution. Evaluation of the efficiency for this estimation was proved via theorems and a simulation study was conducted to compare the coverage probabilities and expected lengths of the three confidence intervals (TestSTAT, Exact and Asymptotic confidence intervals). The results showed that the TestSTAT confidence interval which is derived in this paper uses the same formula as for the Exact confidence interval which is widely used. Additionally, the expected length of the TestSTAT confidence interval is shorter than that of the Asymptotic confidence interval for a small sample size and all levels of the parameter and confidence coefficient. Furthermore, the three confidence interval estimations get systematically closer to the nominal level for all levels of the sample size and the parameter. In addition, the efficiencies of the three confidence interval estimations seem to be no different for a large sample size and all levels of the parameter and confidence coefficient.**

***Index Terms*—Confidence interval, estimation, exponential distribution, coverage probability, parameter**

I. INTRODUCTION

THE one parameter exponential distribution is a continuous distribution and is often used as a model for duration. It is also suitable for the distribution of the time between events when the number of events in any time interval is determined using a Poisson process. This distribution plays an important role in the formation of models in many fields of reliability analysis research, e.g., biological science, environmental research, industrial and systems engineering [1], [2], [3], [4], [5], [6], [7]. In addition, the one parameter exponential distribution is also used in the theory of waiting lines or queues which is applied in many situations, including banking teller queues, airline check-ins and supermarket checkouts [8], [9], [10], [11]. In addition, Sani and Daman [12] applied the one parameter exponential distribution to analyze queuing system with an exponential server and a general server under a controlled queue discipline. Whether these reliability and queuing analysis methods can yield precise and accurate results depend on the methods of parameter estimation. There are two types of parameter estimation from a probability distribution, namely point and interval estimations. In statistics, a point estimation involves the use of observed data from the distribution to calculate a single value as the value of parameter θ ; it is almost certain to be an incorrect estimation as mentioned by Koch and Link [13]. In this research, we investigate the interval estimation of parameter θ that provides a range of values with a known probability of capturing the true parameter θ . The general theory of confidence interval estimation was developed by Neyman [14] who constructed confidence intervals via the inversion of a family of hypothesis tests. The widely used technique of constructing a confidence interval of the parameter for one parameter exponential distribution is based on the pivotal quantities approach which determines what is known as an Exact confidence interval as mentioned by Hogg and Tanis [15], and Casella and Berger [16]. This method is valid for any sample size n as mentioned by Geyer [17], Balakrishnan *et al.* [18] and Cho *et al.* [19]. Where a large sample size n is applied, an Asymptotic confidence interval is mostly used to construct a sequence of the estimator $\hat{\theta}_n$ of θ with a density function $f(\bullet;\theta)$ that is asymptotically normally distributed with mean θ and variance $\sigma_n^2(\theta)$ [4], [20], [21].

In this study, the TestSTAT confidence interval estimation is proposed for one parameter exponential distribution. This confidence interval is derived based on the approach of inverting a test statistic which has a very strong correspondence between hypothesis testing and interval estimation. The TestSTAT method is most helpful in situations where intuition deserts us and we have no idea as to what would constitute a reasonable set as mentioned by Casella and Berger [16]. The efficiency comparisons in terms of the coverage probabilities and the expected lengths of the three confidence intervals are investigated via the theorem proofs. Furthermore, a simulation can be also performed to carry out efficiency comparisons.

II. MATERIALS AND METHODS

A. Criterions for the Efficiency Comparison

The efficiency comparison criteria among the three methods of the $(1-\alpha)100\%$ confidence intervals (Exact, Asymptotic and TestSTAT confidence intervals) are the coverage probability and the expected length of confidence interval.

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Let $CI = (L(\mathbf{X}), U(\mathbf{X}))$ be a confidence interval of a parameter θ based on the data $\mathbf{X}$ having the nominal $(1-\alpha)100\%$ level, where $L(\mathbf{X})$ and $U(\mathbf{X})$, respectively, are the lower and upper endpoints of this confidence interval. The following definitions provide the efficiency comparison criterions in this study:

Definition 2.1 The coverage probability associated with a confidence interval $CI = (L(\mathbf{X}), U(\mathbf{X}))$ for the unknown parameter θ is measured by $P_\theta\{\theta \in (L(\mathbf{X}), U(\mathbf{X}))\}$ (see [4]).

Definition 2.2 The length of a confidence interval, $W = U(\mathbf{X}) - L(\mathbf{X})$, is simply the difference between the upper $U(\mathbf{X})$ and lower $L(\mathbf{X})$ endpoints of a confidence interval $CI = (L(\mathbf{X}), U(\mathbf{X}))$. The expected length of a confidence interval CI is given by $E_\theta(W)$ (see [22], [23], [24]).

B. Confidence Interval Estimations for Parameter θ

Throughout the following discussion, the essential conditions for this work are denoted by (A1) – (A3) as follows:

(A1) Let $X_1, X_2, ..., X_n$ be a random sample of size n from a population of one parameter exponential distribution with parameter $\theta \in \Omega$ where $\Omega = \{\theta : 0 < \theta < \infty\}$. The probability density function of one parameter exponential random variable X is given by (see [4])

$$f(x;\theta) = \begin{cases} \dfrac{1}{\theta} e^{-\frac{x}{\theta}}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(A2) Let $\chi^2_{\alpha/2,\,2n}$ and $\chi^2_{1-\alpha/2,\,2n}$, respectively, be the $\left(\frac{\alpha}{2}\right)^{th}$ and $\left(1-\frac{\alpha}{2}\right)^{th}$ quantiles of the chi-square distribution with 2n degrees of freedom where $n > 0$.

(A3) Let $Z_{\alpha/2}$ be a positive constant which satisfy the relation $\Phi\left(Z_{\alpha/2}\right) - \Phi\left(-Z_{\alpha/2}\right) = 1-\alpha$ and $\Phi(\bullet)$ is the cumulative distribution function of the standard normal distribution.

For $0 < \alpha < 1$, the following three methods of $(1-\alpha)100\%$ confidence intervals are studied for the efficiency comparisons:

1) Exact confidence interval

The confidence set construction with the use of pivotal quantities is called the Exact confidence interval. For $0 < \alpha < 1$, the $(1-\alpha)100\%$ Exact confidence interval for parameter θ is given by (see [16])

$$CI_{Exact} = \left[\frac{2\sum_{i=1}^{n} X_i}{\chi^2_{1-\alpha/2,\,2n}},\ \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{\alpha/2,\,2n}} \right] \tag{1}$$

where $\chi^2_{\alpha/2,\,2n}$ and $\chi^2_{1-\alpha/2,\,2n}$ hold in condition (A2).

