Singularly Perturbed Parabolic Differential Equations with Turning Point and Retarded Arguments

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***Abstract*—In this article we propose an efficient numerical scheme based on a Shishkin mesh for a class of singularly perturbed parabolic convection-diffusion problems with boundary turning point and retarded arguments. The solution of the considered problem exhibit a boundary layer on the left side of the domain. The continuous problem is semidiscretized by means of backward Euler finite difference method in time to get a system of ordinary differential equations at each time level. This system of differential equations is discretized by using the standard upwind finite difference scheme on a non-uniform mesh of Shishkin type. It has been shown theoretically that the numerical solution generated by the method converges uniformly to the solution of the continuous problem with respect to the singular perturbation parameter. Numerical experiments supporting the theoretical results are given.**

***Index Terms*—Singular perturbation, convection diffusion parabolic problems, turning point, differential-difference equations, layer adapted piecewise uniform mesh.**

I. Introduction

Singularly perturbed partial differential equations often occur owing to the nature of certain physical phenomena such as small viscosity in the Navier stoke's equations. They also occur in modeling and analysis of heat and mass transfer process when the thermal conductivity and diffusion coefficients are small and the rate of reaction is large. In biology many singularly perturbed diffusive models have been established to describe the dynamics of some biological systems. The smallness of the diffusive parameter is found in many real life applications, see, for e.g., Murray [9] in which he pointed out that in blood hemoglobin molecules have a diffusion coefficient of the order of 10^{-11} m^2/s while that for the oxygen in the blood is of the order of 10^{-9}m^2/s. As indicated in [1] the dynamics of the solutions of these problems are far better than those of the solution of SPDE-PDEs.

In recent years, many robust numerical methods have been developed for solving the following singularly perturbed ordinary differential equations with delay and advance

$$\varepsilon^2 u''(x) - a(x)u'(x) + b(x)u(x-\delta) + c(x)u(x) + d(x)u(x+\eta) = f(x),$$

where $x \in (0, 1)$ and $\delta,\ \eta$ are delay and advance parameters respectively. Lange and Miura gave a series of papers [2], [3], [4] investigating different classes of problems of the above type using asymptotic analysis. Kadalbajoo and Sharma [11], [12], [13], [14] considered different classes of boundary value problems of the above type giving many robust numerical schemes and illustrating the effect of delay and advance on the solution behavior. Rai and Sharma [16], [17] considered the class of problems of the above type where the coefficient of the convection term vanishes inside the domain and developed ε-uniformly convergent numerical schemes for the solution of such type of problems. The problem considered in this paper is a generalization of the above problem and its study was started by Ramesh and Kadalbajoo [18] where they discussed upwind and midpoint upwind difference methods for singularly perturbed time dependent differential equations. Kumar and Kadalbajoo [5] constructed a numerical scheme comprising of standard implicit finite difference scheme in the temporal direction and a B-spline collocation method in the spatial direction. In [18], [5] the authors restricted their study to the case when the convection coefficient is non-vanishing through out the domain. In this paper we are initiating the study of singularly perturbed time dependent differential equations with boundary turning point and retarded arguments.

Classical numerical methods turn out to be inapplicable for singular perturbation problems. This happens because errors of the numerical solution depend on the perturbation parameter and become small only when the effective mesh-size in the layer is much less than the value of the parameter ε. These methods do not behave uniformly well for each value of singular perturbation parameter ε and in particular give unsatisfactory results when the perturbation parameter ε is quite small. To overcome this drawback the concept of ε-uniform numerical method is developed in which the order of convergence and the error constant are independent of the parameter ε, i.e., numerical methods that converge ε-uniformly. Over the last few decades, many ε-uniform numerical methods have been developed by many researchers for stationary and non-stationary problems [10], [8], [7]. In this paper we construct and analyze a fitted mesh finite difference scheme which utilizes special piecewise uniform mesh condensed in the boundary layer region. Proposed scheme consist of backward Euler method for the time discretization and standard finite difference operator for the spatial discretization. For the theoretical analysis the global error is decomposed into two parts; first due to the time discretization and the second part due to the spatial discretization of the semi-discrete problem obtained after the time discretization.

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Throughout this paper C denotes a generic positive constant independent of the perturbation parameter ε; $N,\ M$ denotes the number of mesh intervals in the spatial and the temporal directions respectively. In the analysis we use the standard supremum norm $||.||_{0,\ D}$ which is defined by $||g||_{0,\ D} = \sup_{\xi\in D}|g(\xi)|$ for a function g defined on some domain D. When the domain is obvious or of no particular significance we simply write $||.||_0$.

II. Problem formulation

We consider the following singularly perturbed two-point boundary value problem

$$\begin{aligned}\frac{\partial y(x,t)}{\partial t} - \varepsilon^2\frac{\partial^2 y(x,t)}{\partial x^2} - a(x,t)\frac{\partial y(x,t)}{\partial x} + \alpha(x,t)y(x-\delta,t)\\ +\quad \beta(x,t)y(x,t) + \omega(x,t)y(x+\eta,t) = f(x,t) \qquad \text{(II.1)}\end{aligned}$$

where

$$a(x,t) = a_0(x,t)x^p,\ p\geq 1,\ \forall\ (x,t)\in\bar{\Omega},\quad a_0(x,t)\geq\alpha>0, \quad \text{(II.2)}$$

$\Omega = D\times\Lambda = (0,1)\times(0,T]$, $S = \bar{\Omega}\setminus\Omega = S_x\cup S_0\cup S_1, S_0 = \{(0,t)|\ 0\leq t\leq T\}$, $S_x = \{(x,0)|\ 0\leq x\leq 1\}$ and $S_1 = \{(1,t)|\ 0\leq t\leq T\}$ in the space time plane for some fixed positive time T subject to the interval conditions

$$y(x,t) = \phi(x,t),\ (x,t)\in\Omega_1 = \{(x,t): -\delta\leq x\leq 0; t\in\Lambda\} \quad \text{(II.3)}$$

$$y(x,t) = \psi(x,t),\ (x,t)\in\Omega_2 = \{(x,t): 1\leq x\leq 1+\eta;\ t\in\Lambda\} \quad \text{(II.4)}$$

$$y(x,0) = y_0(x),\ \forall\ x\in S_x. \quad \text{(II.5)}$$

Here, $0<\varepsilon\ll 1$ is a small positive parameter, $\delta(\varepsilon)$ and $\eta(\varepsilon)$ are also small parameters assumed to be of order $o(\varepsilon)$. The functions $a(x,t),\ \alpha(x,t),\ \beta(x,t),\ \omega(x,t),\ f(x,t),\ \phi(x,t),\ \psi(x,t)$ and $y_0(x)$ are assumed to be smooth, bounded and independent of ε. In the special case when $\delta=\eta=0$, the above problem reduces to singularly perturbed parabolic differential equation. We are concerned with the related class of problems in which delay and advance terms occur in the reaction term and hence making the problem more difficult to tackle.

