Analysis of a Stochastic Competitive Model with Regime Switching

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***Abstract*—This paper is concerned with a stochastic competitive model with Markov switching. Sufficient conditions for stochastic permanence, extinction, global attractivity and stability in distribution are established. Some numerical figures are introduced to validate the theoretical results.**

***Index Terms*—competitive model, Markov switching, permanence, extinction, globally attractive, stability in distribution.**

I. INTRODUCTION

In the natural world, it is a common phenomenon that several species compete for the limited resources, territories, etc. At the same time, the growth of species in the natural world is always affected by some random perturbations. Therefore it is important to study the competitive models with stochastic perturbations. As matter of fact, in recent years many authors have studied the stochastic competitive systems, and we here mention [1]-[12] among many others. Particularly, Mao et al. [6] and [7] revealed that the environmental noise can suppress a potential population explosion in some cases while Mao [8] showed that different structures of environmental noise may have different effects on the population systems. Li and Mao [9] investigated the following stochastic Lotka-Volterra competitive system

$$dx_i = x_i\big[b_i - \sum_{j=1}^{n} a_{ij}x_j\big]dt + \alpha_i x_i dB_i(t),\ i = 1, ..., n, \quad (1)$$

where $x_i = x_i(t)$ represents the population size of ith species at time t, the constant b_i means the intrinsic growth rate of species i, and a_{ij} represents the effect of interspecific (if $i \neq j$) or intraspecific (if $i = j$) interaction, $B_i(t)$ is standard Brownian motion, α_i^2 denotes the intensity of the white noise, $1 \leq i, j \leq n$. The authors [9] considered the permanence, extinction and global attractivity of model (1).

However, it has been noted that (see e.g., [13] and [14]), there are many random perturbations usually cannot be described by the traditional (deterministic or stochastic) Lotka-Volterra models. For example, the intrinsic growth rate b_i in model (1) often vary according to the changes in nutrition and food resources. Another example is that, the intrinsic growth rates of some species in the dry season will be much different from those in the rainy season. Similarly, the interspecific or intraspecific interactions differ in different environments. Usually, the switching between different environments is memoryless and the waiting time for the next switch has an exponential distribution. Therefore we can model the random environments and other random factors in the ecological system by a continuous-time Markov chain $\gamma(t),\ t \geq 0$ with finite-state space $\mathbb{S} = \{1, 2, ..., m\}$. Let Markov chain $\gamma(t)$ be generated by $Q = (q_{ij})$, that is,

$$P\{\gamma(t+\Delta t) = j|\gamma(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t),\ j \neq i; \\ \\ 1 + q_{ii}\Delta t + o(\Delta t),\ j = i, \end{cases} \quad (2)$$

where $q_{ij} \geq 0$ for $i, j = 1, 2, ..., m$ with $j \neq i$ and $\sum_{j=1}^{m} q_{ij} = 0$ for $i = 1, 2..., m$. Then the stochastic competitive ecosystem with regime switching is governed by

$$dx_i = x_i\Big[b_i(\gamma(t)) - \sum_{j=1}^{n} a_{ij}(\gamma(t))x_j\Big]dt + \alpha_i(\gamma(t))x_i dB_i(t), \quad (3)$$

or equivalently, in matrix form

$$dx = \mathrm{diag}(x_1, ..., x_n)\Big\{[b(\gamma) - A(\gamma)x]dt + \alpha(\gamma)dB(t)\Big\}, \quad (4)$$

where $B(t) = (B_1(t), ..., B_n(t))^T$ is an n-dimensional Brownian motion, $b(k) = (b_1(k), ..., b_n(k))^T$, $A(k) = (a_{ij}(k))$, $\alpha(k) = \mathrm{diag}(\alpha_1(k), ..., \alpha_n(k))$, and $b_i(k), a_{ij}(k) \geq 0$ for $k \in \mathbb{S}$, $1 \leq i, j \leq n$. The mechanism of the ecosystem (4) can be explained as follows. Assume that initially, the Markov chain $\gamma(0) = \kappa \in \mathbb{S}$, then the ecosystem (4) obeys the stochastic differential equation

$$dx = \mathrm{diag}(x_1, ..., x_n)\Big\{[b(\kappa) - A(\kappa)x]dt + \alpha(\kappa)dB(t)\Big\}$$

for a random amount of time until the Markov chain $\gamma(t)$ jumps to another state, say, $\varsigma \in \mathbb{S}$. Then the ecosystem obeys the stochastic differential equation

$$dx = \mathrm{diag}(x_1, ..., x_n)\Big\{[b(\varsigma) - A(\varsigma)x]dt + \alpha(\varsigma)dB(t)\Big\}$$

for a random amount of time until the Markov chain $\gamma(t)$ jumps to a new state again.

As matter of fact, in recent years model (3) has received great attention, see e.g. [15]-[20]. Particularly, Zhu and Yin [15], [16] have proposed the following assumption:

(A1) For each $\iota \in \mathbb{S}$ and $i, j = 1, 2, ..., n$ with $j \neq i$,

$$a_{ii}(\iota) > 0, \quad a_{ij}(\iota) \geq 0.$$

The authors [15], [16] have claimed that if Assumption (A1) holds, then

(i) for any initial conditions $x(0) = x_0 \in \mathbb{R}_+^n$ and $\gamma(0)) \in \mathbb{S}$, where

$$\mathbb{R}_+^n := \{(x_1, x_2, ..., x_n) : x_i > 0, i = 1, ..., n\},$$

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there is a unique solution $x(t)$ to (3) on $t \geq 0$, and the solution is continuous and will remain in R_+^n almost surely.

(ii) For any $p > 0$

$$\sup_{t\geq 0} \mathbb{E}\left[\sum_{i}^{n} x_i^p(t)\right] \leq K < \infty.$$

(iii) The solution of (3) is stochastically upper bounded, i.e., for any $\varepsilon > 0$, there is a constant H_ε such that for any initial data $x_0 \in R_+^n$ and $\gamma(0)) \in \mathbb{S}$,

$$\liminf_{t\to+\infty} P\{|x(t)| \leq H_\varepsilon\} \geq 1 - \varepsilon.$$

(iv) The solution $x(t)$ of (3) obeys

$$\limsup_{t\to\infty} \frac{\ln(|x(t)|)}{\ln t} \leq 1.$$

Based on the studies of [15], [16], some interesting topics arise naturally.

(Q1) Note that model (3) is a population model, then it is important and interesting to consider the permanence and extinction of the model.

(Q2) In the study of population models, global attractivity of the solution is also one of the most important topics. Then, is the solution of model (3) globally attractive?

(Q3) In the study of population models, people always seek for the positive equilibrium state and then study its stability. However, model (3) has no positive equilibrium state, then the solution of model (3) can not tends to any positive state. Therefore it is interesting and important to study whether model (3) still has some structural stability.

The aims of this paper are to study these problems. In Section II, we investigate the stochastic permanence of model (3). In Section III, the sufficient conditions for extinction are given. In Section IV, we establish the sufficient conditions for the global attractivity of model (3). In Section V, we show that model (3) can be stable in distribution. In Section VI, some examples and numerical simulations are introduced to validate the main results. The conclusions are given in Section VII.

