Some Discussions on Convergence Concepts of Uncertain Measure

Chunqin Zhang, Hui Zhang

Abstract—**The main purpose of this paper is to discuss the convergence of uncertain variables and the truncation method for uncertain measure. Based on uncertain measure theory, we propose the concepts of convergence with uncertain measure 1 and convergence almost everywhere. Then we investigate some properties of the convergence concepts. Finally, we present and prove the truncation method in uncertain environment. All obtained results are the natural extensions of the classical conclusions to the case where the measure tool is non-additive.**

Index Terms—**uncertain measure, convergence with uncertain measure 1, convergence almost everywhere, truncation method, uncertain variables.**

I. Introduction

CONVERGENCE concepts are basic and important concepts in classical measure theory [1], [2], [3]. With the development of the classical measure theory, some mathematicians felt that additivity is too restrictive in some application contexts. In fact, the additivity requirement of most circumstances cannot be easily satisfied or might not be satisfied at all [4], [5], [6]. Therefore, it is more reasonable to utilize non-additive measure to study the convergence concepts. Some Mathematics workers have explored them for fuzzy measures and uncertain measure, such as Liu Baoding [7], Wang Zhenyuan [8], Zhang Zhiming [9], Gianluca [10] and so forth. Some recent applications of convergence concepts for non-additive measure can be found in [11], [12], [13].

It is known that truncation method is also important in classical measure theory. It is a fundamental technique in proving the strong law of large numbers [1], [2]. When the measure tool is non-additive, this method is very different from additive case. Some mathematics workers have explored the truncation method for non-additive measures such as [14], [15], [16].

Uncertain measure is a kind of non-additive measure, which is a generalization of classical measure. It is essentially to deal with the uncertainty behaves neither randomness nor fuzziness. It was widely applied by some scholars such as Zhang Zhiming [9], Gao Xin [17], Liu Baoding [18], [19], [20] and so on. In this paper, some convergence concepts of uncertain variables and the truncation method for uncertain measure will be investigated. Our work helps to build important theoretical foundations for the development of uncertain measure theory.

Manuscript received February 03, 2015; revised August 21, 2015. This work was supported by National Nature Science Foundation of China (61375075); Natural Science Foundation of Hebei Province (A2014201166); Hebei Education Department, China (QN20131055).

Chunqin Zhang is with College of Mathematics and Information Science Hebei University Baoding, 071002, China. e-mail: hbu_zcq@163. com.

Hui Zhang is with College of Mathematics and Information Science Hebei University Baoding, 071002, China.

II. Preliminary

Uncertain measure is a typical non-additive measure, which is widely applied in some practical application. In this section, the definition and the properties of uncertain measure will be given. Then the expected value for uncertain variables will also be introduced. Some interested readers can refer to [4], [7], [21] for more details on uncertain measure theory.

A. The Definition and Properties of Uncertain Measure

Definition 2.1 let X be a nonempty set, and let $\mathcal{F}$ be a $\sigma-$algebra of X. If for any $A \in \mathcal{F}$, the set function $Un(A)$ satisfies the four axioms as follows:

Axiom 1. (Normality) $Un(X) = 1$.

Axiom 2. (Monotonicity) $\forall A,\ B \in \mathcal{F},\ \ A \subset B$,

$$Un(A) \leq Un(B).$$

Axiom 3. (Self-Duality) $A,\ A^c \in \mathcal{F}$,

$$Un(A) + Un(A^c) = 1.$$

Axiom 4. (Countable-Subadditivity) $\forall \{A_n\} \in \mathcal{F}$, we have

$$Un(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} Un(A_n).$$

Then the set function Un is called an uncertain measure on $\mathcal{F}$. The triplet $(X,\ \mathcal{F},\ Un)$ is called an uncertainty space.

Here A^c is the complement of A.