When the convection coefficient is non-zero the solution of the problem (II.1) exhibit boundary layer behavior and the position of the boundary layer depends upon the sign of the convection coefficient. This case has been discussed by various authors over last few years [18], [1], [5]. In this chapter, we deal with the case when the coefficient of the convection term vanishes at the boundary, i.e., it satisfies (II.2) with $p=1$.

Existence and uniqueness of the solution of (II.1) is established by assuming that the data are Holder continuous and imposing appropriate compatibility conditions at the corner points $(0,0),\ (1,0),\ (-\delta,0),\ (1+\eta,0)$. The required compatibility conditions becomes

$$\begin{aligned}y_0(0) &= \phi(0,0)\\ y_0(1) &= \psi(1,0)\\ \frac{\partial\phi(0,0)}{\partial t} &- \varepsilon\frac{\partial^2 y_0(0,0)}{\partial x^2} + a(0)\frac{\partial y_0(0,0)}{\partial x}\\ +\alpha(0)\phi(-\delta,0) &+ \beta(0)y_0(0,0) + \omega(0)y_0(\eta,0) = f(0,0), \quad \text{(II.6)}\\ \frac{\partial\psi(1,0)}{\partial t} &- \varepsilon\frac{\partial^2 y_0(1,0)}{\partial x^2} + a(1)\frac{\partial y_0(1,0)}{\partial x}\\ +\alpha(1)\phi(1-\delta,0) &+ \beta(1)y_0(1,0) + \omega(1)y_0(1+\eta,0) = f(0,0).\end{aligned}$$

Sufficient conditions for the existence and uniqueness of the solution is given by the following classical theorem(see [15]).

Theorem 2.1: Suppose $\phi(x,t)\in C^{2+\nu,1+\nu/2}(\Omega_1)$, $\psi(x,t)\in C^{2+\nu,1+\nu/2}(\Omega_2)$, $y_0(x)\in C^{2+\nu}(\bar{D})$, $a,b,c,d,f\in C^{\nu}(\bar{D})$, and assume the compatibility conditions (II.6) on the data are satisfied. Then (II.1) has a unique solution $y\in C^{2+\nu,1+\nu/2}(\Omega)$.

III. A priori estimates

Since $\delta\equiv\tau_1\varepsilon,\ \eta\equiv\tau_2\varepsilon$ where $\tau_1,\ \tau_2$ are of $o(1)$ we use Taylor's series to approximate the retarded arguments.

$$\begin{aligned}y(x-\delta,t) &\approx y(x,t) - \delta y_x(x,t) + \frac{\delta^2}{2}y_{xx}(x,t) \quad \text{(III.1)}\\ y(x+\eta,t) &\approx y(x,t) + \eta y_x(x,t) + \frac{\eta^2}{2}y_{xx}(x,t),\end{aligned}$$

where $y_x = \dfrac{\partial y}{\partial x},\quad y_{xx} = \dfrac{\partial^2 y}{\partial x^2}$. Using (III.1) in (II.1) we get

$$\begin{aligned}L_\varepsilon y(x,t)\equiv y_t(x,t) &\ -C_\varepsilon(x,t)y_{xx}(x,t) + Q(x,t)y_x(x,t)\\ &+R(x,t)y(x,t) = f(x,t),\ (x,t)\in\Omega \quad \text{(III.2)}\end{aligned}$$

subject to the conditions

$$\begin{aligned}y(x,0) &= y_0(x),\quad x\in S_x\\ y(0,t) &= \phi(0,t),\ 0\leq t\leq T \qquad \text{(III.3)}\\ y(1,t) &= \psi(1,t),\quad 0\leq t\leq T\end{aligned}$$

where $C_\varepsilon(x,t) = \varepsilon^2 - \alpha(x,t)\dfrac{\delta^2}{2} - \omega(x,t)\dfrac{\eta^2}{2}$, $Q(x,t) = -a(x,t) - \delta\alpha(x,t) + \eta\omega(x,t)$, $R(x,t) = \alpha(x,t) + \beta(x,t) + \omega(x,t)$. Problem (III.2) differ from (II.1) by $O(\delta^3 y_{xxx},\ \eta^3 y_{xxx})$. It is assumed that $0 < C_\varepsilon(x,t) < \varepsilon^2 - \delta^2 M_1 - \eta^2 M_2 = C_\varepsilon$ where $2M_1,\ 2M_2$ are the lower bound for $\alpha(x,t)$ and $\omega(x,t)$ respectively.
It is also assumed that

$$R(x,t) = \alpha(x,t)+\beta(x,t)+\omega(x,t)\geq K_0>0, \quad \text{(III.4)}$$

and

$$\delta\alpha(x,t) - \eta\omega(x,t)\geq 0. \quad \text{(III.5)}$$

Lemma 3.1: Maximum principle: Suppose $\Phi(x,t_{j+1})$ is a smooth function satisfying $\Phi(0,t_{j+1})$,
$\Phi(1,t_{j+1})\geq 0$ and $L_\varepsilon\Phi(x,t_{j+1})\geq 0$ for all $x\in D$ then we have $\Phi(x,t_{j+1})\geq 0$ for all $x\in\bar{D}$.