2) Asymptotic confidence interval

An asymptotic confidence interval is valid only for a sufficiently large sample size. For $0 < \alpha < 1$, the $(1-\alpha)100\%$ asymptotic confidence interval for parameter θ is given by (see [20])

$$CI_{Asymptotic} = \left[\frac{\bar{X}}{1 + \frac{Z_{\alpha/2}}{\sqrt{n}}},\ \frac{\bar{X}}{1 - \frac{Z_{\alpha/2}}{\sqrt{n}}} \right] \tag{2}$$

where $Z_{\alpha/2}$ holds in condition (A3).

3) TestSTAT confidence interval

We propose the TestSTAT confidence interval which is derived from using an inversion of a test statistic as shown in theorem 2.1.

Lemma 2.1 Let $X_1, X_2, ..., X_n$ hold in condition (A1). The acceptance region for an α, $0 < \alpha < 1$, level likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is given by

$$A(\theta_0) = \left\{ (x_1, x_2, ..., x_n) : \left(\frac{\sum_{i=1}^{n} x_i}{\theta_0} \right)^n e^{-\frac{1}{\theta_0}\left(\sum_{i=1}^{n} x_i\right)} \geq k^* \right\},$$

where k^* is a constant chosen to satisfy

$$P_{\theta_0}\left((X_1, X_2, ..., X_n) \in A(\theta_0) \right) = 1 - \alpha .$$

Proof Let $X_1, X_2, ..., X_n$ hold in condition (A1).

Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ with level $\alpha \in (0,1)$ where θ_0 is a fixed positive real number, we have $\omega = \{\theta_0\}$.

The likelihood function is given by

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} = \theta^{-n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}.$$

Observe that

$$\sup_{\theta\in\omega} L(\theta) = \theta_0^{-n} e^{-\frac{1}{\theta_0}\left(\sum_{i=1}^{n} x_i\right)}, \tag{3}$$

since ω has the single element θ_0. On the other hand, one has

$$\sup_{\theta\in\Omega} L(\theta) = n^n \left(\sum_{i=1}^{n} x_i\right)^{-n} e^{-n}, \tag{4}$$

since $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the maximum likelihood estimator of θ, so that the value $\bar{x}$ maximizes $L(\theta)$ over Ω.
Combining equations (3) and (4), the likelihood ratio test statistic is

$$\lambda = \frac{\sup_{\theta\in\omega} L(\theta)}{\sup_{\theta\in\Omega} L(\theta)} = \left(\frac{\sum_{i=1}^{n} x_i}{n\theta_0}\right)^n e^{n-\frac{1}{\theta_0}\left(\sum_{i=1}^{n} x_i\right)}.$$

We do not reject $H_0 : \theta = \theta_0$ if and only if $\lambda \geq k$ where $k > 0$ is a generic constant or

$$\left(\frac{\sum_{i=1}^{n} x_i}{\theta_0}\right)^n e^{-\frac{1}{\theta_0}\left(\sum_{i=1}^{n} x_i\right)} \geq k^* \text{ where } k^* = k\left(\frac{n}{e}\right)^n.$$

Thus, the acceptance region for an α, $0 < \alpha < 1$, level likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is given by

$$A(\theta_0) = \left\{ (x_1, x_2, ..., x_n) : \left(\frac{\sum_{i=1}^{n} x_i}{\theta_0}\right)^n e^{-\frac{1}{\theta_0}\left(\sum_{i=1}^{n} x_i\right)} \geq k^* \right\},$$

where k^* is a constant chosen to satisfy $P_{\theta_0}\left((X_1, X_2, ..., X_n) \in A(\theta_0) \right) = 1-\alpha$.

Lemma 2.2 Let X be a chi-square random variable with n degrees of freedom where $n > 0$. The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, & x > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

There exists a moment generating function of X such that $M_X(t) = \dfrac{1}{(1-2t)^{\frac{n}{2}}}$ for $t < \dfrac{1}{2}$.

Proof Let X be a chi-square random variable with n degrees of freedom where $n > 0$. From (5), we find the moment generating function of X as follows:

$$M_X(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} dx$$

$$= \int_0^\infty \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\left(\frac{x}{2}-tx\right)} dx$$

$$= \int_0^\infty \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\left(\frac{1}{2}-t\right)x} dx$$

$$= \int_0^\infty \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\left(\frac{1-2t}{2}\right)x} dx$$

$$= \frac{\left(\dfrac{1}{(1-2t)^{\frac{n}{2}}}\right)}{\left(\dfrac{1}{(1-2t)^{\frac{n}{2}}}\right)} \int_0^\infty \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2/(1-2t)}} dx$$

$$= \frac{1}{(1-2t)^{\frac{n}{2}}} \int_0^\infty \frac{1}{\Gamma\left(\frac{n}{2}\right) \left(\frac{2}{1-2t}\right)^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2/(1-2t)}} dx$$

$$= \frac{1}{(1-2t)^{\frac{n}{2}}} \times 1 \quad \text{where } 1-2t > 0 \text{ or } t < \frac{1}{2}.$$

Thus, the moment generating function of X is given by $M_X(t) = \dfrac{1}{(1-2t)^{\frac{n}{2}}}$ for $t < \dfrac{1}{2}$.

Lemma 2.3 Let $X_1, X_2, ..., X_n$ hold in condition (A1). Then $W = \frac{2}{\theta}\sum_{i=1}^{n} X_i$ is a chi-square random variable with 2n degrees of freedom where $n > 0$.

Proof Let $X_1, X_2, ..., X_n$ hold in condition (A1). Then the moment generating function of X_i is given by (see [8])

$$M_{X_i}(t) = \frac{1}{1-\theta t},\ t < \frac{1}{2} \text{ where } i = 1, 2, ..., n.$$

The moment generating function of $W = \frac{2}{\theta}\sum_{i=1}^{n} X_i$ is

$$M_W(t) = E\left[e^{tW}\right] = E\left[e^{t\left(\frac{2}{\theta}\sum_{i=1}^{n} X_i\right)}\right] = E\left[e^{\frac{2t}{\theta}\sum_{i=1}^{n} X_i}\right]$$

$$= \prod_{i=1}^{n} E\left[e^{\frac{2t}{\theta}X_i}\right] = \prod_{i=1}^{n} M_{X_i}\left(\frac{2t}{\theta}\right)$$

$$= \prod_{i=1}^{n} \frac{1}{(1-2t)} = \frac{1}{(1-2t)^n}$$

$$= \frac{1}{(1-2t)^{\frac{2n}{2}}} \quad \text{for } t < \frac{1}{2}. \tag{6}$$

We refer from Lemma 2.2, equation (6) is a moment generating function of a chi-square random variable with 2n degrees of freedom where $n > 0$. Thus, $W = \frac{2}{\theta}\sum_{i=1}^{n} X_i$ a chi-square random variable with 2n degrees of freedom where $n > 0$.