Theorem 2.1 Let Un be an uncertain measure. Then for any event A, we have

$$Un(\emptyset) = 0, \quad 0 \leq Un(A) \leq 1.$$

Proof. By using of definition 2.1,

$$Un(\emptyset) = 1 - Un(X) = 1 - 1 = 0,$$

and $\emptyset \subseteq A \subseteq X$, one can see that

$$Un(\emptyset) \leq Un(A) \leq Un(X),$$

that is

$$0 \leq Un(A) \leq 1.$$

Definition 2.2 An uncertain variable is a measurable function ξ from an uncertainty space $(X,\ \mathcal{F},\ Un)$ to the set of real numbers.

Definition 2.3 The uncertain distribution $\Phi:\ R \rightarrow [0,1]$ of an uncertain variable ξ is defined by

$$\Phi(x) = Un\{\xi \leq x\}.$$

That is, $\Phi(x)$ is the uncertainty that the uncertain variable takes a value less than or equal to x.

(Advance online publication: 15 February 2016)

Definition 2.4 The uncertain variables $\xi_1,\ \xi_2,\ ...,\ \xi_n\ ,...$ are said to be independent, if

$$Un\{\bigcap_{i=1}^{n}\{\xi_i \in B_i\}\} = \min_{1\le i\le n} Un\{\xi_i \in B_i\}$$

for any Borel sets $B_1,\ B_2,\ ...,B_n$ of real numbers set R.

Definition 2.5 The uncertain variables $\xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ are said to be identically distribution if

$$Un\{\xi_i \in B\} = Un\{\xi_j \in B\},\ i,j = 1,2,...$$

for any Borel set B of R.

B. Expected Value for Uncertain Variables

Definition 2.6 Suppose that ξ is an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} Un\{\xi \ge r\}dr - \int_{-\infty}^{0} Un\{\xi \le r\}dr,$$

provided that at least one of the two integrals is finite.

Theorem 2.2 Let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$. If

$$\lim_{x\to-\infty}\Phi(x) = 0, \quad \lim_{x\to+\infty}\Phi(x) = 1$$

and the Lebesgue-Stieltjes integral

$$\int_{-\infty}^{+\infty} x d\Phi(x)$$

is finite, then we have

$$E[\xi] = \int_{-\infty}^{+\infty} x d\Phi(x).$$

Theorem 2.3 Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

III. Some Discussions for Uncertain measure

In this section, convergence concepts of uncertain variables sequence and the truncation method will be given on uncertainty space.

A. Convergence Concepts of Uncertain Variables

Definition 3.1 Suppose that $\xi,\ \xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ are uncertain variables defined on the uncertainty space $(X, \mathcal{F}, Un)$. If there exists $E \in \mathcal{F}$ with $Un(E) = 0$ such that $\{\xi_n\}$ converges to ξ on E^c, then we say $\{\xi_n\}$ converges to ξ almost everywhere. Denoted by

$$\xi_n \to \xi \qquad (a.e.).$$

Definition 3.2 Suppose that $\xi_1,\ \xi_2, ..., \xi_n, ...$ is a sequence of uncertain variables. If there exists a uncertain variable ξ, such that

$$Un\{\lim_{n\to\infty}\xi_n = \xi\} = 1,$$

then we say that ξ_n converges with uncertain measure 1 to ξ. Denoted by

$$\lim_{n\to\infty}\xi_n = \xi \qquad (Un - a.s.).$$

Lemma 3.1 **[3]** Suppose that $\xi_1,\ \xi_2, ..., \xi_n, ...$ is sequence of uncertain variables, ξ is a uncertain variable. If

$$A_{n,m} = \{|\xi_n - \xi| \ge \frac{1}{m}\},$$

then

$$\{\lim_{n\to\infty}\xi_n = \xi\} = \bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{k\ge n}\bar{A}_{k,m}.$$

Lemma 3.2 Suppose that $\xi_1,\ \xi_2, ..., \xi_n, ...$ is a sequence of uncertain variables, ξ is a uncertain variable, then the following propositions are equivalent.