Lemma 3.2: The bound on the solution $y(x,t)$ of the problem (III.2), (III.3) is given by

$$|y(x,t)|\leq C,\ (x,t)\in\bar{\Omega}.$$

IV. Time discretization

On the time domain $[0,T]$ we introduce equidistant meshes with uniform time step Δt such that $\bar{\Omega}_t^{\,M} = \{t_j = j\Delta t,\ j = 0,\ldots,M,\ t_0=0,\ t_M=T,\ \Delta t = T/M\}$, where M denotes the number of mesh elements in t-direction.

On the above uniform mesh the time variable is discretized by means of implicit Euler method to get the following system of linear ordinary differential equations.

$$\begin{aligned}&\frac{Y(x,t_{j+1})-Y(x,t_j)}{\Delta t} - C_\varepsilon(x,t_{j+1})\frac{\partial^2 Y(x,t_{j+1})}{\partial x^2}\\ &+Q(x,t_{j+1})\frac{\partial Y(x,t_{j+1})}{\partial x} + R(x,t_{j+1})Y(x,t_{j+1}) = f(x,t_{j+1}),\\ &\text{where } 0\leq j<M \text{ and subject to } Y(x,0)=y_0(x)\quad x\in\bar{D}\\ &Y(0,t_{j+1}) = \phi(0,t_{j+1}),\ S_0^M = \{(0,t_j):\ 0<j\leq M\} \quad \text{(IV.1)}\\ &Y(1,t_{j+1}) = \psi(1,t_{j+1}),\ S_1^M = \{(1,t_j):\ 0<j\leq M\}\end{aligned}$$

where $Y(x, t_{j+1})$ is the solution of the equation (III.2) at $(j+1)$th time level.

Rewriting (IV.1) as

$$
\begin{aligned}
Y(x,0) &= y_0(x), \quad x \in \bar{D} \\
LY(x,t_{j+1}) \equiv -C_\varepsilon(x,t_{j+1})\frac{\partial^2 Y(x,t_{j+1})}{\partial x^2} + &\quad Q(x,t_{j+1}))\frac{\partial Y(x,t_{j+1})}{\partial x} \\
+T(x,t_{j+1}))Y(x,t_{j+1}) = &\quad g(x,t_{j+1}), \; x \in D \\
Y(0,t_{j+1}) = \phi(0,t_{j+1}), \; (0,t_j) &\in S_0^M \\
Y(1,t_{j+1}) = \psi(1,t_{j+1}), \; (1,t_j) &\in S_1^M
\end{aligned} \qquad \text{(IV.2)}
$$

where $T(x,t_{j+1})) = R(x,t_{j+1})) + \dfrac{1}{\Delta t}$, $g(x,t_{j+1}) = f(x,t_{j+1}) + \dfrac{Y(x,t_j)}{\Delta t}$. The local truncation error of the semi-discretization method (IV.1) is given by $e_{j+1} = y(x,t_{j+1}) - Y(x,t_{j+1})$. The local error measures the contribution of each time step to the global error of the time discretization given by E_j at instant t_j. We have the following results for the local error and the global error.

Lemma 4.1: Suppose $\left|\dfrac{\partial^i y(x,t)}{\partial t^i}\right| \leq C, \; \forall \; (x,t) \in \bar{\Omega}, \; i = 0, \; 1, \; 2$. Then the local truncation error in the temporal direction is given by $||e_{j+1}||_0 \leq C(\Delta t)^2$.

Proof: Equation (IV.1) can be written as

$$
\begin{aligned}
Y(x,t_{j+1}) - Y(x,t_j) = \Delta t \left[C_\varepsilon \frac{\partial^2 Y(x,t_{j+1})}{\partial x^2} - Q(x)\frac{\partial Y(x,t_{j+1})}{\partial x} \right] \\
+\Delta t \left[-R(x)Y(x,t_{j+1}) + f(x,t_{j+1}) \right].
\end{aligned} \qquad \text{(IV.3)}
$$

Also

$$
y(x,t_j) = y(x,t_{j+1}) - \Delta t y_t(x,t_{j+1}) + \int_{t_j}^{t_{j+1}} (t_j - \xi) y_{tt}(x,\xi) d\xi. \qquad \text{(IV.4)}
$$

Substituting the value of $y_t(x,t_j)$ from (III.2) we get

$$
\begin{aligned}
y(x,t_{j+1}) - y(x,t_j) = \Delta t \left[C_\varepsilon y_{xx}(x,t_{j+1}) - Q(x) y_x(x,t_{j+1}) \right] \\
+\Delta t \left[-R(x) y(x,t_{j+1}) + f(x,t_{j+1}) \right] + O(\Delta t)^2.
\end{aligned} \qquad \text{(IV.5)}
$$

Subtracting (IV.5) from (IV.3) gives us

$$
\Delta t L(e_{j+1}) = O(\Delta t)^2, \quad e_{j+1}(0) = e_{j+1}(1) = 0 \qquad \text{(IV.6)}
$$

which with the application of the Lemma 3.1 on the operator L gives us the desired estimate. ■

Lemma 4.2: Under the hypothesis of the Lemma 4.1 the global error estimate is given by

$$
||E_j||_0 \leq C\Delta t, \quad \forall j \leq T/\Delta t.
$$

Proof: Using local error estimate upto jth time step given by Lemma 4.1, we get the following global error estimate at $(j+1)^{th}$ time step

$$
\begin{aligned}
||E_{j+1}||_0 &= ||\sum_{l=1}^{j} e_l||_0 \qquad j \leq T/\Delta t \\
&\leq ||e_1||_0 + ||e_2||_0 + ||e_3||_0 + \cdots + ||e_j||_0 \\
&\leq C_1(j\Delta t)\Delta t \\
&\leq C_1 T \Delta t \qquad \text{since } j\Delta t \leq T \\
&\leq C\Delta t
\end{aligned}
$$

where C is a positive constant independent of ε and Δt. ■

Theorem 4.1: The solution $Y(x,t_j)$ of the problem (IV.2) satisfies the following bound on its derivatives

$$
\left| Y^{(k)}(x,t_j) \right| \leq C(C_\varepsilon)^{\frac{-i}{2}}, \quad i = 0, \; 1, \; 2, \; 3.
$$

