Implicit Finite Difference Simulation of Water Pollution Control in a Connected Reservoir System

Witsarut Kraychang and Nopparat Pochai

***Abstract*—In this paper, two mathematical models for simulating water pollutant level and pollution control in a connected reservoir system are proposed. The reservoir system is consisted of two ponds connected by a narrow channel. One pond allows water in from a canal through an entrance gate while the other pond lets the water flow out through an exit gate. The pond water is contaminated with wastewater released from several industrial plants located near the pond. One of the proposed models is a steady-state dispersion model simulating the pollutant level in the connected ponds. The other model is a pollution control model that determines the maximum pollutant level allowed in the wastewater released from each plant in order to achieve a specified pollution level in the ponds as well as to incur a minimum water pre-treatment cost to each plant. The simulation results of these models show that the maximum pollutant level in the two ponds could be effectively controlled at a minimum cost to each plant by optimally limiting the pollutant level in the wastewater it releases.**

***Index Terms*— Water pollution control, Optimization, hydrodynamic model, dispersion model.**

I. INTRODUCTION

NOWADAYS, it is believed that such pollutions as air pollution and water pollution are rather serious and that permanent solutions must be taken into account. Being able to measure the level of pollution, field sampling as well as working in a laboratory is mostly carried out; however, it is impossible to access some important parts within the sight, failing to analyse the very areas. Mathematics, therefore, plays a very crucial role as applied in solving the environmental problem. That is, the problem is replicated into a governing equation, then with a numerical method it is solved, resulting in a plausible finding as it would later explain the issue. In [8], mathematical model is used to air flow and pollutant dispersion in an Urban Street Canyon with fluctuating wind boundary condition (FWBC) and 3D numerical simulations are performed using Large Eddy Simulation (LES). In [10], pollution assessment in Campania region of Southern Italy, analysis air quality by using the Gaussian model ISC. In [9], Large Eddy Simulation (LES) model is utilized to analyze 3D two-phase flow of gas and liquid. Determination of steady-state pollutant levels in a water reservoir causing by wastewater discharge from industrial plants and other external sources can be done accurately by field sampling of the water. However, it is difficult to get samples from every spot in the reservoir and very costly to analyze all of the samples collected. Mathematical simulation is a valuable tool that can be used to simulate the pollutant levels of the water at every spot of the whole reservoir from a relatively small set of collected samples, greatly reducing the total analytical cost. In [1], the authors proposed a mathematical simulation to deal with a lake water quality problem in China. They reported a good match between their calculated and measured pollutant levels. In [2], the authors presented a mathematical model for analyzing the hydrodynamics of and pollutant dispersion in river-type systems. They were able to report changes in the pollutant concentration in the river with time. In a mathematical modeling study of water-quality in Rama-nine reservoir, Pathumthani District, Thailand [3], two mathematical models were used to simulate its pollutant level. The first model was a hydrodynamic model that used the Lax-Wendroff method to provide the velocity vector of water flow and its elevation. The second model was a dispersion model that used a forward-in-time and central-space finite difference scheme to calculate the pollutant concentration. The resultant water velocity vector field, elevation, and pollutant concentration were reported in contour graphs. In [4], two-dimensional hydraulic and pollution models were used to simulate the transport of the pollutant.

After the pollutant level at every location has been mathematically determined, it can be input into an optimization model to find the minimum cost that an industrial plant has to expend to initially treat its wastewater to an acceptably low pollutant level before discharging it into a reservoir. Simplex optimization method is a good mathematical model for determining minimum cost. In [5], a mathematical model was proposed for optimally controlling pollutant level in wastewater discharge that would reduce initial water treatment cost to a minimum. In [6], the authors proposed mathematical models and optimal control techniques for solving some problems in environmental engineering.

In this study, two mathematical models are proposed: a hydrodynamic model and a steady-state pollutant dispersion

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model. They were used to calculate the pollutant level in a connected-pond reservoir system that had an entrance and an exit gate to open water of a canal and to determine the optimal pollutant levels in the wastewaters discharged from nearby industrial plants that would cost the plants minimally to pre-treat.