Theorem 2.1 Let $X_1, X_2, ..., X_n$ hold in condition (A1). For $0 < \alpha < 1$, the $(1-\alpha)100\%$ TestSTAT confidence interval for parameter θ is given by

$$CI_{TestSTAT} = \left[\frac{2\sum_{i=1}^{n} X_i}{\chi^2_{1-\alpha/2,\,2n}}, \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{\alpha/2,\,2n}}\right] \tag{7}$$

where $\chi^2_{\alpha/2,\,2n}$ and $\chi^2_{1-\alpha/2,\,2n}$ hold in condition (A2).

Proof Let $X_1, X_2, ..., X_n$ hold in condition (A1). From Lemma 2.1, we know that the acceptance region gives the $(1-\alpha)100\%$ confidence set

$$C(x_1, x_2, ..., x_n) = \left\{\theta : \left(\frac{\sum_{i=1}^{n} x_i}{\theta}\right)^n e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i\right)} \geq k^*\right\}. \tag{8}$$

The expression of $C(x_1, x_2, ..., x_n)$ in equation (8) depends on $x_1, x_2, ..., x_n$ only throug $\sum_{i=1}^{n} x_i$, thus the confidence interval can be expressed in the form of equation (9).

$$C\left(\sum_{i=1}^{n} x_i\right) = \left\{\theta : L\left(\sum_{i=1}^{n} x_i\right) \leq \theta \leq U\left(\sum_{i=1}^{n} x_i\right)\right\} \tag{9}$$

where L and U are functions which satisfy the constraints in equations (10) and (11) as follows:

$$P_{\theta_0}\left((X_1, X_2, ..., X_n) \in A(\theta_0)\right) = 1-\alpha \tag{10}$$

and

$$\left(\frac{\sum_{i=1}^{n} x_i}{L\left(\sum_{i=1}^{n} x_i\right)}\right)^n e^{-\frac{\sum_{i=1}^{n} x_i}{L\left(\sum_{i=1}^{n} x_i\right)}} = \left(\frac{\sum_{i=1}^{n} x_i}{U\left(\sum_{i=1}^{n} x_i\right)}\right)^n e^{-\frac{\sum_{i=1}^{n} x_i}{U\left(\sum_{i=1}^{n} x_i\right)}}. \tag{11}$$

We set $\frac{\sum_{i=1}^{n} x_i}{L\left(\sum_{i=1}^{n} x_i\right)} = \frac{a}{2}$ and $\frac{\sum_{i=1}^{n} x_i}{U\left(\sum_{i=1}^{n} x_i\right)} = \frac{b}{2}$ (12)

where constants $a > b > 0$.

Substituting equation (12) in equation (11), results in

$$\left(\frac{a}{2}\right)^n e^{-\frac{a}{2}} = \left(\frac{b}{2}\right)^n e^{-\frac{b}{2}}. \tag{13}$$

From equations (9) and (12), the $(1-\alpha)100\%$ TestSTAT confidence interval for parameter θ is given by

$$\left\{\theta : \frac{2\sum_{i=1}^{n} x_i}{a} \leq \theta \leq \frac{2\sum_{i=1}^{n} x_i}{b}\right\} \tag{14}$$

where constants $a > b > 0$ satisfy the following

$$P_\theta\left(\frac{2\sum_{i=1}^{n} X_i}{a} \leq \theta \leq \frac{2\sum_{i=1}^{n} X_i}{b}\right) = P_\theta\left(\frac{1}{a} \leq \frac{\theta}{2\sum_{i=1}^{n} X_i} \leq \frac{1}{b}\right)$$

$$= P_\theta\left(b \leq \frac{2\sum_{i=1}^{n} X_i}{\theta} \leq a\right)$$

$$= P_\theta\left(b \leq W \leq a\right) \text{ where } W = \frac{2\sum_{i=1}^{n} X_i}{\theta}$$

$$= 1-\alpha. \tag{15}$$

From Lemma 2.3, we know that $W = \frac{2\sum_{i=1}^{n} X_i}{\theta}$ is a chi-square random variable with 2n degrees of freedom where $n > 0$. For the equal-tails probability, the constants a and b are equal to $\chi^2_{\alpha/2,\,2n}$ and $\chi^2_{1-\alpha/2,\,2n}$, respectively, which hold in condition (A2).

Therefore, the $(1-\alpha)100\%$ TestSTAT confidence interval for parameter θ is given by

$$CI_{TestSTAT} = \left[\frac{2\sum_{i=1}^{n} X_i}{\chi^2_{1-\alpha/2,\,2n}}, \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{\alpha/2,\,2n}}\right].$$

Remark 2.1 The $(1-\alpha)100\%$ TestSTAT confidence interval for parameter θ uses the same formula as the $(1-\alpha)100\%$ Exact confidence interval.

III. Evaluation of Efficiency for the Proposed Confidence Interval

In this section, we investigate the performances of the three confidence intervals (Exact, Asymptotic and TestSTAT confidence intervals) which are given in the theorems 3.1 – 3.3. As a result of Remark 2.1, the following theorems are proved merely for the Asymptotic and TestSTAT confidence intervals:

Theorem 3.1 Let $X_1, X_2, ..., X_n$ hold in condition (A1). The $CI_{Asymptotic}$ and $CI_{TestSTAT}$ denote the Asymptotic and TestSTAT confidence interval, respectively. The coverage probabilities of $CI_{Asymptotic}$ and $CI_{TestSTAT}$ satisfy equation (16) for all levels of the parameter θ and the confidence coefficient α.

$$P_\theta\left\{\theta \in CI_{Asymptotic}\right\} = P_\theta\left\{\theta \in CI_{TestSTAT}\right\} = 1-\alpha \qquad (16)$$

Proof Let $X_1, X_2, ..., X_n$ hold in condition (A1).

Then, the mean and variance of X_i are given by $E(X_i) = \theta$ and $V(X_i) = \theta$, respectively.