Proof: Let $Q(x,t) = Q_0(x,t)(x-\tau)^p$ where , $Q(\tau,t) = 0$, $Q_0(x) \geq \nu > 0$ and using the streched variable $\hat{x} = (x-\tau)/\sqrt{C_\varepsilon}$ the problem (IV.2) is transformed into

$$
\begin{aligned}
-\frac{\partial^2 Y(\hat{x},t_j)}{\partial \hat{x}^2} \quad + Q_0(\hat{x},t) C_\varepsilon^{\frac{p-1}{2}} \hat{x}^p \frac{\partial Y(\hat{x},t_j)}{\partial \hat{x}} + T(\hat{x},t) Y(\hat{x},t_j) \\
= g(\hat{x},t_{j+1}), \; \hat{x} \in D_0 = (-\tau, (1-\tau)/\sqrt{C_\varepsilon}).
\end{aligned} \qquad \text{(IV.7)}
$$

Now we have two cases

Case 1: When $p > 1$. In this case the second term in the above differential equation become negligible and other terms are independent of ε therefore we have

$$
\left| Y^{(k)}(\hat{x},t_j) \right| \leq C, \; \hat{x} \in D_0 = (-\tau, (1-\tau)/\sqrt{C_\varepsilon}), \; i = 0, \; 1, \; 2, \; 3 \qquad \text{(IV.8)}
$$

which when transformed to the original variable x gives us

$$
\left| Y^{(k)}(x,t_j) \right| \leq C C_\varepsilon^{\frac{-i}{2}}, \; x \in D, \; i = 0, \; 1, \; 2, \; 3.
$$

Case 2: When $p = 1$. In this case the problem become independent of ε and therefore its solution $Y(x,t_j)$ and its partial derivatives satisfy

$$
\left| Y^{(k)}(\hat{x},t_j) \right| \leq C, \; \hat{x} \in D_0 = (-\tau, (1-\tau)/\sqrt{C_\varepsilon}), \; i = 0, \; 1, \; 2, \; 3
$$

and returning back to the original variable, we obtain

$$
\left| Y^{(k)}(x,t_j) \right| \leq C(C_\varepsilon)^{\frac{-i}{2}}, \; x \in D, \; i = 0, \; 1, \; 2, \; 3.
$$

■

To obtain stronger estimates for the bounds on the solution $Y(x,t_j)$ and its derivatives we decompose the solution of (IV.2) into regular and singular component as

$$
Y(x,t_{j+1}) = V(x,t_{j+1}) + W(x,t_{j+1}) \quad x \in \bar{D}
$$

where the regular part $V(x,t_{j+1})$ is the solution of the inhomogeneous problem

$$
LV(x,t_{j+1}) = g(x,t_{j+1}) \quad x \in D \qquad \text{(IV.9)}
$$

$$
V(x,t_{j+1}) = Y(x,t_{j+1}) \quad \text{on } S_x \cup S_1
$$

$$
V(x,t_{j+1}) = h(x,t_{j+1}) \quad \text{on } S_0
$$

and the singular part $W(x,t_{j+1})$ is the solution of the homogeneous problem

$$
LW(x,t_{j+1}) = 0 \quad x \in D \qquad \text{(IV.10)}
$$

$$
W(x,t_{j+1}) = 0 \quad \text{on } S_x \cup S_1
$$

$$
W(x,t_{j+1}) = Y(x,t_{j+1}) - V(x,t_{j+1}) \quad \text{on } S_0.
$$

Theorem 4.2: The solution $V(x,t_{j+1})$ of (IV.9) satisfy the following bound on its derivatives

$$
\left| V^{(k)}(x,t_{j+1}) \right| \leq C(1 + C_\varepsilon^{2-\frac{k}{2}}), \quad \forall x \in \bar{D} \quad k = 0,1,2,3
$$

and the solution $W(x,t_{j+1})$ of (IV.10) satisfy the following bounds

$$
\left| W^{(k)}(x,t_{j+1}) \right| \leq C C_\varepsilon^{-k/2} \exp\left(-\frac{x}{\sqrt{C_\varepsilon}} \right), \quad \forall x \in \bar{D} \quad k = 0,1,2,3.
$$

Proof: The three term asymptotic expansion of the smooth component $V(x,t_{j+1})$ is

$$
V(x,t_{j+1}) = V_0(x,t_{j+1}) + C_\varepsilon V_1(x,t_{j+1}) + C_\varepsilon^2 V_2(x,t_{j+1}) \qquad \text{(IV.11)}
$$

where the function $V_0(x,t_{j+1})$ satisfy the reduced problem

$$\begin{aligned} Q(x)\frac{\partial V_0(x,t_{j+1})}{\partial x} &+ R(x)V_0(x,t_{j+1}) = g(x,t_{j+1}), \\ V_0(1,t_{j+1}) &= \psi(1,t_{j+1}),\ x\in D \end{aligned} \quad \text{(IV.12)}$$

whereas the function $V_1(x,t_{j+1})$ satisfy

$$\begin{aligned} Q(x)\frac{\partial V_1(x,t_{j+1})}{\partial x} + R(x)V_1(x,t_{j+1}) &= -\frac{\partial^2 V_0(x,t_{j+1})}{\partial x^2}, \\ V_1(1,t_{j+1}) &= 0 \end{aligned} \quad \text{(IV.13)}$$

and finally, the function $V_2(x,t_{j+1})$ satisfy

$$\begin{aligned} &C_\varepsilon\frac{\partial^2 V_2(x,t_{j+1})}{\partial x^2} - Q(x,t_{j+1})\frac{\partial V_2(x,t_{j+1})}{\partial x} \\ &-R(x,t_{j+1})V_2(x,t_{j+1}) = \frac{\partial^2 V_1(x,t_{j+1})}{\partial x^2} \\ &V_1(0,t_{j+1}) = 0, \qquad V_1(1,t_{j+1}) = 0. \end{aligned} \quad \text{(IV.14)}$$

Since V_0, V_1 are independent of C_ε we have

$$\left|\frac{\partial^i V_0}{\partial x^i}\right| \le C, \quad \left|\frac{\partial^i V_1}{\partial x^i}\right| \le C \quad \text{(IV.15)}$$

for all the integers i and j such that $0 \le 2i + j$.
As (IV.14) has same form as (IV.1) therefore we have

$$\left|\frac{\partial^i V_2}{\partial x^i}\right| \le CC_\varepsilon^{\frac{-i}{2}}. \quad \text{(IV.16)}$$