1) $\lim_{n\to\infty}\xi_n = \xi \quad (Un - a.s.)$;

2) $Un\{\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{k\ge n}[|\xi_k - \xi| < \frac{1}{m}]\} = 1,$

namely

$$Un\{\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k\ge n}[|\xi_k - \xi| \ge \frac{1}{m}]\} = 0;$$

3) $\forall\varepsilon > 0,\ \ Un\{\bigcup_{n=1}^{\infty}\bigcap_{k\ge n}[|\xi_k - \xi| < \varepsilon]\} = 1,$

namely

$$Un\{\bigcap_{n=1}^{\infty}\bigcup_{k\ge n}[|\xi_k - \xi| \ge \varepsilon\} = 0.$$

Proof. It follows from lemma 3.1 and 1) that

$$Un\{\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{k\ge n}[|\xi_k - \xi| < \frac{1}{m}]\} = 1.$$

Since uncertain measure is self-dual,

$$Un\{\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k\ge n}[|\xi_k - \xi| \ge \frac{1}{m}]\} = 0$$

is true. So 1) is equivalent to 2). Now we prove that 2) is equivalent to 3). $\forall\varepsilon > 0$, when $m > \frac{1}{\varepsilon}$, we have

$$\bigcap_{n=1}^{\infty}\bigcup_{k\ge n}[|\xi_k - \xi| \ge \varepsilon] \subset \bigcap_{n=1}^{\infty}\bigcup_{k\ge n}[|\xi_k - \xi| \ge \frac{1}{m}]$$

$$\subset \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k\ge n}[|\xi_k - \xi| \ge \frac{1}{m}],$$

thus

$$Un\{\bigcap_{n=1}^{\infty}\bigcup_{k\ge n}[|\xi_k - \xi| \ge \varepsilon]\}$$

$$\le Un\{\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k\ge n}[|\xi_k - \xi| \ge \frac{1}{m}]\}.$$

According to 2), we know 3) is true. Conversely, for any given positive integer m, assume that

$$T_m = \bigcap_{n=1}^{\infty}\bigcup_{k\ge n}[|\xi_k - \xi| \ge \frac{1}{m}],$$

by virtue of 3), $Un(T_m) = 0$. Because Un is countable-subadditivity, we have

$$0 \leq Un\{\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}[|\xi_k-\xi| \geq \frac{1}{m}]\}$$

$$\leq \sum_{m=1}^{\infty} Un\{\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}[|\xi_k-\xi| \geq \frac{1}{m}]\}$$

$$= \sum_{m=1}^{\infty} Un\{T_m\} = 0,$$

this implies that

$$Un\{\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}[|\xi_k-\xi| \geq \frac{1}{m}]\} = 0,$$

thus 2) is true.

Lemma 3.3 Suppose that X is a nonempty set, $\rho(X)$ is the power set of X. Let $A_k \in \rho(X)$,

$$u_k = Un\{A_k\}, \quad k = 1, 2, ...,$$

if

$$\sum_{k=1}^{\infty} u_k < \infty,$$

then

$$Un\{\bigcap_{n=1}^{\infty}\bigcup_{k\geq n} A_k\} = 0.$$

Proof. When

$$\sum_{k=1}^{\infty} u_k < \infty,$$

for $\forall\varepsilon > 0$, there exists a positive integer N, such that

$$0 \leq Un\{\bigcap_{n=1}^{\infty}\bigcup_{k\geq n} A_k\} \leq Un\{\bigcup_{k\geq N} A_k\}$$

$$\leq \sum_{k\geq N} Un\{A_k\} = \sum_{k\geq N} u_k \leq \varepsilon,$$

one can see that

$$Un\{\bigcap_{n=1}^{\infty}\bigcup_{k\geq n} A_k\} = 0.$$

Lemma 3.4 **([2])** Suppose that $\xi_n,\ \xi \in \mathcal{F}$, for any given $\varepsilon_k > 0$,

$$\lim_{n\to\infty} \varepsilon_k = 0,$$

we have

$$(1)\quad \{\xi_n \to \xi\} = \bigcap_{\varepsilon>0}\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\{|\xi_n-\xi| < \varepsilon\}$$

$$= \bigcap_{k=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\{|\xi_n-\xi| < \varepsilon_k\};$$

$$(2)\quad \{|\xi_n-\xi_m| \to 0\} = \bigcap_{\varepsilon>0}\bigcup_{n=1}^{\infty}\bigcap_{v=1}^{\infty}\{|\xi_{n+v}-\xi_n| < \varepsilon\}$$

$$= \bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{v=1}^{\infty}\{|\xi_{n+v}-\xi_n| < \varepsilon_k\}.$$