This paper is divided into 8 sections: the first section is the introduction; the second section describes the unsteady-state hydrodynamic model; the third section describes the steady-state dispersion model; the fourth section describes the finite difference numerical method for solving the steady-state dispersion model; the fifth section shows an example of the detailed numerical calculation procedure; the sixth section describes the results of steady state dispersion calculation; the seventh section describes the pollution control and cost optimization calculation and their results; and the last section is the discussion and conclusion.

II. Unsteady State Hydrodynamic Model

In this section, two mathematical models are described. They were used to simulate time-varying pollutant levels causing by wastewater discharges from several plants into a connected reservoir system. The first model is a hydrodynamic model that determined the velocity and elevation of the water at any locations in the two connected ponds of the reservoir system, while the second model is a pollutant dispersion model that determined the pollutant level at any points in the ponds.

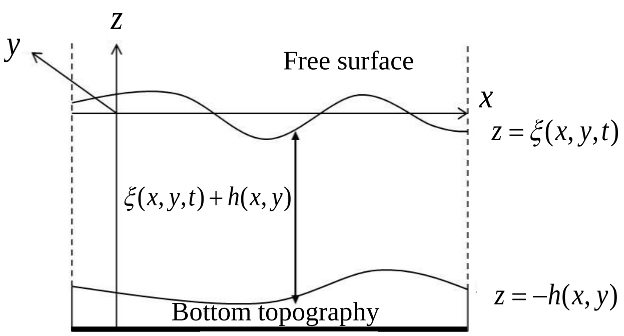


Fig. 1. Vertical cross-section of the water in the ponds.

The two-dimensional unsteady flow of water into and out of the ponds of the connected reservoir system can be determined by a system of shallow water equations that takes into account mass and momentum conservation. This system of equations can be derived by depth-averaging the Navier-Stokes equations in the vertical direction, neglecting the diffusion of momentum due to turbulence and discarding the terms expressing the effects of friction and wind. The continuity equation [7] is then expressed as follows:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (h+\zeta)u}{\partial x} + \frac{\partial (h+\zeta)v}{\partial y} = 0, \tag{1}$$

and the momentum equations [5] are expressed as below:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial \zeta}{\partial y} = 0, \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial \zeta}{\partial x} = 0, \tag{3}$$

Where h is the depth measured from the mean surface water level to the bed of the pond (m),
ζ is the elevation of water surface level from the mean water level (m),
g is acceleration due to gravity (m/s^2),
u, v are velocity components (m/s), and
f is Coriolis factor.

For our problem, the following terms in the equations above were discarded–Coriolis factor, surface wind, and shearing stresses at the pond's bed–and the elevation of the water surface level was assumed to be much smaller than the depth of the pond: $\zeta << h$ or $h := h + \zeta$, where h is a constant representing flat pond bottom. Consequently, the governing equations for our study were as follows:

$$\frac{\partial \zeta}{\partial t} + h\frac{\partial u}{\partial x} + h\frac{\partial v}{\partial y} = 0, \tag{4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0, \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0. \tag{6}$$

By linearly transforming Eqs. (4)-(6) via the following transformations: $U = u/\sqrt{gh}, V = v/\sqrt{gh}, X = x/l, Y = y/l$, $Z = \zeta/h$, and $T = t\sqrt{gh}/l$, as described in [3], our governing equations became non-dimensional as expressed below:

$$\frac{\partial Z}{\partial t} + \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{7}$$

$$\frac{\partial U}{\partial T} + \frac{\partial Z}{\partial X} = 0, \tag{8}$$

$$\frac{\partial V}{\partial T} + \frac{\partial Z}{\partial Y} = 0, \tag{9}$$

The initial conditions for our problem were as follows: the x and y-velocity components were zero as well as the water elevation: $u = 0, v = 0,$ and $z = 0$, while the boundary conditions were as follows: (i) $u = 0, \frac{\partial v}{\partial y} = 0, z = 0$ for the horizontal edges of the rectangular pond; (ii) $\frac{\partial u}{\partial x} = 0, v = 0, z = 0$ for the vertical edges; and (iii) $z = f(x, y)$ for the water flows into the entrance gate, similar to those reported in [3].