Thus, combining (IV.15) and (IV.16) we get

$$\left|\frac{\partial^i V}{\partial x^i}\right| \le C(1 + C_\varepsilon^{2-\frac{i}{2}}).$$

For finding estimates on $W(x,t_{j+1})$ and its higher order derivatives we define two barrier functions

$$\Phi^\pm(x,t_{j+1}) = C\exp\left(-\frac{x}{\sqrt{C_\varepsilon}}\right) \pm W(x,t_{j+1}) \quad \text{(IV.17)}$$

where C is a constant such that $\Phi^\pm(0,t_{j+1}),\ \Phi^\pm(1,t_{j+1}) \ge 0$, for all $j = 0,\dots,M-1$.
Also

$$\begin{aligned} L\Phi^\pm(x,t_{j+1}) &= C\left(-1 + \frac{Q(x,t_{j+1})}{\sqrt{C_\varepsilon}} + T(x,t_{j+1})\right)\exp\left(-\frac{x}{\sqrt{C_\varepsilon}}\right) \\ &\ge 0, \quad \forall x\in D \quad (\because T-1>0, Q(x,t)\ge 0). \end{aligned}$$

Then using Lemma 3.1 we get

$$\Phi(x,t_{j+1}) \ge 0, \quad \text{for all } x\in\bar{D} \quad \text{(IV.18)}$$

which when substituted in (IV.17) gives us

$$|W(x,t_{j+1})| \le C\exp\left(-\frac{x}{\sqrt{C_\varepsilon}}\right), \quad x\in\bar{D}. \quad \text{(IV.19)}$$

Now using the transformation $\hat{x} = x/\sqrt{C_\varepsilon}$ for the problem (IV.10) and the same technique which was used for finding the bounds on the transformed problem (IV.7) we obtain

$$\left|\frac{\partial^i W(\hat{x},t_{j+1})}{\partial\hat{x}^i}\right| \le C|W(x,t_{j+1})|.$$

Returning back to the original variable x and using (IV.19), we obtain the desired estimate. ∎

V. Spatial discretization

In this section we construct a piecewise uniform mesh in such a way that more mesh points are generated in the boundary layer region rather than outside this region. Consider the spatial domain $\bar{D} = [0,1]$ and let $N \ge 4$ be a positive integer. The given domain is divided into two subintervals $[0,\sigma]$, $[\sigma,1]$ and on each subinterval a uniform mesh with $N/2$ mesh intervals is placed such that $\bar{D}_N = \{0 = x_1, x_2, \dots, x_{N/2} = \sigma, \dots, x_N = 1\}$ where the transition parameter σ is defined as $\sigma = \min(0.5,\ \sqrt{C_\varepsilon}\ln N)$. Mesh elements are given by

$$x_i = \begin{cases} ih_i & \text{if } i = 0,1,\dots N/2 \\ \sigma + (i - N/2)h_i & \text{if } i = N/2+1,\dots,N \end{cases}$$

where

$$h_i = \begin{cases} 2\sigma/N & 1 \le i \le N/2 \\ 2(1-\sigma)/N & N/2+1 \le i \le N. \end{cases}$$

On the set of grid points $\bar{\Omega}^N = \bar{D}_N \times [0,T]_{\Delta t}$ the parabolic operator is now discretized by means of upwind finite difference operator defined as

$$D_x^+D_x^-Y_{i,j} = \frac{2}{h_i + h_{i+1}}\left(\frac{Y_{i+1,j} - Y_{i,j}}{h_{i+1}} - \frac{Y_{i,j} - Y_{i-1,j}}{h_i}\right)$$

$$D_x^-Y_{i,j} = \frac{Y_{i,j} - Y_{i-1,j}}{h_i} \quad D_x^+Y_{i,j} = \frac{Y_{i+1,j} - Y_{i,j}}{h_{i+1}}$$

The discrete analogue of (IV.2) is thus defined as

$$\begin{aligned} L^*Y_{i,j} \equiv\ & -C_\varepsilon(x_i)D_x^+D_x^-Y_{i,j} + Q(x_i)D_x^+Y_{i,j} + T(x_i)Y_{i,j} \\ &= g(x_i,t_j), \quad i = 1,2,\dots N-1,\ j = 0,\dots,M-1 \\ Y_{0,j+1} &= \phi(0,j+1),\ Y_{N,j+1} = \psi(1,j+1),\ 0 \le j < M. \end{aligned} \quad \text{(V.1)}$$

The operator L^* satisfy the following comparison principle

Lemma 5.1: Let $\Psi_{i,j} = \Psi(x_i,t_j)$ be any mesh function such that $\Psi_{i,j} \ge 0$ on S. Then $L^*\Psi_{i,j} \ge 0$ on Ω^N implies $\Psi_{i,j} \ge 0$ on $\bar{\Omega}^N$.
This Lemma is used to prove the following property of the finite difference operator L^*.

Lemma 5.2: Let $Z_{i,j} = Z(x_i,t_j)$ be any mesh function such that $Z_{i,0} = 0,\ \forall\ i = 0,\dots N,\ Z_{0,j} = 0 = Z_{N,j},\ \forall\ j = 1,\dots,M$. Then

$$|Z_{i,j}| \le \frac{1}{K_0}\max_{0<k<N}|L^*Z_{k,j}|, \quad \forall\ i = 0,1\dots,N, j = 0,1,\dots,M.$$

Theorem 5.1: The solution $Y(x_i,t_{j+1})$ of the problem (IV.2) and $Y_{i,j+1}$ of (V.1) satisfy the following error estimate for all N

$$|Y(x_i,t_{j+1}) - Y_{i,j+1}| \le CN^{-1}(\ln N)^2,\ \forall\ i = 0,1,\dots,N$$

where C is a constant independent of ε, δ and η.