Theorem 3.1 Suppose that $\xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ are uncertain variables defined on the uncertainty space $(X, \mathcal{F}, Un)$. Then $\{\xi_n\}$ converges with uncertain measure 1 to 0 if and only if $\forall c \in (0, \infty)$,

$$\sum_{k=1}^{\infty} Un(|\xi_k| \geq c) < \infty.$$

Proof. By virtue of lemma 3.2,

$$\lim_{n\to\infty} \xi_n = 0 \quad (Un - a.s.)$$

if and only if $\forall c > 0$,

$$Un\{\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}[|\xi_k| \geq c]\} = 0.$$

According to lemma 3.3, we know that theorem 3.1 holds.

Theorem 3.2 Suppose that $\xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ are uncertain variables, then

(1) $\xi_n \to \xi \quad (a.e.)$, if and only if

$$Un(\bigcap_{n=1}^{\infty}\bigcup_{v=1}^{\infty}\{|\xi_{n+v}-\xi_n| \geq \varepsilon\} = 0, \quad \forall\varepsilon > 0.$$

(2) $|\xi_n - \xi_m| \to 0 \quad (a.e.)$, if and only if

$$Un(\bigcap_{n=1}^{\infty}\bigcup_{v=1}^{\infty}\{|\xi_{n+v}-\xi| \geq \varepsilon\} = 0, \quad \forall\varepsilon > 0.$$

Proof. (1) If

$$\xi_n \to \xi \qquad (a.e.),$$

then $\forall\varepsilon > 0$, by virtue of lemma 3.4,

$$Un(\bigcap_{n=1}^{\infty}\bigcup_{v=1}^{\infty}\{|\xi_{n+v}-\xi| \geq \varepsilon\}$$

$$\leq Un(\bigcup_{\varepsilon>0}\bigcap_{n=1}^{\infty}\bigcup_{v=1}^{\infty}\{|\xi_{n+v}-\xi| \geq \varepsilon\}$$

$$= Un(\{\xi_n \to \xi\}^c) = 0.$$

On the other hand, if

$$Un(\bigcap_{n=1}^{\infty}\bigcup_{v=1}^{\infty}\{|\xi_{n+v}-\xi| \geq \varepsilon\} = 0, \quad \forall\varepsilon > 0,$$

then for any given $\varepsilon_k > 0$,

$$\lim_{k\to\infty} \varepsilon_k = 0,$$

it follows from lemma 3.4 that

$$Un(\{\xi_n \to \xi\}^c) = Un(\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{v=1}^{\infty}\{|\xi_{n+v}-\xi| \geq \varepsilon_k\})$$

$$\leq \sum_{k=1}^{\infty} Un(\bigcap_{n=1}^{\infty}\bigcup_{v=1}^{\infty}\{|\xi_{n+v}-\xi| \geq \varepsilon_k\})] = 0,$$

that is

$$\xi_n \to \xi \quad (a.e.).$$

(2) In the similar way, we can prove the second conclusion.

B. Truncation Method for Uncertain Variables

Definition 3.3 Suppose that $\xi_1,\ \xi_2,...,\xi_n,...$ is a sequence of uncertain variables, ξ_n, $n=1,2,...$, have finite expected values $E[\xi_n]$. Assume that

$$\bar{\xi}_n=\frac{1}{n}\sum_{i=1}^{n}\xi_i,$$

if

$$\lim_{n\to\infty}(\bar{\xi}_n-E(\bar{\xi}_n))=0 \qquad (Un-a.s.),$$

then we say that $\{\xi_n\}$ obeys the strong law of large numbers.