II. Stochastic permanence of model (3)

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in R_+}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in R_+}$ satisfying the usual conditions(i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_l(t), ..., B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Assume that $B(t)$ and Markov chain $\gamma(t)$ are independent. Without loss of generality, we also assume that the initial conditions $x(0)$ and $\gamma(0)$ are non-random. From now on, we assume (A1) always holds

For the sake of convenience and simplicity, we define the following notations:

$$a^u = \max_{\iota\in\mathbb{S}}\{a(\iota)\}, \quad a^l = \min_{\iota\in\mathbb{S}}\{a(\iota)\}.$$

For any constant sequence $\{c_{ij}\}, \quad (1 \leq i, j \leq n)$, define

$$(\check{c_{ij}}) = \max_{1\leq i,j\leq n} c_{ij}, \quad (\hat{c_{ij}}) = \min_{1\leq i,j\leq n} c_{ij}.$$

Suppose that $f(t)$ is a continuous function on $[0, +\infty)$, define

$$[f(t)]^+ = \begin{cases} f(t), & f(t) > 0; \\ 0, & f(t) \leq 0, \end{cases}$$

$$[f(t)]^- = \begin{cases} -f(t), & f(t) < 0; \\ 0, & f(t) \geq 0, \end{cases}$$

If $x \in R^n$, its norm is denoted by $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$.

In order to give our results, let us now introduce another hypothesis.

(A2) $(\hat{r_i^l}) > 0$, where

$$r_i(\gamma(t)) = b_i(\gamma(t)) - \frac{1}{2}\alpha_i(\gamma(t)), \quad t \geq 0, \quad 1 \leq i \leq n.$$

From a biological point of view, this assumption means that each species in model (3) owns sufficiently large intrinsic growth rate or sufficiently small intensity of the noise.

Lemma 1. *Let Assumptions (A1) and (A2) hold. For any initial conditions $x(0) = x_0 \in \mathbb{R}_+^n$ and $\gamma(0)) \in \mathbb{S}$, the solution $x(t)$ of model (3) obeys*

$$\limsup_{t\to+\infty} \mathbb{E}\left(\frac{1}{|x(t)|^\theta}\right) \leq M \tag{5}$$

and

$$\liminf_{t\to+\infty} \frac{\ln\ (|x(t)|)}{\ln\ t} \geq -\frac{(\check{\alpha_i^u})^2}{2(\hat{r_i^l})} \quad a.s., \tag{6}$$

where θ is an arbitrary positive constant satisfying

$$\theta(\check{\alpha_i^u})^2 < 2(\hat{r_i^l}) \tag{7}$$

and M is a constant.

Proof: Define

$$U(x) = \frac{1}{\sum_{i=1}^{n} x_i}, \ x \in R_+^n; \ V_1(x(t)) = U(x(t)), \quad t \geq 0.$$

It then follows from the generalized Itô's formula (see, e.g. [21] and [22]) that

$$\begin{aligned} dV_1(x) \quad &= \Big\{ - V_1^2(x)\sum_{i=1}^{n} x_i\Big(b_i(\gamma) - \sum_{j=1}^{n} a_{ij}(\gamma)x_j\Big) \\ &\quad + V_1^3(x)\sum_{i=1}^{n} \alpha_i^2(\gamma)x_i^2\Big\}dt \\ &\quad - V_1^2(x)\sum_{i=1}^{n} \alpha_i(\gamma)x_i dB_i(t). \end{aligned}$$

where we drop t from $x_i(t)$ and $b_i(\gamma(t))$ etc. From Assumption (A2), we can choose a positive constant θ such that it obeys (7). Define

$$V_2(x(t)) = (1 + V_1(x(t)))^\theta.$$

Making use of the generalized Itô's formula again gives

$$\begin{aligned}
&dV_2(x(t))\\
&=\theta(1+V_1(x))^{\theta-2}\Big\{-(1+V_1(x))V_1^2(x)\\
&\times\sum_{i=1}^{n}x_i\Big(b_i(\gamma)-\sum_{j=1}^{n}a_{ij}(\gamma)x_j\Big)\\
&+V_1^3(x)\sum_{i=1}^{n}\alpha_i^2(\gamma)x_i^2+\frac{\theta+1}{2}V_1^4(x)\sum_{i=1}^{n}\alpha_i^2(\gamma)x_i^2\Big\}dt\\
&-\theta(1+V_1(x))^{\theta-1}V_1^2(x)\sum_{i=1}^{n}\alpha_i(\gamma)x_idB_i(t)\\
&=:\theta(1+V_1(x))^{\theta-2}F(x,\iota)dt\\
&-\theta(1+V_1(x))^{\theta-1}V_1^2(x)\sum_{i=1}^{n}\alpha_i(\gamma)x_idB_i(t).
\end{aligned}\tag{8}$$

Clearly, for $t\geq 0$ and $\iota\in\mathbb{S}$,

$$\begin{aligned}
F(x,\iota)\ &\leq-\frac{1}{2n}\Big[2(\hat{r_i^l})-\theta(\check{\alpha_i^u})^2\Big]V_1^2(x)\\
&+\Big[(\check{a_{ij}^u})+(\check{\alpha_i^u})^2\Big]V_1(x)+(\check{a_{ij}^u}).
\end{aligned}$$

Substituting the above inequality into (8) yields

$$\begin{aligned}
&dV_2(x(t))\\
&\leq\theta(1+V_1(x))^{\theta-2}\Big\{-\frac{1}{2n}\Big[2(\hat{r_i^l})-\theta(\check{\alpha_i^u})^2\Big]V_1^2(x)\\
&+\Big[(\check{a_{ij}^u})+(\check{\alpha_i^u})^2\Big]V_1(x)+(\check{a_{ij}^u})\Big\}dt\\
&-\theta(1+V_1(x))^{\theta-1}V_1^2(x)\sum_{i=1}^{n}\alpha_i(\gamma)x_idB_i(t).
\end{aligned}\tag{9}$$

Now, let κ be sufficiently small satisfying

$$0<\frac{2n\kappa}{\theta}<2(\hat{r_i^l})-\theta(\check{\alpha_i^u})^2$$

Define $V_3(x(t))=e^{\kappa t}V_2(x(t))$. An application of the generalized Itô's formula results in

$$\begin{aligned}
&dV_3(x(t))=\kappa\ e^{\kappa t}V_2(x)dt+e^{\kappa t}dV_2(x)\\
&\leq e^{\kappa t}(1+V_1(x))^{\theta-2}\Big\{\kappa(1+V_1(x))^2\\
&-\frac{\theta}{2n}\Big[2(\hat{r_i^l})-\theta(\check{\alpha_i^u})^2\Big]V_1^2(x)\\
&+\theta[(\check{a_{ij}^u})+(\check{\alpha_i^u})^2]V_1(x)+\theta(\check{a_{ij}^u})\Big\}dt\\
&-\theta e^{\kappa t}(1+V_1(x))^{\theta-1}V_1^2(x)\sum_{i=1}^{n}\alpha_i(\gamma)x_i(t)dB_i(t)\\
&=e^{\kappa t}\frac{\theta}{2n}(1+V_1(x))^{\theta-2}\Big\{\\
&-V_1^2(x)\Big[2(\hat{r_i^l})-\theta(\check{\alpha_i^u})^2-\frac{2n\kappa}{\theta}\Big]\\
&+2n\Big[(\check{a_{ij}^u})+(\check{\alpha_i^u})^2+\frac{2\kappa}{\theta}\Big]V_1+2n(\check{a_{ij}^u})+\frac{2n\kappa}{\theta}\Big\}dt\\
&-\theta e^{\kappa t}(1+V_1(x))^{\theta-1}V_1^2(x)\sum_{i=1}^{n}\alpha_i(\gamma)x_idB_i(t)\\
&=:e^{\kappa t}J(x,\iota)dt\\
&-\theta e^{\kappa t}(1+V_1(x))^{\theta-1}V_1^2(x)\sum_{i=1}^{n}\alpha_i(\gamma)x_idB_i(t).
\end{aligned}$$