First, we consider the coverage probability associated with a confidence interval $CI_{TestSTAT}$ which is denoted by

$$P_\theta\left\{\theta \in CI_{TestSTAT}\right\} = P_\theta\left\{\theta \in \left[\frac{2\sum_{i=1}^{n} X_i}{\chi^2_{1-\alpha/2,\,2n}}, \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{\alpha/2,\,2n}}\right]\right\}$$

$$= P_\theta\left(\frac{2\sum_{i=1}^{n} X_i}{\chi^2_{1-\alpha/2,\,2n}} \le \theta \le \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{\alpha/2,\,2n}}\right)$$

$$= P_\theta\left(\chi^2_{\alpha/2,\,2n} \le \frac{2\sum_{i=1}^{n} X_i}{\theta} \le \chi^2_{1-\alpha/2,\,2n}\right)$$

From Lemma 2.3, we know that $\dfrac{2\sum_{i=1}^{n} X_i}{\theta}$ is a chi-square distributed with 2n degrees of freedom.

$$\text{Then, } P_\theta\left\{\theta \in CI_{TestSTAT}\right\} = 1-\alpha \qquad (17)$$

Likewise, the coverage probability associated with a confidence interval $CI_{Asymptotic}$ is given by

$$P_\theta\left\{\theta \in CI_{Asymptotic}\right\} = P_\theta\left\{\theta \in \left[\frac{\bar{X}}{1+\frac{Z_{\alpha/2}}{\sqrt{n}}}, \frac{\bar{X}}{1-\frac{Z_{\alpha/2}}{\sqrt{n}}}\right]\right\}$$

$$= P_\theta\left(\frac{\bar{X}}{1+\frac{Z_{\alpha/2}}{\sqrt{n}}} \le \theta \le \frac{\bar{X}}{1-\frac{Z_{\alpha/2}}{\sqrt{n}}}\right)$$

$$= P_\theta\left\{\theta - \frac{\theta Z_{\alpha/2}}{\sqrt{n}} \le \bar{X} \le \theta + \frac{\theta Z_{\alpha/2}}{\sqrt{n}}\right\}$$

$$= P_\theta\left\{-Z_{\alpha/2} \le \frac{\bar{X}-\theta}{\theta/\sqrt{n}} \le Z_{\alpha/2}\right\}$$

Using the central limit theorem, we know that $\dfrac{\bar{X}-\theta}{\theta/\sqrt{n}}$ is approximately standard normally distributed.

$$\text{Then, } P_\theta\left\{\theta \in CI_{Asymptotic}\right\} = 1-\alpha \qquad (18)$$

From equations (17) and (18), we obtain the following result:

$P_\theta\left\{\theta \in CI_{Asymptotic}\right\} = P_\theta\left\{\theta \in CI_{TestSTAT}\right\} = 1-\alpha$ for all levels of the parameter θ and the confidence coefficient α.

Proposition 3.1 Let $X_1, X_2, ..., X_n$ hold in condition (A1). The expected length of an Asymptotic confidence interval is given by

$E_\theta\left[W_{Asymptotic}\right] = C\theta$ where $C = \dfrac{\sqrt{n}}{\sqrt{n} - Z_{\alpha/2}} - \dfrac{\sqrt{n}}{\sqrt{n} + Z_{\alpha/2}}$

and $Z_{\alpha/2} > 0$ for all levels of the parameter θ and the confidence coefficient α.

Proof Let $CI_{Asymptotic} = \left[\dfrac{\bar{X}}{1+\frac{Z_{\alpha/2}}{\sqrt{n}}}, \dfrac{\bar{X}}{1-\frac{Z_{\alpha/2}}{\sqrt{n}}}\right]$ be the Asymptotic confidence interval. Using the Definition 2.2, the length of $CI_{Asymptotic}$ is denoted by

$$W_{Asymptotic} = \frac{\bar{X}}{1-\frac{Z_{\alpha/2}}{\sqrt{n}}} - \frac{\bar{X}}{1+\frac{Z_{\alpha/2}}{\sqrt{n}}}$$

$$= \left(\frac{1}{1-\frac{Z_{\alpha/2}}{\sqrt{n}}} - \frac{1}{1+\frac{Z_{\alpha/2}}{\sqrt{n}}}\right)\bar{X}$$

$$= \left(\frac{\sqrt{n}}{\sqrt{n} - Z_{\alpha/2}} - \frac{\sqrt{n}}{\sqrt{n} + Z_{\alpha/2}} \right) \bar{X}$$

The expected length of an Asymptotic confidence interval is given by

$$E_\theta\left[W_{Asymptotic} \right] = E_\theta\left[\left(\frac{\sqrt{n}}{\sqrt{n} - Z_{\alpha/2}} - \frac{\sqrt{n}}{\sqrt{n} + Z_{\alpha/2}} \right) \bar{X} \right]$$

$$= \left(\frac{\sqrt{n}}{\sqrt{n} - Z_{\alpha/2}} - \frac{\sqrt{n}}{\sqrt{n} + Z_{\alpha/2}} \right) E_\theta\left[\bar{X} \right] = C E_\theta\left[\bar{X} \right]$$

where $C = \frac{\sqrt{n}}{\sqrt{n} - Z_{\alpha/2}} - \frac{\sqrt{n}}{\sqrt{n} + Z_{\alpha/2}}$ and $Z_{\alpha/2} > 0$.

Since $E_\theta\left[\bar{X} \right] = \theta$ where $\theta > 0$, then $E_\theta\left[W_{Asymptotic} \right] = C\theta$ for all levels of the parameter θ and the confidence coefficient α.

Proposition 3.2 Let $X_1, X_2, ..., X_n$ hold in condition (A1). The expected length of a TestSTAT confidence interval is given by

$E_\theta\left[W_{TestSTAT} \right] = D\theta$ where $D = \frac{2n}{\chi^2_{\alpha/2,\,2n}} - \frac{2n}{\chi^2_{1-\alpha/2,\,2n}}$

for all levels of the parameter θ and the confidence coefficient α.