III. STEADY STATE DISPERSION MODEL

A. *Governing equation*

The steady-state advection-diffusion equation of the pollutant level in the ponds is below,

$$\bar{u}\frac{\partial C}{\partial x}+\bar{v}\frac{\partial C}{\partial y}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right), \tag{10}$$

for all $(x,y)\in\Omega\subseteq\mathbb{R}^2$ where $\bar{u},\bar{v}$ are the average velocities (m/s) in the x and y directions, respectively, determined by the hydrodynamic model in the first section and previously reported in [3].

B. *Boundary conditions*

Three boundary conditions were used for determining the steady-state dispersion:

a) the rate of change of pollutant level at the edge of the reservoir system was zero, i.e.,

$$\frac{\partial C}{\partial n}=0.; \tag{11}$$

b) the rate of change of pollutant concentration coming in at the entrance gate with respect to x-coordinate was constant,

$$\frac{\partial C}{\partial x}=c_1, \tag{12}$$

and the rate of change of pollutant concentration going out at the exit gate with respect to y-coordinate was constant,

$$\frac{\partial C}{\partial y}=-c_2.; \tag{13}$$

c) the pollutant level in the wastewater discharge from each industrial plant at the edge of the ponds was as follows,

$$C=d_i \tag{14}$$

for all $i=1,2,3,...,P$ where P is number of industrial plants and d_i is a constant.

IV. FINITE DIFFERENCE METHOD OF THE STEADY STATE DISPERSION MODEL

To discretize the steady-state dispersion Eq. (10), the $\Omega\in\mathbb{R}^2$ of x and y-coordinates was divided into M and N steps such that $M\Delta x=L_x$ where L_x is the maximum length of Ω in the x- coordinate and $N\Delta y=L_y$ where L_y is the maximum length of Ω in y-coordinate. We could then approximate a $C(x_l,y_m)$ value by the $C_{l,m}$ value of the difference approximation of $C(x,y)$ at the points $l\Delta x$ and $m\Delta y$, where $0\le l\le M$ and $0\le m\le N$. The grid-points (x_l,y_m) were the points at which $x_l=l\Delta x$ for all $l=0,1,2,...,M$ and $y_m=m\Delta y$ for all $m=0,1,2,...,N$, where M and N are positive integer. Three schemes of Taylor's series expansion for approximating terms in Eq. (10) were used to check whether the backward and forward difference schemes for the first-order derivatives yielded different results than the widely-used central difference scheme or not. These schemes are described below.

A. *Backward-in-space finite difference scheme*

Applying the backward-in-space technique to the first-order derivatives and the central-in-space technique to the second-order derivatives of Eq. (10), we get the following discretized equations:

$$\frac{\partial C}{\partial x}\approx\frac{C_{l,m}-C_{l-1,m}}{\Delta x}, \tag{15}$$

$$\frac{\partial C}{\partial y}\approx\frac{C_{l,m}-C_{l,m-1}}{\Delta y}, \tag{16}$$

$$\frac{\partial^2 C}{\partial x^2}\approx\frac{C_{l+1,m}-2C_{l,m}+C_{l-1,m}}{\Delta x^2}, \tag{17}$$

$$\frac{\partial^2 C}{\partial y^2}\approx\frac{C_{l,m+1}-2C_{l,m}+C_{l,m-1}}{\Delta y^2}. \tag{18}$$

Substituting Eqs. (15)-(18) into Eq. (10), we have

$$\begin{aligned}&\bar{u}\left(\frac{C_{l,m}-C_{l-1,m}}{\Delta x}\right)+\bar{v}\left(\frac{C_{l,m}-C_{l,m-1}}{\Delta y}\right)=\\&D\left(\frac{C_{l+1,m}-2C_{l,m}+C_{l,m-1}}{\Delta x^2}+\frac{C_{l,m+1}-2C_{l,m}+C_{l,m-1}}{\Delta y^2}\right).\end{aligned} \tag{19}$$