Note that $J(x,\iota)$ is upper bounded in $R_+^n\times\mathbb{S}$, namely

$$M_1:=\sup_{(x,\iota)\in R_+^n\times\mathbb{S}}J(x,\iota)<+\infty.\tag{10}$$

Consequently,

$$\begin{aligned}
dV_3(x)\ &\leq M_1e^{\kappa t}dt\\
&-\theta\ e^{\kappa t}(1+V_1(x))^{\theta-1}V_1^2(x)\sum_{i=1}^{n}\alpha_i(\gamma)x_idB_i(t).
\end{aligned}$$

Integrating both sides of the above inequality and then taking expectations gives

$$\mathbb{E}[V_3(x)]=\mathbb{E}[e^{\kappa t}(1+V_1(x))^{\theta}]\leq(1+V_1(x_0))^{\theta}+\frac{M_1}{\kappa}e^{\kappa t}.$$

That is to say

$$\limsup_{t\to+\infty}\mathbb{E}[V_1^{\theta}(x)]\leq\limsup_{t\to+\infty}\mathbb{E}[(1+V_1(x))^{\theta}]\leq\frac{M_1}{\kappa}.\tag{11}$$

For $x(t)\in R_+^n$, note that

$$\Big(\sum_{i=1}^{n}x_i\Big)^{\theta}\leq\Big(n\max_{1\leq i\leq n}x_i\Big)^{\theta}\leq n^{\theta}|x|^{\theta}.\tag{12}$$

Therefore,

$$\limsup_{t\to+\infty}\mathbb{E}\Big[\frac{1}{|x(t)|^{\theta}}\Big]\leq n^{\theta}\frac{M_1}{\kappa}=:M,$$

which is the required assertion (5).

Now, we will prove (6). In fact, making use of (10), we observe from (9) that

$$\begin{aligned}
dV_2(x)\ &\leq M_1dt\\
&-\theta(1+V_1(x))^{\theta-1}V_1^2(x)\sum_{i=1}^{n}\alpha_i(\gamma)x_idB_i(t).
\end{aligned}$$

That is to say

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\leq\tau\leq t+1}V_2(x(\tau))\Big]\leq\mathbb{E}\Big[V_2(x(t))\Big]+M_1\\
&+\mathbb{E}\Big[\sup_{t\leq\tau\leq t+1}\Big|\int_t^{\tau}\theta(1+V_1(x(s)))^{\theta-1}\\
&\times V_1^2(x(s))\sum_{i=1}^{n}\alpha_i(\gamma(s))x_i(s)dB_i(s)\Big|\Big].
\end{aligned}\tag{13}$$

By virtue of the Burkholder-Davis-Gundy inequality (see, e.g. [23]) and the Hölder inequality, we can see that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\leq\tau\leq t+1}\Big|\int_t^{\tau}\theta(1+V_1(x(s)))^{\theta-1}\\
&\times V_1^2(x(s))\sum_{i=1}^{n}\alpha_i(\gamma(s))x_i(s)dB_i(s)\Big|\Big]\\
&\leq 0.5\mathbb{E}\Big[\sup_{t\leq\tau\leq t+1}V_2(x(\tau))\Big]\\
&+9\theta^2(\check{\alpha_i^u})^2\mathbb{E}\Big[\int_t^{t+1}V_2(x(s))ds\Big].
\end{aligned}$$

Substituting this inequality into (13), we can show that

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\leq\tau\leq t+1}V_2(x(\tau))\Big]&\leq 2\mathbb{E}[V_2(x(t))]+2M_1\\
&+18\theta^2(\check{\alpha_i^u})^2\mathbb{E}\Big[\int_t^{t+1}V_2(x(s))ds\Big].
\end{aligned}$$

Letting $t \to \infty$ and making us of (11) gives

$$\limsup_{t\to+\infty} \mathbb{E}\bigg[\sup_{t\le\tau\le t+1} V_2(x(\tau))\bigg] \le \frac{2M_1}{\kappa} + 18\theta^2(\check{\alpha_i^u})^2\frac{M_1}{\kappa} + 2M_1.$$

It follows from (12) and the definition of $V_2(x(t))$ that

$$\limsup_{t\to+\infty} \mathbb{E}\bigg[\sup_{t\le\tau\le t+1} \frac{1}{|x(\tau)|^\theta}\bigg] \le 2n^\theta\bigg\{\frac{M_1}{\kappa} + 9\theta^2(\check{\alpha_i^u})^2\frac{M_1}{\kappa} + M_1\bigg\} =: M_2.$$

Then for arbitrarily small $\varepsilon > 0$, by the Chebyshev inequality, we get

$$P\bigg\{\sup_{k\le\tau\le k+1} \frac{1}{|x(\tau)|^\theta} > t^{\varepsilon+(\theta(\check{\alpha_i^u})^2)/(2(\hat{r_i^l}))}\bigg\} \le \frac{M_2}{t^{\varepsilon+(\theta(\check{\alpha_i^u})^2)/(2(\hat{r_i^l}))}}.$$

In view of the Borel-Cantelli lemma (see e.g. [23]), one can obtain that for almost all $\omega \in \Omega$,

$$\sup_{k\le\tau\le k+1} \frac{1}{|x(\tau)|^\theta} \le t^{\varepsilon+(\theta(\check{\alpha_i^u})^2)/(2(\hat{r_i^l}))} \tag{14}$$

holds for all but finitely many k. Therefor there exists a $k_0(\omega)$ excluding a P-null set, for which (14) holds whenever $k \ge k_0$. Consequently, for almost all $\omega \in \Omega$, if $k \le t \le k+1$ and $k \ge k_0$, then

$$-\frac{\ln\ |x(t)|^\theta}{\ln t} \le \frac{\big(\varepsilon + \theta\frac{(\check{\alpha_i^u})^2}{2(\hat{r_i^l})}\big)\ln k}{\ln k} = \varepsilon + \theta\frac{(\check{\alpha_i^u})^2}{2(\hat{r_i^l})}$$

Therefore

$$\frac{\ln\ |x(t)|^\theta}{\ln t} \ge -\varepsilon - \frac{\theta(\check{\alpha_i^u})^2}{2(\hat{r_i^l})}.$$

Thereby

$$\liminf_{t\to+\infty} \frac{\ln\ |x(t)|}{\ln t} \ge -\frac{\varepsilon}{\theta} - \frac{(\check{\alpha_i^u})^2}{2(\hat{r_i^l})}, \quad a.s.$$

Letting $\varepsilon \to 0$ we obtain the desired assertion (6). This competes the proof. ■

Now, we are in the position to show the stochastic permanence whose definition is given below.

Definition 1. *Model (3) is said to be stochastic permanence if for any $\varepsilon \in (0,1)$, there exists a pair of positive constants $\delta = \delta(\varepsilon)$ and $\chi = \chi(\varepsilon)$ such that for any initial value $x(0) \in R_+^n$ and $\gamma(0) \in \mathbb{S}$, the solution obeys*

$$\liminf_{t\to+\infty} P\{|x(t)| \le \chi\} \ge 1-\varepsilon,$$

$$\liminf_{t\to+\infty} P\{|x(t)| \ge \delta\} \ge 1-\varepsilon.$$

Theorem 1. *Under assumptions (A1) and (A2), model (3) is stochastically permanent.*

Proof: From the works of [16], it is easy to see that we need only to show

$$\liminf_{t\to+\infty} P\{|x(t)| \ge \delta\} \ge 1-\varepsilon.$$

For any $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then by the Chebyshev inequality

$$P\{|x(t)| < \delta\} = P\{1/|x(t)| > 1/\delta\} \le \delta\mathbb{E}[1/|x(t)|].$$

That is to say,

$$\limsup_{t\to+\infty} P\{|x(t)| < \delta\} \le \delta M = \varepsilon.$$

In other words,

$$\liminf_{t\to+\infty} P\{|x(t)| \ge \delta\} \ge 1-\varepsilon.$$

This completes the proof. ■

III. Extinction

In the previous section, we have shown that under some condition, model (3) is stochastic permanence which is one of most important topics in biomathematics. In this section, we will investigate another important topic — extinction.