Proof Let $CI_{TestSTAT} = \left[\frac{2\sum_{i=1}^{n} X_i}{\chi^2_{1-\alpha/2,\,2n}}, \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{\alpha/2,\,2n}} \right]$ be the TestSTAT confidence interval. Using the Definition 2.2, the length of $CI_{TestSTAT}$ is denoted by

$$W_{TestSTAT} = \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{\alpha/2,\,2n}} - \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{1-\alpha/2,\,2n}}$$

$$= \left(\frac{2n}{\chi^2_{\alpha/2,\,2n}} - \frac{2n}{\chi^2_{1-\alpha/2,\,2n}} \right) \frac{\sum_{i=1}^{n} X_i}{n}$$

$$= \left(\frac{2n}{\chi^2_{\alpha/2,\,2n}} - \frac{2n}{\chi^2_{1-\alpha/2,\,2n}} \right) \bar{X}$$

The expected length of a TestSTAT confidence interval is given by

$$E_\theta\left[W_{TestSTAT} \right] = E_\theta\left[\left(\frac{2n}{\chi^2_{\alpha/2,\,2n}} - \frac{2n}{\chi^2_{1-\alpha/2,\,2n}} \right) \bar{X} \right]$$

$$= \left(\frac{2n}{\chi^2_{\alpha/2,\,2n}} - \frac{2n}{\chi^2_{1-\alpha/2,\,2n}} \right) E_\theta\left[\bar{X} \right] = DE_\theta\left[\bar{X} \right]$$

where $D = \frac{2n}{\chi^2_{\alpha/2,\,2n}} - \frac{2n}{\chi^2_{1-\alpha/2,\,2n}}$.

Since $E_\theta\left[\bar{X} \right] = \theta$ where $\theta > 0$, then $E_\theta\left[W_{TestSTAT} \right] = D\theta$ for all levels of the parameter θ and the confidence coefficient α.

Theorem 3.2 Let $\chi^2_{\alpha/2,\,2n}$ and $\chi^2_{1-\alpha/2,\,2n}$ hold in condition (A2), and $Z_{\alpha/2}$ holds in condition (A3). If there exist a sample size n such that $n > Z^2_{\alpha/2}$ and

$\frac{1}{\chi^2_{\alpha/2,\,2n}} - \frac{1}{\chi^2_{1-\alpha/2,\,2n}} < \frac{Z_{\alpha/2}}{\sqrt{n}\left(n - Z^2_{\alpha/2} \right)}$ for all levels of the parameter θ and the confidence coefficient α,

then $E_\theta\left[W_{TestSTAT} \right] < E_\theta\left[W_{Asymptotic} \right]$.

Proof Let $0 < \chi^2_{\alpha/2,\,2n} < \chi^2_{1-\alpha/2,\,2n}$ and $Z_{\alpha/2} > 0$ for all levels of the confidence coefficient α.

Supposing there exist a sample size n such that $n > Z^2_{\alpha/2}$ and $\frac{1}{\chi^2_{\alpha/2,\,2n}} - \frac{1}{\chi^2_{1-\alpha/2,\,2n}} < \frac{Z_{\alpha/2}}{\sqrt{n}\left(n - Z^2_{\alpha/2} \right)}$.

From Propositions 3.1 and 3.2, the difference between the expected of $W_{TestSTAT}$ and that of $W_{Asymptotic}$ can be written in the form of equation (19).

$$E_\theta\left[W_{TestSTAT} \right] - E_\theta\left[W_{Asymptotic} \right] = D\theta - C\theta$$

$$= (D - C)\theta \qquad (19)$$

The term $(D - C)$ in equation (19) becomes

$$D - C = \left(\frac{2n}{\chi^2_{\alpha/2,\,2n}} - \frac{2n}{\chi^2_{1-\alpha/2,\,2n}} \right) - \left(\frac{\sqrt{n}}{\sqrt{n} - Z_{\alpha/2}} - \frac{\sqrt{n}}{\sqrt{n} + Z_{\alpha/2}} \right)$$

$$= 2n\left(\frac{1}{\chi^2_{\alpha/2,\,2n}} - \frac{1}{\chi^2_{1-\alpha/2,\,2n}} \right) - \left(\frac{2\sqrt{n} Z_{\alpha/2}}{n - Z^2_{\alpha/2}} \right)$$

$$= 2\sqrt{n}\left\{\sqrt{n}\left(\frac{1}{\chi^2_{\alpha/2,\,2n}}-\frac{1}{\chi^2_{1-\alpha/2,\,2n}}\right)-\left(\frac{Z_{\alpha/2}}{n-Z^2_{\alpha/2}}\right)\right\} \quad (20)$$

In equation (20), we know that $2\sqrt{n} > 0$ and

$$\sqrt{n}\left(\frac{1}{\chi^2_{\alpha/2,\,2n}}-\frac{1}{\chi^2_{1-\alpha/2,\,2n}}\right)-\left(\frac{Z_{\alpha/2}}{n-Z^2_{\alpha/2}}\right) < 0,$$

then $D-C < 0$ (21)

Since $\theta > 0$ and substituting equation (21) in equation (19), then we obtain $E_\theta\left[W_{TestSTAT}\right]-E_\theta\left[W_{Asymptotic}\right] < 0$. That is $E_\theta\left[W_{TestSTAT}\right] < E_\theta\left[W_{Asymptotic}\right]$.

Lemma 3.1 Let W be a chi-square random variable with 2n degrees of freedom where $n > 0$. The values of $\chi^2_{\alpha/2,\,2n}$ and $\chi^2_{1-\alpha/2,\,2n}$ which hold in condition (A2). They can be written in the forms of

(i) $\chi^2_{\alpha/2,\,2n} \approx 2n-2\sqrt{n}Z_{\alpha/2}$ when $n > Z^2_{\alpha/2}$ (22)

(ii) $\chi^2_{1-\alpha/2,\,2n} \approx 2n+2\sqrt{n}Z_{\alpha/2}$ when $n > Z^2_{\alpha/2}$ (23)

when $Z_{\alpha/2} > 0$ holds in condition (A3) as a sample size n increases for all levels of the confidence coefficient α.