Theorem 3.3 (Truncation method) Let $\xi_1,\xi_2,...,\xi_n,...$ be a sequence of independent and identically distributed uncertain variables whose distribution function is $\Phi(x)$, $\xi_n,\ n=1,2,...$, have the same expected values a (a is finite), and

$$\xi_n^*=\xi_n\chi_{\{|\xi_n|<n\}}(\omega),\ n=1,2,...,$$

if

$$\sum_{n=1}^{\infty}Un\{\xi_n^*\neq\xi_n\}<\infty,$$

then $\{\xi_n^*\}$ obeys the strong law of large numbers if and only if $\{\xi_n\}$ obeys the strong law of large numbers.

Proof. Let

$$\bar{\xi}_n=\frac{1}{n}\sum_{i=1}^{n}\xi_i,\quad \bar{\xi}_n^*=\frac{1}{n}\sum_{i=1}^{n}\xi_i^*,\quad E(\xi_i)=a,$$

then

$$|\bar{\xi}_n-a|=|\bar{\xi}_n-\bar{\xi}_n^*+\bar{\xi}_n^*-E(\bar{\xi}_n^*)+E(\bar{\xi}_n^*)-a|$$
$$\leq|\bar{\xi}_n-\bar{\xi}_n^*|+|\bar{\xi}_n^*-E(\bar{\xi}_n^*)|+|E(\bar{\xi}_n^*)-a|$$

it follows from lemma 3.2, lemma 3.3 and

$$\sum_{n=1}^{\infty}Un\{\xi_n^*\neq\xi_n\})<\infty$$

that

$$\lim_{n\to\infty}|\bar{\xi}_n-\bar{\xi}_n^*|=0 \qquad (Un-a.s.).$$

Since $\{\xi_n^*\}$ obeys the strong law of large numbers,

$$\lim_{n\to\infty}|\bar{\xi}_n^*-E(\bar{\xi}_n^*)|=0 \qquad (Un-a.s.)$$

holds true. And

$$\lim_{n\to\infty}E(\xi_n^*)=\lim_{n\longrightarrow\infty}\int_{|x|<n}xd\Phi(x)=E(\xi_n)=a.$$

When $n\to\infty$,

$$|E(\bar{\xi}_n^*)-a|=|\frac{1}{n}E(\sum_{i=1}^{n}\xi_i^*)-a|$$
$$=|\frac{1}{n}\sum_{i=1}^{n}E(\xi_i^*)-a|\longrightarrow 0,$$

(If $\lim_{n\to\infty}a_n=a$, then $\frac{1}{n}\sum_{k=1}^{n}a_k\to a$). Thus

$$\lim_{n\to\infty}|E(\bar{\xi}_n^*)-a|=0 \qquad (Un-a.s.)$$

is valid. Finally, we conclude that

$$Un\{\lim_{n\to\infty}|\bar{\xi}_n-a|=0\}=1.$$

Now we give the proof of the second part of the theorem. Since

$$|\bar{\xi}_n^*-a|=|\bar{\xi}_n^*-\bar{\xi}_n+\bar{\xi}_n-E(\bar{\xi}_n)+E(\bar{\xi}_n)-a|$$
$$\leq|\bar{\xi}_n^*-\bar{\xi}_n|+|\bar{\xi}_n-E(\bar{\xi}_n)|+|E(\bar{\xi}_n)-a|,$$

from the proof above, one can see that

$$\lim_{n\to\infty}|\bar{\xi}_n^*-\bar{\xi}_n|=0\ \ (Un-a.s.).$$

Since $\{\xi_n\}$ obeys the strong law of large numbers,

$$\lim_{n\to\infty}|\bar{\xi}_n-E(\bar{\xi}_n)|=0\ \ (Un-a.s.)$$

holds true. Noting that $E(\bar{\xi}_n)-a=0$, we have

$$\lim_{n\to\infty}|E(\bar{\xi}_n)-a|=0\ \ (Un-a.s.).$$

Finally,

$$Un\{\lim_{n\to\infty}|\bar{\xi}_n^*-a|=0\}=1,$$

that is, $\{\xi_n^*\}$ obeys the strong law of large numbers.