Definition 2. *$x(t)$ is said to go to extinction if $\lim_{t\to+\infty} x(t) = 0$.*

Now we give our main result of this section.

Theorem 2. *For any given initial value $x(0) \in R_+^n$ and $\gamma(t) \in \mathbb{S}$, the solution $x(t)$ of model (3) has the property that for every $1 \le i \le n$,*

$$\limsup_{t\to+\infty} \frac{\ln x_i(t)}{t} \le \sum_{k=1}^m \rho_k r_i(k), \tag{15}$$

where $\rho = (\rho_1, ..., \rho_k)$ is the stationary distribution of the Markovian chain $\gamma(t)$. Particularly, if $\sum_{k=1}^m \rho_k r_i(k) < 0$, then x_i goes to extinction.

Proof: Making use of the generalized Itô's formula yields

$$d\ \ln x_i(t) = \Big[r_i(\gamma(t)) - \sum_{j=1}^n a_{ij}(\gamma(t))x_j(t)\Big]dt + \alpha_i(\gamma(t))dB_i(t).$$

That is to say

$$\begin{aligned}
&\ln x_i(t) = \ln x_i(0) + \int_0^t r_i(\gamma(s))ds \\
&- \int_0^t \sum_{j=1}^n a_{ij}(\gamma(s))x_j(s)ds + \int_0^t \alpha_i(\gamma(s))dB_i(s) \\
&\le \ln x_i(0) + \int_0^t r_i(\gamma(s))ds + \int_0^t \alpha_i(\gamma(s))dB_i(s) \\
&=: \ln x_i(0) + \int_0^t r_i(\gamma(s))ds + U(t).
\end{aligned} \tag{16}$$

Clearly, $U(t)$ is a martingale with quadratic variation

$$\langle U, U\rangle_t = \int_0^t \alpha_i^2(\gamma(s))ds \le (\alpha_i^u)^2\ t.$$

Making use of the strong law of large numbers for martingales gives

$$\lim_{t\to+\infty} \frac{U(t)}{t} = 0, \quad a.s.$$

Dividing t on the both sides of (16) and then letting $t \to \infty$ results in

$$\limsup_{t\to+\infty} \frac{\ln x_i(t)}{t} \le \limsup_{t\to+\infty} \frac{1}{t}\int_0^t r_i(\gamma(s))ds,$$

which is the required assertion (15). ■

Theorem 2 shows that if one species in model (3) owns sufficiently small intrinsic growth rate or sufficiently large intensity of the noise, then the survival of this species is threatened.

IV. Global attractivity

In this section, we will establish sufficient conditions for the global attractivity to model (3).

Definition 3. *Let $x(t)$, $y(t)$ be two arbitrary solutions of model (3) with initial data $(x(0),\gamma(0)) \in R_+^n \times \mathbb{S}$ and $(y(0),\gamma(0)) \in R_+^n \times \mathbb{S}$, respectively. If*

$$\lim_{t\to+\infty} |x(t)-y(t)| = 0, \quad a.s.$$

then system (3) is said to be globally attractive.

A3 There exist positive constants $\lambda_1,\ \lambda_2, ...,\ \lambda_n$ and λ such that

$$\lambda_i a_{ii}(\iota) - \sum_{j=1,j\neq i}^{n} \lambda_j a_{ji}(\iota) > \lambda$$

for all $1 \le i \le n$ and $\iota \in \mathbb{S}$. The biological interpretation of this assumption is that, each species in model (3) owns sufficiently large intraspecific interaction coefficients or sufficiently small small interspecific interaction coefficients.

Theorem 3. *Under Assumption (A3), model (3) is globally attractive.*

Proof: Let $x(t)$, $y(t)$ be two arbitrary solutions of model (3) with initial data $(x(0),\gamma(0)) \in R_+^n \times \mathbb{S}$ and $(y(0),\gamma(0)) \in R_+^n \times \mathbb{S}$, respectively. It then follows from the generalized Itô's formula that

$$\begin{aligned} d\ln x_i(t) &= \Big[r_i(\gamma(t)) - \sum_{j=1}^{n} a_{ij}(\gamma(t))x_j(t)\Big]dt \\ &\quad +\alpha_i(\gamma(t))dB_i(t), \end{aligned}$$

$$\begin{aligned} d\ln y_i(t) &= \Big[r_i(\gamma(t)) - \sum_{j=1}^{n} a_{ij}(\gamma(t))y_j(t)\Big]dt \\ &\quad +\alpha_i(\gamma(t))dB_i(t). \end{aligned}$$

Thus

$$d\,(\ln x_i(t) - \ln y_i(t)) = -\sum_{j=1}^{n} a_{ij}(\gamma(t))(x_j(t)-y_j(t))dt. \tag{17}$$

Define

$$V(t) = \sum_{i=1}^{n} \lambda_i |\ln x_i(t) - \ln y_i(t)|, \quad t \ge 0.$$

In view of the generalized Itô's formula, one can see that

$$\begin{aligned} d^+V(t) &= \sum_{i=1}^{n} \lambda_i \mathrm{sgn}(x_i(t)-y_i(t)) d(\ln x_i(t) - \ln y_i(t)) \\ &= -\sum_{i=1}^{n} \lambda_i \mathrm{sgn}(x_i(t)-y_i(t)) \sum_{j=1}^{n} a_{ij}(\gamma)(x_j(t)-y_j(t)) \\ &\le -\sum_{i=1}^{n} \lambda_i a_{ii}(\gamma(t)) \Big|x_i(t)-y_i(t)\Big| dt \\ &\quad + \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \lambda_i a_{ij}(\gamma(t)) \Big|x_j(t)-y_j(t)\Big| dt \\ &= -\sum_{i=1}^{n} \lambda_i a_{ii}(\gamma(t)) \Big|x_i(t)-y_i(t)\Big| dt \\ &\quad + \sum_{j=1}^{n} \sum_{i=1,i\neq j}^{n} \lambda_j a_{ji}(\gamma(t)) \Big|x_i(t)-y_i(t)\Big| dt \\ &= -\sum_{i=1}^{n} \lambda_i a_{ii}(\gamma) \Big|x_i(t)-y_i(t)\Big| dt \\ &\quad + \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \lambda_j a_{ji}(\gamma) |x_i(t)-y_i(t)| dt \\ &\le -\sum_{i=1}^{n} \Big(\lambda_i a_{ii}(\gamma) - \sum_{j=1,j\neq i}^{n} \lambda_j a_{ji}(\gamma)\Big) \Big|x_i - y_i\Big| dt \\ &\le -\lambda \sum_{i=1}^{n} \Big|x_i(t)-y_i(t)\Big| dt. \end{aligned} \tag{18}$$

Integrating (18) from 0 to t yields

$$V(t) + \lambda \int_0^t \sum_{i=1}^{n} |x_i(s)-y_i(s)|ds \le V(0) < \infty.$$

Letting $t \to \infty$, one can observe that

$$\int_0^{\infty} |x(s)-y(s)|ds \le \int_0^{\infty} \sum_{i=1}^{n} |x_i(s)-y_i(s)|ds < \infty. \tag{19}$$