Proof: Similar to what we did for the semidiscrete problem, the solution $Y = \{Y_{i,j}\}_{i=0,\ j=0}^{N,\ M}$ of (V.1) can be decomposed into regular and singular component as

$$Y_{i,j+1} = V_{i,j+1} + W_{i,j+1} \quad \text{(V.2)}$$

where the regular part $V_{i,j+1}$ is the solution of the inhomogeneous problem

$$\begin{aligned} L^*V_{i,j+1} &= g_{i,j+1} \\ V_{0,j+1} &= V(0,t_{j+1}) \\ V_{N,j+1} &= V(1,t_{j+1}) \end{aligned} \quad \text{(V.3)}$$

and the singular part $W_{i,j+1}$ is the solution of the homogeneous problem

$$\begin{aligned} L^* W_{i,j+1} &= 0 \\ W_{0,j+1} &= W(0, t_{j+1}) \\ W_{N,j+1} &= W(1, t_{j+1}). \end{aligned} \tag{V.4}$$

The nodal error is given by

$$Y_{i,j+1} - Y(x_i, t_{j+1}) = V_{i,j+1} - V(x_i, t_{j+1}) + W_{i,j+1} - W(x_i, t_{j+1}). \tag{V.5}$$

For computing the nodal error of the regular component we have

$$\begin{aligned} L^*(V_{i,j+1} - V(x_i, t_{j+1})) = -C_\varepsilon(x_i, t_{j+1})\Big(D_x^+ D_x^- - \frac{\partial^2}{\partial x^2}\Big)V(x_i, t_{j+1}) \\ + Q(x_i, t_{j+1})\Big(D_x^+ - \frac{\partial}{\partial x}\Big)V(x_i, t_{j+1}). \end{aligned}$$

Let $x_i \in \bar{D}_N$. Then for any function $\Psi(x) \in C^2(\bar{D}_N)$ we have

$$\begin{aligned} \left|\left(\frac{\partial^2}{\partial x^2} - D_x^+ D_x^-\right)\Psi(x_i)\right| &\leq (x_{i+1} - x_{i-1})||\Psi^{(3)}||_3 \quad \text{(V.6)} \\ \left|\left(\frac{\partial}{\partial x} - D^+\right)\Psi(x_i)\right| &\leq (x_i - x_{i-1})||\Psi^{(2)}||_2. \end{aligned}$$

For the proofs of the above results refer [10].

Using (V.6), Theorem 4.2 and the fact that $x_{i+1} - x_{i-1} \leq 2N^{-1}$, $x_i - x_{i-1} \leq N^{-1}$ in (V.6) followed by simplification we get

$$\Big|L^*\left(V_{i,j+1} - V(x_i, t_{j+1})\right)\Big| \leq CN^{-1}. \tag{V.7}$$

Use of Lemma 5.2 for the function $(V_{i,j+1} - V(x_i, t_{j+1}))$ results into

$$|V_{i,j+1} - V(x_i, t_{j+1})| \leq CN^{-1}, \quad \forall\ i. \tag{V.8}$$

The derivation of the error on the singular component depend upon the mesh parameter σ. We have following cases depending upon the value of σ.

Case 1 : $\sigma = \frac{1}{2}$, i.e., uniform mesh.
In this case we have $\frac{1}{2} < \sqrt{C_\varepsilon}\ln N \implies C_\varepsilon^{-1} < 4(\ln N)^2$. The error estimation for this case is similar to that of the regular component. Using the fact that $x_{i+1} - x_{i-1} \leq 2N^{-1}$ and $x_i - x_{i-1} \leq N^{-1}$ we get

$$\begin{aligned} |L^*(W_{i,j+1} - W(x_i, t_{j+1}))| &\leq C(N^{-1}C_\varepsilon^{-1/2} + N^{-1}C_\varepsilon^{-1}) \\ &\leq CN^{-1}(\ln N)^2 \quad x_i \in D_N. \end{aligned} \tag{V.9}$$

Applying Lemma 5.2 to the mesh function $(W_{i,j+1} - W(x_i, t_{j+1}))$ results into

$$|W_{i,j+1} - W(x_i, t_{j+1})| \leq CN^{-1}(\ln N)^2 \quad x_i \in \bar{D}_N. \tag{V.10}$$

Case 2 : $\sigma < 1/2$, i.e., piecewise uniform mesh.
In this case the mesh is piecewise uniform and the mesh spacing is $h = 2\sigma/N$ in the subinterval $(0, \sigma)$ whereas in the subinterval $(\sigma, 1)$ it is $H = 2(1-\sigma)/N$.
Conside ring the subinterval $[\sigma, 1)$ and using triangle inequality we have

$$|W_{i,j+1} - W(x_i, t_{j+1})| \leq |W_{i,j+1}| + |W(x_i, t_{j+1})|. \tag{V.11}$$

Now the bound on $|W(x_i, t_{j+1})|$ given by Theorem 4.2 is

$$|W(x_i, t_{j+1})| \leq C\exp\left(-\frac{\sigma}{\sqrt{C_\varepsilon}}\right) = CN^{-1}. \tag{V.12}$$

For the subinterval $(0, \sigma)$ the truncation error becomes

$$|L^*(W_{i,j+1} - W(x_i, t_{j+1}))| \leq C(N^{-1}\ln N + Q(x_i)N^{-1}\ln N\ C_\varepsilon^{-1/2}).$$

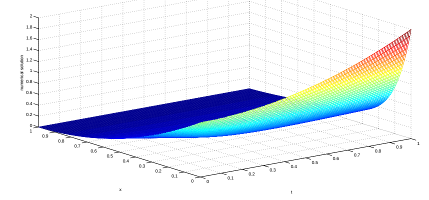

Fig. 1. The numerical solution profiles of example 1 for $\varepsilon = 2^{-4}$, $\delta = 0.6\varepsilon$, $\eta = 0.5\varepsilon$ and $M = N = 128$.