Proof Let W be a chi-square random variable with 2n degrees of freedom for $n > 0$, the mean and variance of W are given by $E(W) = 2n$ and $V(W) = 4n$, respectively. The condition (A2) holds. Then, the proof of (i) is as follows:

$$P\left(W < \chi^2_{\alpha/2,\,2n}\right) = \frac{\alpha}{2} \quad (24)$$

From equation (24), the chi-square distribution (after standardization) tends to the standard normal distribution as n increases [2]; that is,

$$\lim_{n\to\infty} P\left(\frac{W-E(W)}{\sqrt{V(W)}} < \frac{\chi^2_{\alpha/2,\,2n}-E(W)}{\sqrt{V(W)}}\right)$$

$$= \Phi\left(\frac{\chi^2_{\alpha/2,\,2n}-E(W)}{\sqrt{V(W)}}\right) = \lim_{n\to\infty}\frac{\alpha}{2}. \quad (25)$$

By substituting $E(W) = 2n$ and $V(W) = 4n$ in equation (25), then we obtain

$$\lim_{n\to\infty} P\left(Z < \frac{\chi^2_{\alpha/2,\,2n}-2n}{2\sqrt{n}}\right) = \Phi\left(\frac{\chi^2_{\alpha/2,\,2n}-2n}{2\sqrt{n}}\right) = \frac{\alpha}{2} \quad (26)$$

Also, by using condition (A3), we have

$$P\left(Z < -Z_{\alpha/2}\right) = \Phi\left(-Z_{\alpha/2}\right) = \frac{\alpha}{2} \quad (27)$$

and $Z_{\alpha/2} > 0$. As n increases, equation (26) is approximately equal to equation (27), we obtain equation (28).

$$-Z_{\alpha/2} \approx \frac{\chi^2_{\alpha/2,\,2n}-2n}{2\sqrt{n}} \quad (28)$$

After rearranging equation (28), we conclude that $\chi^2_{\alpha/2,\,2n} \approx 2n-2\sqrt{n}Z_{\alpha/2}$ as n increases for all levels of α.

As a result of $\chi^2_{\alpha/2,\,2n} > 0$, we also obtain $2n-2\sqrt{n}Z_{\alpha/2} > 0$. That is, the sample size n must be greater than $Z^2_{\alpha/2}$.

Likewise, the proof of (ii) is as follows:

$$P\left(W < \chi^2_{1-\alpha/2,\,2n}\right) = 1-\frac{\alpha}{2} \quad (29)$$

From equation (29), the chi-square distribution (after standardization) tends to the standard normal distribution as n increases [2]; that is,

$$\lim_{n\to\infty} P\left(\frac{W-E(W)}{\sqrt{V(W)}} < \frac{\chi^2_{1-\alpha/2,\,2n}-E(W)}{\sqrt{V(W)}}\right)$$

$$= \Phi\left(\frac{\chi^2_{1-\alpha/2,\,2n}-E(W)}{\sqrt{V(W)}}\right) = \lim_{n\to\infty}\left(1-\frac{\alpha}{2}\right). \quad (30)$$

By substituting $E(W) = 2n$ and $V(W) = 4n$ in equation (30), then we obtain

$$\lim_{n\to\infty} P\left(Z < \frac{\chi^2_{1-\alpha/2,\,2n}-2n}{2\sqrt{n}}\right) = \Phi\left(\frac{\chi^2_{1-\alpha/2,\,2n}-2n}{2\sqrt{n}}\right) = 1-\frac{\alpha}{2} \quad (31)$$

Also, by using condition (A3), we have

$$P\left(Z < Z_{\alpha/2}\right) = \Phi\left(Z_{\alpha/2}\right) = 1-\frac{\alpha}{2} \quad (32)$$

As n increases, equation (31) is approximately equal to equation (32), we obtain $Z_{\alpha/2} \approx \dfrac{\chi^2_{1-\alpha/2,\,2n}-2n}{2\sqrt{n}}$ (33).

After rearranging equation (33), we conclude that $\chi^2_{1-\alpha/2,\,2n} \approx 2n+2\sqrt{n}Z_{\alpha/2}$ as n increases for all levels of α.

As a result of $\chi^2_{1-\alpha/2,\,2n} > 0$, we also obtain $2n+2\sqrt{n}Z_{\alpha/2} > 0$. That is, the sample size n must be greater than $Z^2_{\alpha/2}$.

Theorem 3.3 If conditions (A1) – (A3) hold and there exists a sample size n such that $n > Z^2_{\alpha/2}$, then

(i) $E_\theta\left[W_{TestSTAT}\right] \approx E_\theta\left[W_{Asymptotic}\right]$,

(ii) $E_\theta\left[W_{TestSTAT}\right]$ tends to decrease and

(iii) $E_\theta\left[W_{Asymptotic}\right]$ tends to decrease

when sample size n increases for all levels of the parameter θ and the confidence coefficient α.

Proof Assume conditions (A1) - (A3) hold. When a sample size n such that $n > Z^2_{\alpha/2}$ increases for all levels of the confidence coefficient α, the proofs of (i) – (iii) are as follows:

(i) To prove $E_\theta\left[W_{TestSTAT}\right] \approx E_\theta\left[W_{Asymptotic}\right]$, we use Proposition 3.2 that

$$E_\theta\left[W_{TestSTAT}\right] = D\theta \tag{34}$$

where

$$D = \frac{2n}{\chi^2_{\alpha/2,\,2n}} - \frac{2n}{\chi^2_{1-\alpha/2,\,2n}} \tag{35}$$

Substituting equations (22) and (23) from Lemma 3.1 in equation (35), we obtain that

$$D \approx \frac{2n}{2n - 2\sqrt{n}Z_{\alpha/2}} - \frac{2n}{2n + 2\sqrt{n}Z_{\alpha/2}}$$

$$\approx \frac{\sqrt{n}}{\sqrt{n} - Z_{\alpha/2}} - \frac{\sqrt{n}}{\sqrt{n} + Z_{\alpha/2}} \tag{36}$$

Substituting equations (36) in equation (34), we obtain

$E_\theta\left[W_{TestSTAT}\right] \approx \left(\frac{\sqrt{n}}{\sqrt{n} - Z_{\alpha/2}} - \frac{\sqrt{n}}{\sqrt{n} + Z_{\alpha/2}}\right)\theta$ which is the approximate formula as for $E_\theta\left[W_{Asymptotic}\right]$ in Proposition 3.1.

Thus, $E_\theta\left[W_{TestSTAT}\right] \approx E_\theta\left[W_{Asymptotic}\right]$ as n increases for all levels of θ and α.

(ii) We know that $0 < \chi^2_{\alpha/2,\,2n} < \chi^2_{1-\alpha/2,\,2n}$ from condition (A2). Then, there exist a constant D which is shown in Proposition 3.2 such that

$$D = \frac{2}{\dfrac{\chi^2_{\alpha/2,\,2n}}{n}} - \frac{2}{\dfrac{\chi^2_{1-\alpha/2,\,2n}}{n}} > 0 \text{ for all levels of n and } \alpha.$$

Consider equations (22) and (23) from Lemma 3.1 such that

$$\chi^2_{\alpha/2,\,2n} \approx 2n - 2\sqrt{n}Z_{\alpha/2} = 2\sqrt{n}\left(\sqrt{n} - Z_{\alpha/2}\right) > 0 \tag{37}$$

$$\chi^2_{1-\alpha/2,\,2n} \approx 2n + 2\sqrt{n}Z_{\alpha/2} = 2\sqrt{n}\left(\sqrt{n} + Z_{\alpha/2}\right) > 0 \tag{38}$$

where $Z_{\alpha/2}$ is a positive constant which depends on α and $Z^2_{\alpha/2} < n$ for condition (A3) holds. From equations (37) and (38), the values of $\chi^2_{\alpha/2,\,2n}$ and $\chi^2_{1-\alpha/2,\,2n}$ tend to increase when n increases. Therefore, the value of $D = \frac{2}{\chi^2_{\alpha/2,\,2n}/n} - \frac{2}{\chi^2_{1-\alpha/2,\,2n}/n}$ tends to decrease as n increases.