Moreover, one can see that

$$\mathbb{E}\int_0^{\infty} |x(s)-y(s)|ds < \infty. \tag{20}$$

Now set $v(t) = x(t) - y(t)$. Then it is obvious that $v \in C(R_+, R)$. Clearly, it follows from (19) that

$$\liminf_{t\to+\infty} |v(t)| = 0, \quad a.s. \tag{21}$$

We now claim that

$$\lim_{t\to+\infty} |v(t)| = 0, \quad a.s. \tag{22}$$

If this statement is not true, then

$$P\{\limsup_{t\to+\infty} |v(t)| > 0\} > 0.$$

Fixing a number $\varepsilon > 0$ such that

$$P(\Omega_1) \ge 2\varepsilon, \tag{23}$$

where

$$\Omega_1 = \{\limsup_{t\to+\infty} |v(t)| > 2\varepsilon\}.$$

Define the stopping times

$$\sigma_1 = \inf\{t \ge 0 : |v(t)| \ge 2\varepsilon\},$$
$$\sigma_{2k} = \inf\{t \ge \sigma_{2k-1} : |v(t)| \le \varepsilon\},$$

$$\sigma_{2k+1} = \inf\{t \geq \sigma_{2k} : |v(t)| \geq 2\varepsilon\}, \quad k = 1, 2, ...$$

By (21) and the definition of Ω_1 one sees that

$$\sigma_k < \infty \text{ for } \forall\ k \geq 1\ if\ \omega \in \Omega_1. \tag{24}$$

Using (20) one then derive that

$$\begin{aligned} \infty &> \mathbb{E}\int_0^{\infty} |v(s)|ds \\ &\geq \sum_{k=1}^{\infty} \mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty, \sigma_{2k}<\infty\}} \int_{\sigma_{2k-1}}^{\sigma_{2k}} |v(s)|ds\bigg] \\ &\geq \varepsilon \sum_{k=1}^{\infty} \mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty\}}(\sigma_{2k-1} - \sigma_{2k})\bigg], \end{aligned} \tag{25}$$

where I_A stands for the indicator function of set A. Note that (21) implies $\sigma_{2k} < \infty$ provided $\sigma_{2k-1} < \infty$.

At the same time, rewriting equation (3) gives

$$\begin{aligned} x_i(t) &= x_i(0) + \int_0^t f_i(x(s), s, \gamma(s))ds \\ &\quad + \int_0^t g_i(x(s), s, \gamma(s))dB_i(s), \end{aligned}$$

where

$$f_i(x(s), s, \gamma(s)) = x_i(s)\bigg[b_i(\gamma(s)) - \sum_{j=1}^{n} a_{ij}(\gamma(s))x_j(s)\bigg],$$

$$g_i(x(s), s, \gamma(s)) = \alpha_i(\gamma(s))x_i(s).$$

Compute that

$$\begin{aligned} &\mathbb{E}\bigg(\bigg|f_i(x(s), s, \gamma)\bigg|^2\bigg) \\ &= \mathbb{E}\bigg(x_i^2(s)\bigg|b_i(\gamma) - \sum_{j=1}^{n} a_{ij}(\gamma)x_j(s)\bigg|^2\bigg) \\ &\leq 0.5\mathbb{E}(x_i^4(s)) + 0.5\mathbb{E}\bigg[\bigg|b_i(\gamma) - \sum_{j=1}^{n} a_{ij}(\gamma)x_j(s)\bigg|^4\bigg] \\ &\leq 0.5\mathbb{E}(x_i^4(s)) + 0.5(n+1)^3 \\ &\times\mathbb{E}\bigg(b_i^4(\gamma) + \sum_{j=1}^{n} a_{ij}^4(\gamma)x_j^4(s)\bigg) \\ &\leq 0.5\mathbb{E}(x_i^4(s)) + 0.5(n+1)^3 \\ &\times\bigg[(b_i^u)^4 + \sum_{j=1}^{n}(a_{ij}^u)^4\mathbb{E}(x_j^4(s))\bigg] \\ &=: F_i(2, x(0)), \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}\bigg(\bigg|g_i(x(s), s, \gamma(s))\bigg|^2\bigg) = \mathbb{E}\bigg(|\alpha_i(\gamma(s))|^2 x_i^2(s)\bigg) \\ &\leq (\alpha_i^u)^2\mathbb{E}(x_i^2(s)) =: G_i(2, x(0)). \end{aligned}$$

It then follows from the Hölder inequality and the moment inequality of stochastic integrals that

$$\begin{aligned} &\mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\leq t\leq T}\bigg|x_i(\sigma_{2k-1}+t) - x_i(\sigma_{2k-1})\bigg|^2\bigg] \\ &\leq 2\mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\leq t\leq T} \\ &\bigg|\int_{\sigma_{2k-1}}^{\sigma_{2k-1}+t} f_i(x(s), \gamma(s), s)ds\bigg|^2\bigg] \\ &+2\mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\leq t\leq T} \\ &\bigg|\int_{\sigma_{2k-1}}^{\sigma_{2k-1}+t} g_i(x(s), \gamma(s), s)dB_i(s)\bigg|^2\bigg] \\ &\leq 2T\mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty\}} \int_{\sigma_{2k-1}}^{\sigma_{2k-1}+T} |f_i(x(s), \gamma(s), s)|^2 ds\bigg] \\ &+8\mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty\}} \int_{\sigma_{2k-1}}^{\sigma_{2k-1}+T} |g_i(x(s), \gamma(s), s)|^2 d(s)\bigg] \\ &\leq 2(T+4)T[F_i(2, x(0)) + G_i(2, x(0))] \end{aligned}.$$

This implies that

$$\begin{aligned} &\mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\leq t\leq T}\bigg|x(\sigma_{2k-1}+t) - x(\sigma_{2k-1})\bigg|^2\bigg] \\ &= \mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\leq t\leq T} \\ &\sum_{i=1}^{n}\bigg|x_i(\sigma_{2k-1}+t) - x_i(\sigma_{2k-1})\bigg|^2\bigg] \\ &\leq \sum_{i=1}^{n}\mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\leq t\leq T} \\ &\bigg|x_i(\sigma_{2k-1}+t) - x_i(\sigma_{2k-1})\bigg|^2\bigg] \\ &\leq 2(T+4)T\sum_{i=1}^{n}\bigg[F_i(2, x(0)) + G_i(2, x(0))\bigg]. \end{aligned} \tag{26}$$

Similarly, we can show that

$$\begin{aligned} &\mathbb{E}\bigg[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\leq t\leq T}\bigg|y(\sigma_{2k-1}+t) - y(\sigma_{2k-1})\bigg|^2\bigg] \\ &\leq 2(T+4)T\sum_{i=1}^{n}\bigg[F_i(2, y(0)) + G_i(2, y(0))\bigg]. \end{aligned} \tag{27}$$

Let

$$F_i(2) = \max\{F_i(2, x(0)), F_i(2, y(0))\},$$

$$G_i(2) = \max\{G_i(2, x(0)), G_i(2, y(0))\}.$$

By (ii), we can let $T = T(\varepsilon) > 0$ be sufficiently small such that

$$16(T+4)T\sum_{i=1}^{n}[F_i(2) + G_i(2)] \leq \varepsilon^3.$$

Applying (26) and (27) yields

$$\begin{aligned} &P\bigg\{\{\sigma_{2k-1} < \infty\}\bigcap\Omega_k^1\bigg\} \\ &\leq \frac{2(T+4)T\sum_{i=1}^{n}[F_i(2) + G_i(2)]}{0.25\varepsilon^2} \leq \frac{\varepsilon}{2}. \end{aligned} \tag{28}$$

$$\begin{aligned} &P\bigg\{\{\sigma_{2k-1} < \infty\}\bigcap\Omega_k^2\bigg\} \\ &\leq \frac{2(T+4)T\sum_{i=1}^{n}[F_i(2) + G_i(2)]}{0.25\varepsilon^2} \leq \frac{\varepsilon}{2}, \end{aligned} \tag{29}$$

where

$$\Omega_k^1 = \left\{ \sup_{0\le t\le T} |x(\sigma_{2k-1}+t) - x(\sigma_{2k-1})| \ge \frac{\varepsilon}{2} \right\},$$

$$\Omega_k^2 = \left\{ \sup_{0\le t\le T} |y(\sigma_{2k-1}+t) - y(\sigma_{2k-1})| \ge \frac{\varepsilon}{2} \right\}.$$