Now consider the barrier functions $\Psi_{i,j+1}^{\pm}$ defined by

$$\Psi_{i,j+1}^{\pm} = C(N^{-1}\ln N + N^{-1}\ln N\ C_\varepsilon^{-1/2}(\sigma - x_i) + N^{-1}) \pm (W_{i,j+1} - W(x_i, t_{j+1})).$$

Applying Lemma 5.1 for the barrier function $\Psi_{i,j+1}^{\pm}$ we get

$$\Psi_{i,j+1}^{\pm} \geq 0, \quad \forall\ x_i \in (0, \sigma).$$

This implies

$$\begin{aligned} |W_{i,j+1} - W(x_i, t_{j+1})| &\leq C(N^{-1}\ln N + N^{-1}\ln N\ C_\varepsilon^{-1/2}\sigma + N^{-1}) \\ &\leq C(N^{-1}(\ln N)^2). \end{aligned}$$

Thus combining the estimates in each of the subregions we have

$$|W_{i,j+1} - W(x_i, t_{j+1})| \leq CN^{-1}(\ln N)^2, \quad \forall\ x_i \in \bar{D}_N.$$

■

Theorem 5.2: Let $y(x,t)$ be the solution of problem (III.2), (III.3) and $Y_{i,j+1}$ be the solution of (V.1) then we have

$$||Y(x,t) - Y_{i,j+1}||_0 \leq C(M^{-1} + N^{-1}(\ln N)^2).$$

Proof: The proof follows from Lemma 4.2 and Theorem 3.1. ■

VI. Numerical experiments

To illustrate the theory given in the present study and examine the performance of the proposed numerical scheme a set of numerical experiments is carried out. Since exact solution is not known for the considered problem a double mesh principle [6] is used to tabulate the maximum pointwise error and the order of convergence of the proposed method. Maximum pointwise error $E_\varepsilon^{N,\Delta t}$ at all the mesh points are evaluated using the formula

$$\begin{aligned} E_\varepsilon^{N,\Delta t} &= \max_{0 \leq i,j \leq N,M}|Y^{N,\Delta t}(x_i, t_j) - Y^{2N,\Delta t/2}(x_i, t_j)|, \\ E^{N,\Delta t} &= \max_\varepsilon E_\varepsilon^{N,\Delta t} \end{aligned}$$

where the superscript N indicates the number of mesh points used in the spatial direction and $t_j = j\Delta t$ where Δt is the time step. Rates of uniform convergence $p_\varepsilon^{N,\delta t}$ are

$$p_\varepsilon^{N,\Delta t} = \frac{\log(E_\varepsilon^{N,\Delta t}/E_\varepsilon^{2N,\Delta t/2})}{\log 2}.$$

Example 1:

$$\begin{aligned} \varepsilon u_{xx}(x,t) + xu_x(x,t) &- u_t(x,t) - u(x,t) - 2u(x-\delta,t) \\ &- u(x+\eta,t) = x^2 - 1 \\ u(0,t) &= 1 + t^2,\ u(1,t) = 0,\ \text{at } T = 1 \\ u(x,0) &= (1-x)^2,\ 0 \leq x \leq 1. \end{aligned}$$

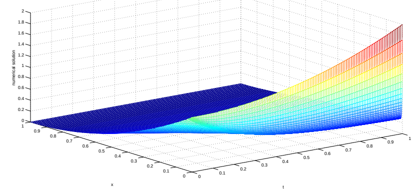

Fig. 2. The numerical solution profile of example 1 for $\varepsilon = 2^{-15}$, $\delta = 0.6\varepsilon$, $\eta = 0.5\varepsilon$ and $M = N = 128$.

TABLE I
THE MAXIMUM POINTWISE ERROR ($E_\varepsilon^{N,\Delta t}$) AND RATE OF CONVERGENCE p_ε^N, WHEN APPLIED TO EXAMPLE 1 FOR VARIOUS VALUES OF ε, M, N AND $\delta = 0.5\varepsilon$, $\eta = 0$.

ε	$M = N = 8$	$M = N = 16$	$M = N = 32$	$M = N = 64$	$M = N = 128$	$M = N = 256$
2^{-2}	$3.265E-02$	$1.869E-02$	$1.014E-02$	$5.296E-03$	$2.706E-03$	$1.368E-03$
	0.66	0.76	0.84	0.94	1.0	1.1
2^{-6}	$4.703E-02$	$3.164E-02$	$2.240E-02$	$1.384E-02$	$7.268E-03$	$3.745E-03$
	0.58	0.63	0.70	0.8	0.89	0.99
2^{-10}	$4.487E-02$	$3.014E-02$	$2.088E-02$	$1.331E-02$	$8.051E-03$	$4.716E-03$
	0.58	0.63	0.7	0.77	0.85	0.93
2^{-14}	$4.441E-02$	$2.980E-02$	$2.052E-02$	$1.308E-02$	$7.899E-03$	$4.627E-03$
	0.58	0.63	0.7	0.77	0.85	0.93
2^{-18}	$4.43E-02$	$2.972E-02$	$2.043E-02$	$1.302E-02$	$7.861E-03$	$4.604E-03$
	0.58	0.63	0.7	0.77	0.85	0.93

TABLE II
THE MAXIMUM POINTWISE ERROR ($E_\varepsilon^{N,\Delta t}$) AND RATE OF CONVERGENCE p_ε^N, WHEN APPLIED TO EXAMPLE 1 FOR VARIOUS VALUES OF ε, M, N AND $\delta = 0.9\varepsilon$, $\eta = 0.5\varepsilon$

ε	$M = N = 8$	$M = N = 16$	$M = N = 32$	$M = N = 64$	$M = N = 128$	$M = N = 256$
2^{-2}	$3.824E-02$	$2.328E-02$	$1.271E-02$	$6.687E-03$	$3.429E-03$	$1.737E-03$
	0.63	0.72	8.1	0.91	1.01	1.12
2^{-6}	$4.813E-02$	$3.259E-02$	$2.315E-02$	$1.435E-02$	$7.566E-03$	$3.895E-03$
	0.57	0.63	0.70	0.79	0.89	0.99
2^{-10}	$4.514E-02$	$3.032E-02$	$2.107E-02$	$1.342E-02$	$8.125E-03$	$4.759E-03$
	0.58	0.63	0.7	0.77	0.85	0.93
2^{-14}	$4.447E-02$	$2.985E-02$	$2.056E-02$	$1.311E-02$	$7.917E-03$	$4.637E-03$
	0.58	0.63	0.7	0.77	0.85	0.93
2^{-18}	$4.431E-02$	$2.974E-02$	$2.044E-02$	$1.303E-02$	$7.866E-03$	$4.607E-03$
	0.58	0.63	0.7	0.77	0.85	0.93