A simple form of Eq. (19) is expressed below,

$$C_{l,m}-S_1^BC_{l-1,m}-S_2^BC_{l,m-1}-S_3^BC_{l+1,m}-S_4^BC_{l,m+1}=0 \tag{20}$$

where

$$S_1^B=\left[\frac{\bar{u}}{\Delta x}+\frac{D}{\Delta x^2}\right]\Big/\left[\frac{\bar{u}}{\Delta x}+\frac{\bar{v}}{\Delta y}+\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right], \tag{21}$$

$$S_2^B=\left[\frac{\bar{v}}{\Delta y}+\frac{D}{\Delta y^2}\right]\Big/\left[\frac{\bar{u}}{\Delta x}+\frac{\bar{v}}{\Delta y}+\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right], \tag{22}$$

$$S_3^B=\left[\frac{D}{\Delta x^2}\right]\Big/\left[\frac{\bar{u}}{\Delta x}+\frac{\bar{v}}{\Delta y}+\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right], \tag{23}$$

$$S_4^B=\left[\frac{D}{\Delta y^2}\right]\Big/\left[\frac{\bar{u}}{\Delta x}+\frac{\bar{v}}{\Delta y}+\frac{2D}{\Delta x^2}+\frac{2D}{\Delta y^2}\right]. \tag{24}$$

If $C_{l,m}$ lies at the boundary of the pond, it is calculated by applying the backward difference scheme to Eqs. (11)-(13) to approximate the boundary conditions,

$$\frac{\partial C}{\partial x}\approx\frac{C_{i,j}-C_{i-1,j}}{\Delta x}, \tag{25}$$

$$\frac{\partial C}{\partial y}\approx\frac{C_{i,j}-C_{i,j-1}}{\Delta y}. \tag{26}$$

B. *Forward-in-space finite difference scheme*

Applying the forward-in-space technique to the first-order derivatives and the central-in-space technique to the second-order derivatives of Eq. (10), we get the following discretized equations:

$$\frac{\partial C}{\partial x} \approx \frac{C_{l+1,m} - C_{l,m}}{\Delta x}, \tag{27}$$

$$\frac{\partial C}{\partial y} \approx \frac{C_{l,m+1} - C_{l,m}}{\Delta y}, \tag{28}$$

$$\frac{\partial^2 C}{\partial x^2} \approx \frac{C_{l+1,m} - 2C_{l,m} + C_{l-1,m}}{\Delta x^2}, \tag{29}$$

$$\frac{\partial^2 C}{\partial y^2} \approx \frac{C_{l,m+1} - 2C_{l,m} + C_{l,m-1}}{\Delta y^2}. \tag{30}$$

Substituting Eqs. (27)-(30) into Eq. (10), we have

$$\begin{aligned} &\bar{u}\left(\frac{C_{l+1,m} - C_{l,m}}{\Delta x}\right) + \bar{v}\left(\frac{C_{l,m+1} - C_{l,m}}{\Delta y}\right) = \\ &D\left(\frac{C_{l+1,m} - 2C_{l,m} + C_{l,m-1}}{\Delta x^2} + \frac{C_{l,m+1} - 2C_{l,m} + C_{l,m-1}}{\Delta y^2}\right). \end{aligned} \tag{31}$$