It follows from (28) and (29) that

$$P\Big\{\Big\{\sigma_{2k-1} < \infty\Big\} \bigcap \{\Omega_k^1 \bigcup \Omega_k^2\Big\}\} \le \varepsilon.$$

Making use of (24) gives

$$\begin{aligned}
&P\Big\{\Big\{\sigma_{2k-1} < \infty\Big\} \cap \Big\{(\Omega_k^1)^c \bigcap (\Omega_k^2)^c\Big\}\Big\} \\
&= P\Big\{\sigma_{2k-1} < \infty\Big\} \\
&-P\Big\{\Big\{\sigma_{2k-1} < \infty\Big\} \cap \Big\{\Omega_k^1 \bigcup \Omega_k^2\Big\}\Big\} \\
&\ge 2\varepsilon - \varepsilon = \varepsilon,
\end{aligned}$$

where A^c means the complementary set of A. We further compute that

$$\begin{aligned}
&P\Big\{\Big\{\sigma_{2k-1} < \infty\Big\} \\
&\bigcap \Big\{ \sup_{0\le t\le T} \Big|\nu(\sigma_{2k-1}+t) - \nu(\sigma_{2k-1})\Big| < \varepsilon \Big\}\Big\} \\
&\ge P\Big\{\{\sigma_{2k-1} < \infty\Big\} \cap \Big\{(\Omega_k^1)^c \bigcap (\Omega_k^2)^c\Big\}\Big\} \ge \varepsilon.
\end{aligned} \tag{30}$$

Set

$$\Omega_k^3 = \left\{ \sup_{0\le t\le T} \Big|\nu(\sigma_{2k-1}+t) - \nu(\sigma_{2k-1})\Big| < \varepsilon \right\}.$$

It is easy to see that if $\omega \in \{\sigma_{2k-1} < \infty\} \bigcap \Omega_k^3$, then

$$\sigma_{2k}(\omega) - \sigma_{2k-1}(\omega) \ge T.$$

Using (25) and (30) gives

$$\begin{aligned}
\infty \quad &> \varepsilon \sum_{k=1}^{\infty} \mathbb{E}\Big[I_{\{\sigma_{2k-1}<\infty\}}(\sigma_{2k-1} - \sigma_{2k})\Big] \\
&\ge \varepsilon \sum_{k=1}^{\infty} \mathbb{E}\Big[I_{\{\sigma_{2k-1}<\infty\}\bigcap \Omega_k^3}(\sigma_{2k-1} - \sigma_{2k})\Big] \\
&\ge \varepsilon T \sum_{k=1}^{\infty} P\Big\{\{\sigma_{2k-1} < \infty\} \bigcap \Omega_k^3\Big\} \\
&\ge \varepsilon T \sum_{k=1}^{\infty} \varepsilon \\
&= \infty.
\end{aligned}$$

which is a contraction. Therefore (22) must hold and this completes the proof. ■

V. Stability in distribution

In this section, let us consider the stability in distribution of the model (3). Let $y(t)$ denote the $R_+^n \times \mathbb{S}$-valued process $(x(t), \gamma(t))$ which is the solution of model (3). Let $p(t, x_0, \iota, dy \times \chi)$ stand for the transition probability of the process $y(t)$ and $P(t, x_0, \iota, \mathfrak{B} \times \mathcal{S})$ represent the transition probability of the event $\{y(t) \in \mathfrak{B} \times \mathcal{S}\}$ with initial data $y(0) = (x_0, \iota)$, where $\mathfrak{B}$ is a Borel set of R_+^n, $\mathcal{S}$ is a subset of $\mathbb{S}$. Therefore

$$P(t, x_0, \iota, \mathfrak{B} \times \mathcal{S}) = \sum_{\chi \in \mathcal{S}} \int_{\mathfrak{B}} p(t, x_0, \iota, dy \times \chi).$$

Let $\mathscr{P}(R_+^n \times \mathbb{S})$ be the space of all probability measures on $R_+^n \times \mathbb{S}$. For any P_1, P_2, define

$$\begin{aligned}
d_L(P_1, P_2) \quad &= \sup_{g\in L} \Big| \sum_{\iota\in\mathbb{S}} \int_{R_+^n} g(x_0, \iota) P_1(dx_0, \iota) \\
&\quad - \sum_{\iota\in\mathbb{S}} \int_{R_+^2} g(x_0, \iota) P_2(dx_0, \iota) \Big|,
\end{aligned}$$

where

$$\begin{aligned}
L = \Big\{ g : R_+^n \times \mathbb{S} \quad &\to R \Big| |g(x_0, \iota) - g(y_0, \chi)| \le \\
&|x_0 - y_0| + |\iota + \chi|, |g(\cdot)| \le 1 \Big\}.
\end{aligned}$$

For the sake of convenience, let $\lambda_\iota(t)$ stand for the Markov chain starting from $\iota \in \mathbb{S}$ at $t = 0$ and let $x^{x_0,\iota}(t)$ represent the solution of Eq. (4) with initial data $x(0) = x_0 \in R_+^n$ and $\gamma(0) = \iota \in \mathbb{S}$.

Definition 4. *If there exists a unique probability measure $\pi(\cdot \times \cdot)$ on $R_+^n \times \mathbb{S}$ such that for any $(x_0 \times \iota) \in R_+^n \times \mathbb{S}$, the transition probability $p(t, x_0, \iota, dy \times \{\chi\})$ of $x(t)$ converges weakly to $\pi(dy \times \{\chi\})$ when $t \to +\infty$, then System (4) is said to be asymptotically stable in distribution (ASD).*

Now we are in the position to state and prove our main results of this section.

Theorem 4. *Let Assumptions (A1), (A2) and (A3) hold, then the model (4) is ASD.*

Proof: Define $\overline{K}_{a,R} = \{x \in R_+^n | a \le |x| \le R\}$ and $\overline{K}_{a,R}^c = R_+^n - \overline{K}_{a,R}$ for a sufficiently large positive number R and a sufficiently small positive number a. Hence by (ii) and the tightness of transition probability density of $x(t)$ we have for any $\varepsilon_0 > 0$,

$$p(s, x_0, \iota, \overline{K}_{a,R}^c \times \mathbb{S}) \le \varepsilon_0. \tag{31}$$

For any $f \in L$, there is a $T_1 > 0$ such that

$$\begin{aligned}
&\Big|\mathbb{E}g(x^{x_0,\iota}(t), \lambda_\iota(t)) - \mathbb{E}g(y^{y_0,\chi}(t), \lambda_\chi(t))\Big| \\
&\le 2P\{\tau_{\iota\chi} > T_1\} \\
&+\mathbb{E}(I_{\tau_{\iota\chi}>T_1})\Big|g(x^{x_0,\iota}(t), \lambda_\iota(t)) - g(y^{y_0,\chi}(t), \lambda_\chi(t))\Big|,
\end{aligned} \tag{32}$$

where $\tau_{\iota\chi} = \inf\{t \ge 0, \lambda_\iota(t) = \lambda_\chi(t)\}$ for $\iota, \chi \in \mathbb{S}$. Note that the Markov chain is ergodic, then $\tau_{\iota\chi} < \infty$. Therefore for such T_1 and any $\varepsilon_1 > 0$, we have

$$P\{\tau_{\iota\chi} > T_1\} \le \frac{\varepsilon_1}{8}, \quad \iota, \chi \in \mathbb{S}. \tag{33}$$