That is, $E_\theta\left[W_{TestSTAT}\right] = D\theta$ tends to decrease as n increases for all levels of n, θ and α.

(iii) There exists a constant C which is shown in Proposition 3.1 such that

$$C = \frac{\sqrt{n}}{\sqrt{n} - Z_{\alpha/2}} - \frac{\sqrt{n}}{\sqrt{n} + Z_{\alpha/2}} = \frac{2}{\dfrac{n - Z^2_{\alpha/2}}{\sqrt{n}Z_{\alpha/2}}}$$

where $Z_{\alpha/2}$ is a positive constant which depends on α for condition (A3) holds. Therefore, the value of $C = \frac{2}{\frac{n - Z^2_{\alpha/2}}{\sqrt{n}Z_{\alpha/2}}}$ tends to decrease as n increases. That is, $E_\theta\left[W_{Asymptotic}\right] = C\theta$ tends to decrease as n increases for all levels of n, θ and α.

IV. Simulation Results

This section provides a simulation study for the coverage probabilities and expected lengths of the three confidence intervals (Exact, Asymptotic and TestSTAT confidence intervals). Nine populations were each generated of size N = 100,000 in the form of a one parameter exponential distribution with θ = 0.5, 1, 2, 5, 7, 10, 30, 50 and 100. For each population, sample sizes of n = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 500 and 1,000 were randomly generated 5,000 times. From each set of samples, we then used the three methods to construct the 95% confidence interval for the parameter θ. In this section, the case of sample size $n > Z^2_{\alpha/2}$ where $Z^2_{\alpha/2} = 3.8416$ is conducted to guarantee that the results of the simulation study conform to the results in theorems 3.1–3.3 which are mentioned in section III. The results from the simulation are presented in Figs 1 and 2.

Fig 1 shows that the Exact, Asymptotic and TestSTAT confidence intervals achieve coverage closest to the nominal level (0.95) on average for all levels of θ and sample size n. This simulation result conforms to the results in theorem 3.1. In addition, Fig 2 shows that the Asymptotic confidence interval has the widest expected length of confidence interval for a small sample size. This conclusion conforms to the results in theorem 3.2. Furthermore, the expected lengths of the three confidence intervals do not seem to be different for a large sample size and all levels of θ. In addition, they tend to decrease when the sample size increases for all levels of θ. This simulation result conforms to the results in theorem 3.3.

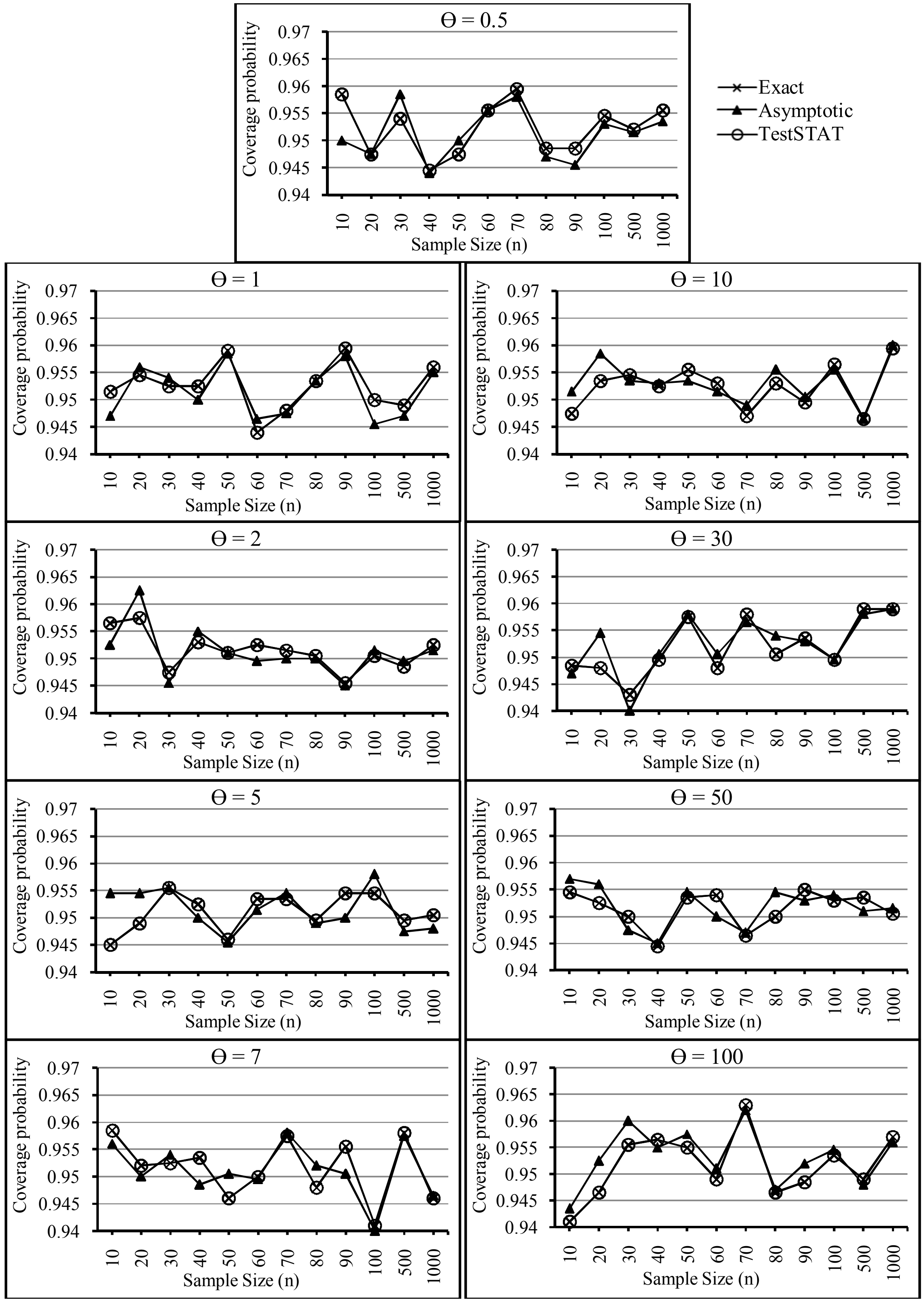

Fig 1. Coverage probabilities of the three confidence intervals for θ = 0.5, 1, 2, 5, 7, 10, 30, 50 and 100.