A simple form of Eq. (31) is expressed below,

$$C_{l,m} - S_1^F C_{l-1,m} - S_2^F C_{l,m-1} - S_3^F C_{l+1,m} - S_4^F C_{l,m+1} = 0 \tag{32}$$

where

$$S_1^F = \left[\frac{D}{\Delta x^2}\right] \Big/ \left[-\frac{\bar{u}}{\Delta x} - \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{33}$$

$$S_2^F = \left[\frac{D}{\Delta y^2}\right] \Big/ \left[-\frac{\bar{u}}{\Delta x} - \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{34}$$

$$S_3^F = \left[-\frac{\bar{u}}{\Delta x} + \frac{D}{\Delta x^2}\right] \Big/ \left[-\frac{\bar{u}}{\Delta x} - \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{35}$$

$$S_4^F = \left[-\frac{\bar{v}}{\Delta y} + \frac{D}{\Delta y^2}\right] \Big/ \left[-\frac{\bar{u}}{\Delta x} - \frac{\bar{v}}{\Delta y} + \frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right]. \tag{36}$$

If $C_{l,m}$ lies at the boundary of the pond, it is calculated by applying the forward difference scheme to Eqs. (11)-(13) to approximate the boundary conditions,

$$\frac{\partial C}{\partial x} = \frac{C_{i+1,j} - C_{i,j}}{\Delta x}, \tag{37}$$

$$\frac{\partial C}{\partial y} = \frac{C_{i,j+1} - C_{i,j}}{\Delta y}. \tag{38}$$

C. *Central-in-space finite difference scheme*

Applying the central-in-space technique to both the first-order derivatives and the second-order derivatives of Eq. (10), we get the following discretized equations:

$$\frac{\partial C}{\partial x} = \frac{C_{l+1,m} - C_{l-1,m}}{2\Delta x}, \tag{39}$$

$$\frac{\partial C}{\partial y} = \frac{C_{l,m+1} - C_{l,m-1}}{2\Delta y}, \tag{40}$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{C_{l+1,m} - 2C_{l,m} + C_{l-1,m}}{\Delta x^2}, \tag{41}$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{C_{l,m+1} - 2C_{l,m} + C_{l,m-1}}{\Delta y^2}. \tag{42}$$

Substituting Eqs. (39)-(42) into Eq. (10), we have

$$\begin{aligned} &\bar{u}\left(\frac{C_{l+1,m} - C_{l-1,m}}{2\Delta x}\right) + \bar{v}\left(\frac{C_{l,m+1} - C_{l,m-1}}{2\Delta y}\right) = \\ &D\left(\frac{C_{l+1,m} - 2C_{l,m} + C_{l,m-1}}{\Delta x^2} + \frac{C_{l,m+1} - 2C_{l,m} + C_{l,m-1}}{\Delta y^2}\right). \end{aligned} \tag{43}$$

A simple form of Eq. (44) is expressed below,

$$C_{l,m} - S_1^C C_{l-1,m} - S_2^C C_{l,m-1} - S_3^C C_{l+1,m} - S_4^C C_{l,m+1} = 0 \tag{44}$$

where

$$S_1^C = \left[\frac{\bar{u}}{2\Delta x} + \frac{D}{\Delta x^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{45}$$

$$S_2^C = \left[\frac{\bar{v}}{2\Delta y} + \frac{D}{\Delta y^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{46}$$

$$S_3^C = \left[-\frac{\bar{u}}{2\Delta x} + \frac{D}{\Delta x^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right], \tag{47}$$

$$S_4^C = \left[-\frac{\bar{v}}{2\Delta y} + \frac{D}{\Delta y^2}\right] \Big/ \left[\frac{2D}{\Delta x^2} + \frac{2D}{\Delta y^2}\right]. \tag{48}$$

If $C_{l,m}$ lies on the boundary of the pond, it is calculated by applying the central difference scheme to Eqs. (11)-(13) to approximate the boundary conditions,

$$\frac{\partial C}{\partial x} = \frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x}, \tag{49}$$

$$\frac{\partial C}{\partial y} = \frac{C_{i,j+1} - C_{i,j-1}}{2\Delta y}. \tag{50}$$

V. Matrix-Form of the Intermediate Results of an Example Dispersion Problem

Consider an example with a simple square domain $\Omega = [0,1] \times [0,1]$ shown in Fig. 2. The boundary conditions of this domain were set to be as follows: a non-absorbing boundary $\frac{\partial C}{\partial n} = 0$; a pollutant level discharged from each plant of $C = C_1$ (kg/m^3); and velocities of water flow in the x and y-direction of $\bar{u} = -0.025$ (m/s) and $\bar{v} = -0.025$ (m/s), respectively. The gridding size is taken to be $\Delta x = \Delta y = 0.25$.