Let $u = x^{x_0,\iota}(\tau_{\iota\chi})$, $v = y^{y_0,\iota}(\tau_{\iota\chi})$ and $k = \lambda_\iota(\tau_{\iota\chi}) = \lambda_\chi(\tau_{\iota\chi})$. Compute the second part of (32), one can observe that

$$\begin{aligned}&\mathbb{E}(I_{\tau_{\iota\chi}\le T_1})\Big|g(x^{x_0,\iota}(t),\lambda_\iota(t))-g(y^{y_0,\chi}(t),\lambda_\chi(t))\Big|\\&\le\mathbb{E}\Big[I_{\tau_{\iota\chi}\le T_1}\mathbb{E}\Big(\Big|g(x^{x_0,\iota}(t),\lambda_\iota(t))\\&\qquad-g(y^{y_0,\chi}(t),\lambda_\chi(t))\Big||\mathcal{F}_{\tau_{\iota\chi}}\Big)\Big]\\&\le\mathbb{E}\Big[I_{\tau_{\iota\chi}\le T_1}\mathbb{E}\Big(\Big|g(x^{u,k}(t-\tau_{\iota\chi}),\lambda_k(t-\tau_{\iota\chi}))\\&\qquad-g(y^{v,k}(t-\tau_{\iota\chi}),\lambda_k(t-\tau_{\iota\chi}))\Big|\Big)\Big]\\&\le\mathbb{E}\Big[I_{\tau_{\iota\chi}\le T_1}\mathbb{E}\Big(2\bigwedge\Big|x^{u,k}(t-\tau_{\iota\chi})-y^{v,k}(t-\tau_{\iota\chi})\Big|\Big)\Big]\end{aligned}\tag{34}$$

Define

$$\Omega_1=\{\omega\in\Omega|a\le|x^{x_0,\iota}|\le R\},\quad t\in[0,T_1],$$

where $R>0$ is a sufficiently large number and $a>0$ is a sufficiently small number. Then (ii) and (5) shows that

$$P(\Omega_1)>1-\frac{\varepsilon_1}{16},\quad (x_0,\iota)\in\overline{K}_{a,R}\times\mathbb{S}.\tag{35}$$

It then follows from Theorem 3 and Chebyshev's inequality that there exists a $T_2>0$ such that for all $t\ge T_2$,

$$\mathbb{E}\Big(2\bigwedge\Big|x^{x_0,\iota}(t)-y^{y_0,\iota}(t)\Big|\Big)<\frac{\varepsilon_1}{2}.\tag{36}$$

Consequently, in view of (35) and (36), we can see that

$$\begin{aligned}&\mathbb{E}\Big[I_{\tau_{\iota\chi}\le T_1}\mathbb{E}\Big(2\bigwedge\Big|x^{u,k}(t-\tau_{\iota\chi})-y^{v,k}(t-\tau_{\iota\chi})\Big|\Big)\Big]\\&\le 2P(\Omega-\Omega_1)+\mathbb{E}\Big[I_{\Omega_1\cap\{\tau_{\iota\chi}\le T_1\}}\\&\times\mathbb{E}\Big(2\bigwedge\Big|x^{u,k}(t-\tau_{\iota\chi})-y^{v,k}(t-\tau_{\iota\chi})\Big|\Big)\Big]\\&\le\tfrac{\varepsilon_1}{4}+\tfrac{\varepsilon_1}{2}=\tfrac{3\varepsilon_1}{4}.\end{aligned}\tag{37}$$

Substituting (33) and (37) into (32) results in

$$\Big|\mathbb{E}g(x^{x_0,\iota}(t),\lambda_\iota(t))-\mathbb{E}g(y^{y_0,\chi}(t),\lambda_\chi(t))\Big|\le\varepsilon_1.\tag{38}$$

Now for any $g\in L$ and $t,s>0$, fix any $(x_0,\iota)\in R_+^n\times\mathbb{S}$, one can obtain that

$$\begin{aligned}&\Big|\mathbb{E}g(x^{x_0,\iota}(t+s),\lambda_\iota(t+s))-\mathbb{E}g(x^{x_0,\chi}(t),\lambda_\iota(t))\Big|\\&=\Big|\mathbb{E}\Big[\mathbb{E}\Big(g(x^{x_0,\iota}(t+s),\lambda_\iota(t+s))|\mathcal{F}_s\Big)\Big]\\&\quad-\mathbb{E}g(x^{x_0,\iota}(t),\lambda_\iota(t))\Big|\\&=\Big|\sum_{i\in\mathbb{S}}\int_{R_+^n}\mathbb{E}g(x^{z_0,i}(t),\lambda_i(t))p(s,x_0,\iota,dz\times\{i\})\\&\quad-\mathbb{E}g(x^{x_0,\iota}(t),\lambda_\iota(t))\Big|\\&\le\sum_{i\in\mathbb{S}}\int_{R_+^n}\Big|\mathbb{E}g(x^{z_0,i}(t),\lambda_i(t))-\mathbb{E}g(x^{x_0,\iota}(t),\lambda_\iota(t))\Big|\\&\quad\times p(s,x_0,\iota,dz\times\{i\})\\&\le 2p(s,x_0,\iota,\overline{K}_{a,R}^c\times\mathbb{S})+\sum_{i\in\mathbb{S}}\int_{\overline{K}_{a,R}^c}\Big|\mathbb{E}g(x^{z_0,i},\lambda_i)\\&\quad-\mathbb{E}g(x^{x_0,\iota}(t),\lambda_\iota(t))\Big|p(s,x_0,\iota,dz\times\{i\}).\end{aligned}$$

An application of (31) and (38) gives that for $t>T,\ s>0$

$$\Big|\mathbb{E}g(x^{x_0,\iota}(t+s),\lambda_\iota(t+s))-\mathbb{E}g(x^{x_0,\chi}(t),\lambda_\iota(t))\Big|\le\varepsilon,$$

where $\varepsilon=\varepsilon_0+\varepsilon_1$. It then follows from the arbitrariness of g that

$$\sup_{g\in L}\big|\mathbb{E}g(x^{x_0,\iota}(t+s),\lambda_\iota(t+s))-\mathbb{E}g(x^{x_0,\iota}(t))\big|\le\varepsilon.$$

In other words

$$d_L(p(t+s,x_0,\iota,\cdot\times\cdot),p(t,x_0,\iota,\cdot\times\cdot))\le\varepsilon,\ \ \forall t\ge T,\ s>0.$$

Hence $\{p(t,0,1,\cdot\times\cdot):t\ge0\}$ is Cauchy in $\mathscr{P}$ with metric d_L. Consequently there is a unique $\pi(\cdot\times\cdot)\in\mathscr{P}$ such that

$$\lim_{t\to0}d_L(p(t,0,1,\cdot\times\cdot),\pi(\cdot\times\cdot))=0.$$

In view of Theorem 3, one can see that

$$\lim_{t\to0}d_L(p(t,x_0,\iota,\cdot\times\cdot),p(t,0,1,\cdot\times\cdot))=0.$$

Thereby

$$\begin{aligned}&\lim_{t\to0}d_L(p(t,x_0,\iota,\cdot\times\cdot),\pi(\cdot\times\cdot))\\&\le\lim_{t\to0}d_L(p(t,x_0,\iota,\cdot\times\cdot),p(t,0,1,\cdot\times\cdot))\\&+\lim_{t\to0}d_L(p(t,0,1,\cdot\times\cdot),\pi(\cdot\times\cdot))=0,\end{aligned}$$