The proof of the theorem is complete.

IV. Conclusions

Convergence concepts and the truncation method are very important in classical measure theory. This paper discussed the convergence concepts of uncertain variables and the truncation method for uncertain measure. First, the properties of uncertain measure were further discussed. Second, some relevant conclusions of convergence concepts for uncertain measure were introduced. Finally, the truncation method was given and proven on uncertainty space. These results are extensions of the corresponding conclusions of classical theory. All investigations helped to build important theoretical foundations for the systematic and comprehensive development of uncertain measure theory.

References

[1] Chung Kailai, *A course in probability theory*, 3rd ed, China Machine Press, Beijing, 2010.

[2] Yan Shijian and Liu Xiufang, *Measure and probability*, 2nd ed, Beijing Normal Universiy Press, Beijing, 2011.

[3] Liang Zhishun , Deng Jixian et al, *Probability theory and mathematical statistics*, Higher Education Press, Beijing, 1988.

[4] Liu Baoding, *Uncertainty theory*, 4th ed, Uncertianty Theory Laboratory, Beijing, 2012.

[5] Tian Jingfeng, Zhang Zhiming, Tian Dazeng, ”Moment eatimation inequalities based on g_λ-random variable on Sugeno measure space”, *Journal of Inequalities and Applications*, **11** (2010), 1-10.

[6] Ha Minghu, Yang Lanzhen and Wu Congxi, *Introduction to General Fuzzy Set Value Measure* , Science Press, Beijing, 2009.

[7] Liu Baoding, *Uncertainty theory*, 3rd ed, Uncertainty Theory Laboratory, 2009.

[8] Wang Zhenyuan and George J. Klir, *Generalized measure theory*, Spingger-Verlag, New York, 2008.

[9] Zhang Zhiming, *Some discussions on uncertain measure*, Fuzzy Optimization and Decision Making, **10** (2011), 31-43.

[10] Gianluca and Cassese, *Convergence in measure under finite additivity*, The Indian Journal of Statistics, **75-A** (2013), 171-193.

[11] Wang Shuming, Liu Yankui, Modeling renewal processes in fuzzy decision system, Applied Mathematical Modelling, (2014) 1-18.

[12] Liu Baoding, *Uncertainty theory*, 2nd ed. Springer-Verlag, Berlin, 2007.

[13] Ha Minghu, Hong Zhang, Witold Pedrycz, Hongjie Xing, “The expected value models on Sugeno measure space,”*International Journal of Approximate Reasoning*, **50** (2009), 1022-1035.

[14] Zhang Chunqin and Yang Fang, “Strong law of large numbers on quasi-probability space,”*Journal of Mathematics*, **32**(6) (2012), 999-1004.

[15] Zhang Chunqin and Li Chunguo, “Truncation method for Sugeno measure,”*Fuzzy Systems and Mathematics*, **28**(4) (2014), 115-119.
[16] Yuan, D. ”The truncaton of random variables and a strong deviation theorem”, *Journal of Mathematics* **23**(3) (2003), 349–353.
[17] Xin, G. ” Some properties of continuous uncertain measure”, *International Journal of Uncertainty Fuzziness and Knowledge-Based Systems* **17**(3) (2009), 419–426.
[18] Liu Baoding, ” Fuzzy process, hybrid process and uncetain process”, *Journal of Uncertain Systems* **2**(1) (2008), 3–16.
[19] Liu Baoding, ”Hybrid logic and uncertain logic”, *Journal of Uncertain Systems* **3**(2) (2009), 83–94.
[20] Liu Baoding, ”Some research problems in uncertainty theory”, *Journal of Uncertain Systems* **3**(1) (2009), 3–10.
[21] Liu Baoding, *Uncertainty Theory*, Studies in fuzziness and soft computing, Springer London, Limited, 2007.