TABLE III
THE MAXIMUM POINTWISE ERROR ($E_\varepsilon^{N,\Delta t}$) AND RATE OF CONVERGENCE p_ε^N, WHEN APPLIED TO EXAMPLE 1 FOR VARIOUS VALUES OF ε, M, N AND $\delta = 0.4\varepsilon$, $\eta = 0.8\varepsilon$

ε	$M = N = 8$	$M = N = 16$	$M = N = 32$	$M = N = 64$	$M = N = 128$	$M = N = 256$
2^{-2}	$2.623E-02$	$1.453E-02$	$7.724E-03$	$4.005E-03$	$2.038E-03$	$1.029E-03$
	0.69	0.79	0.88	0.98	1.08	1.18
2^{-6}	$4.426E-02$	$2.983E-02$	$2.057E-02$	$1.275E-02$	$6.683E-03$	$3.430E-03$
	0.59	0.64	0.72	0.81	0.91	1.01
2^{-10}	$4.417E-02$	$2.968E-02$	$2.043E-02$	$1.303E-02$	$7.867E-03$	$4.609E-03$
	0.58	0.64	0.70	0.78	0.85	0.94
2^{-14}	$4.423E-02$	$2.969E-02$	$2.040E-02$	$1.301E-02$	$7.853E-03$	$4.599E-03$
	0.58	0.63	0.7	0.77	0.85	0.93
2^{-18}	$4.425E-02$	$2.97E-02$	$2.040E-02$	$1.300E-02$	$7.85E-03$	$4.598E-03$
	0.58	0.63	0.7	0.77	0.85	0.93

VII. CONCLUSION

In the current work a numerical scheme is proposed to examine the singularly perturbed time dependent differential-difference equation with turning point in one space dimension on a rectangular domain. The solution of the considered problem exhibit boundary layer on the left hand side of the domain. Euler implicit finite difference method is used fin time and the resulting set of ordinary differential equations at each time level are discretized by using standard upwind finite difference scheme on a non-uniform mesh of Shishkin type. An extensive amount of analysis is carried out in order to obtain uniform convergence. There are difficulties to approximate the solution of the problem due to presenceof the perturbation parameter, retarded arguments and the turning point. Theoretical analysis is carried out in order to obtain the stability and error estimate. It is proved that the method proposed is unconditionally stable and the convergence obtained is parameter uniform.

REFERENCES

[1] A. R. Ansari, S. A. Bakr, G. I. Shishkin, "A parameter uniform finite difference method for singularly perturbed delay parabolic partial differential equations", *Journal of Computational and Applied Mathematics*, vol. 205 pp. 552-566, 2007.

[2] C. G. Lange, R. M. Miura, "Singular perturbation analysis of boundary value problems for differential-difference equations. V. Small shifts with layer behavior", *SIAM Journal on Applied Mathematics*, vol. 54, pp. 249-272, 1994.

[3] C. G. Lange, R. M. Miura, "Singular perturbation analysis of boundary value problems for differential-difference equations. VI. Small shifts with rapid oscillations, *SIAM Journal on Applied Mathematics*, vol. 54, pp. 273-283, 1994.

[4] C. G. Lange, R. M. Miura, Singular perturbation analysis of boundary value problems for differential-difference equations. III. Turning point problems", *SIAM Journal on Applied Mathematics* vol. 45, pp. 708-734, 1985.

[5] D. Kumar, M. K. Kadalbajoo, "A parameter uniform numerical method for time-dependent singularly perturbed differential-difference equations", *Applied Mathematical Modelling*, vol.35, pp. 2805-2819, 2011.

[6] E. P. Doolan, J. J. H. Miller, W. H. A. Schilders, "Uniform numerical methods for problems with initial and boundary layers",*Boole Press, Dublin*, 1980.

[7] F. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, G. I. Shishkin, "Robust computational techniques for boundary layers", *Chapman-Hall/CRC, New York*, 2000.

[8] H. G. Roos, M. Stynes, L. Tobiska, "Numerical methods for singularly perturbed differential equations, convection-diffusion and flow problems",Springer Series in Computational Mathematics 24,*Springer-Verlag, Berlin*, 2008.

[9] J. D. Murray, "Mathematical biology I: An introduction third edition", *Springer-Verlag, Berlin*, 2001.

[10] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, "Fitted numerical methods for singular perturbation problems (revised edition)", *World Scientific*, Singapore, 2012.

[11] M. K. Kadalbajoo, K. K. Sharma, "Numerical analysis of boundary value problem for singularly perturbed delay differential equations with layer behavior", *Applied Mathematics and Computation*, vol. 151, no.1, pp. 11-28, 2004.

[12] M. K. Kadalbajoo, K. K. Sharma, "ε-Uniform fitted mesh method for singularly perturbed differential-difference equations: Mixed type of shifts with layer behavior", *International Journal of Computer Mathematics*, vol. 81, pp. 49-62, 2004.

[13] M. K. Kadalbajoo, K. K. Sharma, "Numerical treatment of a mathematical model arising from a model of neuronal variability", *Journal of Mathematical Analysis and Applications* vol. 307, pp. 606-627, 2005.

[14] M. K. Kadalbajoo, K. K. Sharma, "A numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations", *Applied Mathematics and Computation*, vol.197 pp. 692-707, 2008.

[15] O. A. Ladyzhenskaja, V. A. Solonnikov, N. N. Uralceva, "Linear and quasilinear equations of parabolic type", *American Mathematical Society*, Providence, R. I., 1968.

[16] P. Rai, K. K. Sharma, "Parameter uniform numerical method for singularly perturbed differential-difference equations with interior layer", *International Journal of Computer Mathematics*, vol. 88, no. 16, pp. 3416-3435, 2011.

[17] P. Rai, K. K. Sharma, "Numerical study of singularly perturbed differential-difference equation arising in the modeling of neuronal variability", *Computers & Mathematics with Applications* vol. 63, pp. 118-132, 2012.

[18] V. P. Ramesh, M. K. Kadalbajoo, "Upwind and midpoint upwind difference methods for time dependent differential-difference equations with layer behaviour",*Applied Mathematics and Computation*, vol. 202, pp. 453-471, 2008.