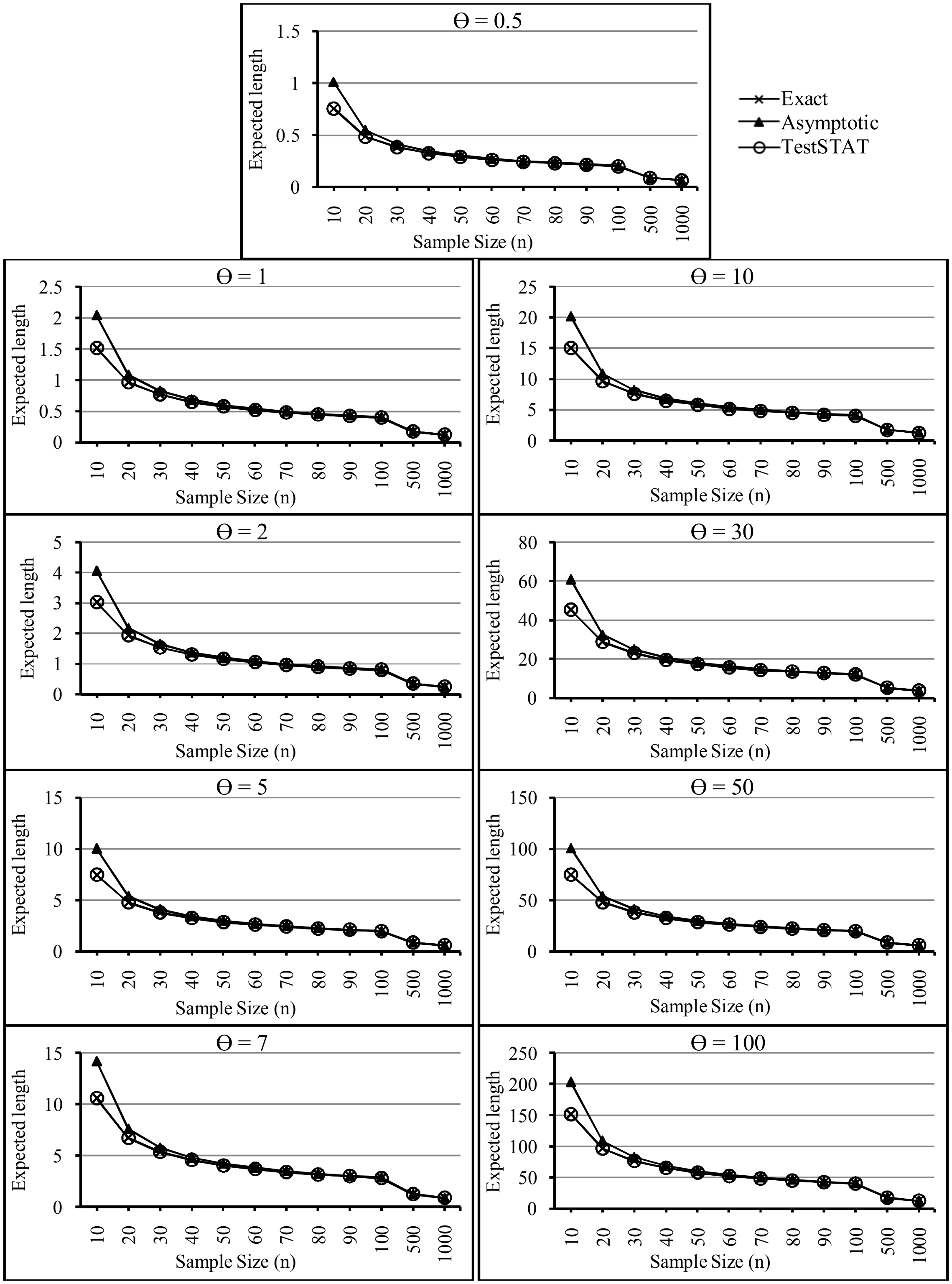

Fig 2. Expected lengths of the three confidence intervals for θ = 0.5, 1, 2, 5, 7, 10, 30, 50 and 100.

V. Discussion

The TestSTAT confidence interval which is derived in this paper uses the same formula as for the Exact confidence interval which is widely used. This simulation study and the proof in theorem 3.1 found that the coverage probabilities of the TestSTAT and Exact confidence intervals were close to the nominal level for all levels of sample size as mentioned by Geyer [17], Balakrishnan *et al*. [18], Cho *et al*. [19] and Jiang and Wong [25]. In addition, the study using the simulation dataset and the proof in theorem 3.3 found that the Asymptotic confidence interval was very efficient (short expected length of confidence interval) for a large sample size, as mentioned by Mukhopadhyay [4], Cho *et al*. [19], Mood *et al*. [20] and Shawiesh [21].

VI. CONCLUSION

The TestSTAT confidence interval is derived based on inverting a test statistic approach which is most helpful in situations where we have no good idea about a parameter θ. After the proof, we found that the TestSTAT confidence interval gave the same formula as for the Exact confidence interval which is most commonly used.

A simulation study was performed to guarantee the theoretical results 3.1 – 3.3 that are presented in this article and to compare the efficiencies of the three methods—Exact, Asymptotic and TestSTAT confidence intervals—in terms of the coverage probabilities and expected lengths of confidence interval. The comprehensive comparison results showed that for all levels of θ and α, the expected length of TestSTAT confidence interval is shorter than that of the Asymptotic confidence interval for a small sample size n which satisfy the conditions that $n > Z^2_{\alpha/2}$ and

$$\frac{1}{\chi^2_{\alpha/2,\,2n}} - \frac{1}{\chi^2_{1-\alpha/2,\,2n}} < \frac{Z_{\alpha/2}}{\sqrt{n}\left(n - Z^2_{\alpha/2}\right)}.$$

For a large sample size n such that $n > Z^2_{\alpha/2}$, there seemed to be no difference in the efficiency of the three methods for all levels of θ and α.

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