This completes the proof. ∎

VI. Examples and numerical simulations

In this section, let us work out some numerical figures to illustrate the main results by use the the Milstein methods given in [29] (see also [30], [31]). Consider the following two-species model:

$$\begin{cases}dx_1(t)=x_1(t)\Big[b_1(\gamma(t))-a_{11}(\gamma(t))x_1(t)\\\qquad\qquad-a_{12}(\gamma(t))x_2(t)\Big]dt+\alpha_1(\gamma(t))x_1(t)dB_1(t),\\dx_2(t)=x_2(t)\Big[b_2(\gamma(t))-a_{21}(\gamma(t))x_1(t)\\\qquad\qquad-a_{22}(\gamma(t))x_2(t)\Big]dt+\alpha_2(\gamma(t))x_2(t)dB_2(t),\end{cases}\tag{39}$$

where $\gamma(t)$ is a Markovian chain with states $\mathbb{S}=\{1,2\}$, $a_{11}(\gamma(t))\equiv0.8$, $a_{12}(\gamma(t))\equiv0.4$, $a_{21}(\gamma(t))\equiv0.5$, $a_{22}(\gamma(t))\equiv0.7$, $b_1(\gamma(t))\equiv0.6$, $b_2(\gamma(t))\equiv0.5$. It is easy to see that Assumptions (A1) and (A3) hold ($\lambda_1=\lambda_2=1$).

In Fig.1, we choose $\alpha_1^2(1)=0.3$, $\alpha_1^2(2)=0.1$, $\alpha_2^2(1)=0.25$, $\alpha_2^2(2)=0.15$, $\rho_1=0.5$. Then $\widehat{r_i^l}=0.375$. By Theorem 1, the model (39) is stochastically permanent. Fig.1 confirms this.

In Fig.2, we choose $\alpha_1^2(1)=2.5$, $\alpha_1^2(2)=0.1$, $\alpha_2^2(1)=0.95$, $\alpha_2^2(2)=0.15$, $\rho_1=0.5$. Then $\rho_1r_1(1)+\rho_2r_1(2)=-0.05$ and $\rho_1r_2(1)+\rho_2r_2(2)=-0.05$. By Theorem 2, both x_1 and x_2 go to extinction. See Fig.2.

In Fig.3, the parameters are the same with Fig.1. It then follows from Theorem 3 that model (39) is globally attractive. Fig.3 confirms this.

In Fig.4, the parameters are the same with Fig.1. By Theorem 4, model (39) is asymptotically stable in distribution. See Fig.4.

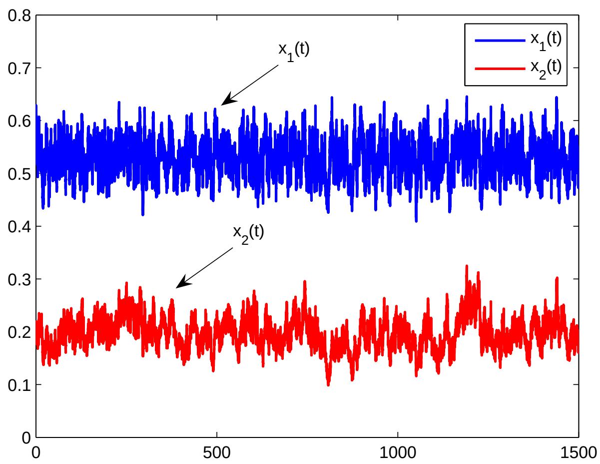

Fig. 1: Plot of a solution trajectory for model (39) with initial conditions $x_1(0) = 0.6$, $x_2(0) = 0.2$. This figure shows that model (39) is stochastically permanent.

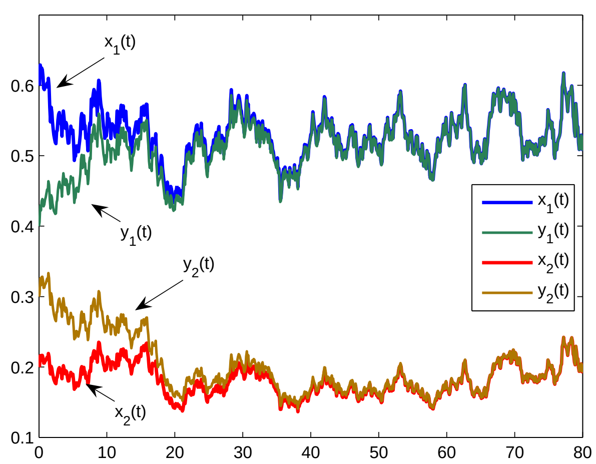

Fig. 3: Plot of two solution trajectories for model (39) with two sets of initial conditions $x_1(0) = 0.6$, $x_2(0) = 0.2$, $y_1(0) = 0.4$ $y_2(0) = 0.3$. This figure shows that model (39) is globally attractive.

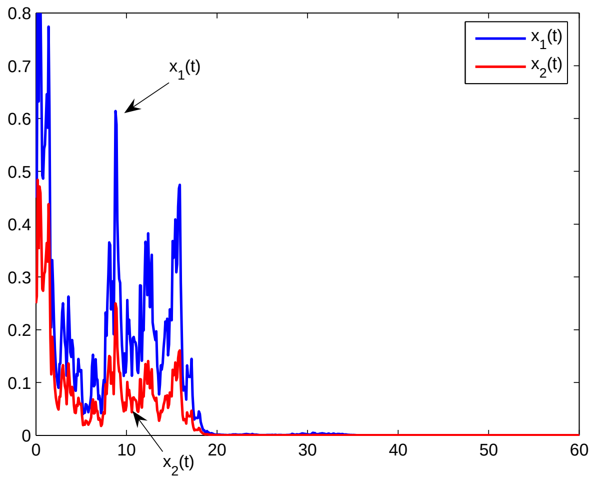

Fig. 2: Plot of a solution trajectory for model (39) with initial conditions $x_1(0) = 0.6$, $x_2(0) = 0.2$. This figure shows that both x_1 and x_2 go to extinction.

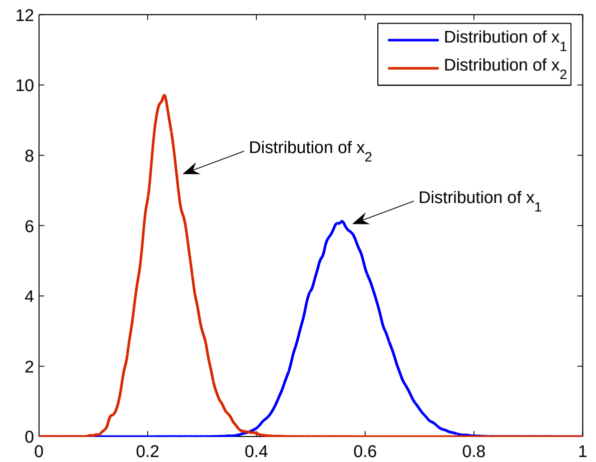

Fig. 4: Distribution for model (39) with initial conditions $x_1(0) = 0.6$, $x_2(0) = 0.2$. This figure shows that model (39) is asymptotically stable in distribution.

VII. Conclusion and Further Research

Recently, owing to their theoretical and practical significance, stochastic differential equations with Markovian switching have received great attention and have been studied extensively (see, for example,[15]-[20],[24]-[28]). This paper has been devoted to an n-dimensional stochastic competitive model with Markovian switching. Sufficient conditions for stochastic permanence, extinction, global attractivity and stability in distribution were obtained.

Some interesting questions deserve further investigation. The classical competitive exclusion principle (see, for example, [32]) tells us that competitive coefficients play a very important role in determining persistence-extinction of populations in deterministic competitive model, then it is an interesting and important topic to find out whether competitive coefficients also play an important role in determining persistence-extinction of populations in model (3). It is also interesting to consider other population models (see, for example, [33]-[38]) with Markovian switching.

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