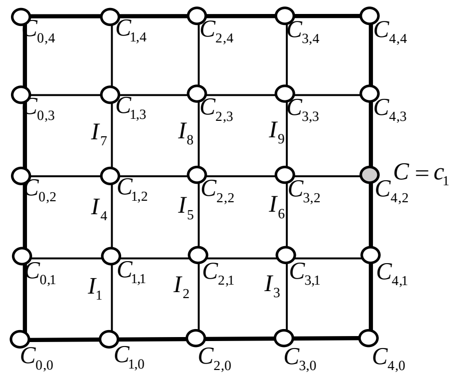


Fig. 2. The simple domain of our example.

Numerically solving the governing Eqs. (20), (32), and (44) with these simple boundary conditions, we obtain intermediate results in the form of matrices. These matrix forms are employed as described in the cost optimization model in the seventh section. Each matrix of our example in the matrix equation of the optimization model, expressed as Eq. (51),

$$[A]\{C\} = \{B\}, \tag{51}$$

is in the form below:

$$[A] = \left[\begin{array}{ccccccccccccccccccccccccc}
0 & -s_2 & 0 & 0 & 0 & -s_1 & 1 & -s_3 & 0 & 0 & 0 & -s_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & -s_2 & 0 & 0 & 0 & -s_1 & 1 & -s_3 & 0 & 0 & 0 & -s_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & -s_2 & 0 & 0 & 0 & -s_1 & 1 & -s_3 & 0 & 0 & 0 & -s_4 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & -s_2 & 0 & 0 & 0 & -s_1 & 1 & -s_3 & 0 & 0 & 0 & -s_4 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_2 & 0 & 0 & 0 & -s_1 & 1 & -s_3 & 0 & 0 & 0 & -s_4 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_2 & 0 & 0 & 0 & -s_1 & 1 & -s_3 & 0 & 0 & 0 & -s_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_2 & 0 & 0 & 0 & -s_1 & 1 & -s_3 & 0 & 0 & 0 & -s_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_2 & 0 & 0 & 0 & -s_1 & 1 & -s_3 & 0 & 0 & 0 & -s_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_2 & 0 & 0 & 0 & -s_1 & 1 & -s_3 & 0 & 0 & 0 & -s_4 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & & & & & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & -1 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0
\end{array}\right]$$

$$\{C\} = \{c_{0,0} \quad c_{1,0} \quad c_{2,0} \quad c_{3,0} \quad c_{4,0} \quad c_{0,1} \quad c_{1,1} \quad c_{2,1} \quad c_{3,1} \quad c_{4,1} \quad c_{0,2} \quad c_{1,2} \quad c_{2,2} \quad c_{3,2} \quad c_{4,2} \quad c_{0,3} \quad c_{1,3} \quad c_{2,3} \quad c_{3,3} \quad c_{4,3} \quad c_{0,4} \quad c_{1,4} \quad c_{2,4} \quad c_{3,4} \quad c_{4,4}\}^T$$

$$\{B\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad c_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^T$$

VI. RESULTS OF STEADY STATE DISPERSION CALCULATION

In this section, various results are reported in a table, several surface and contour plots, and a comparison graph. Simulation runs of the dispersion model were performed with these following settings: five plants– F_1, F_2, F_3, F_4, and F_5 –at locations shown in Fig. 4(b). discharging wastewater at various pollutant levels shown in Table 1; rate of change of incoming pollutant level with respect to x-coordinate at the entrance gate of c_1 (kg/m^4); rate of change of outgoing pollutant level with respect to y-coordinate at the exit gate of c_2 (kg/m^4); the same average water velocities in x and y-directions of -0.025 (m/s), which was also employed in [3]; and diffusion coefficient of $D = 10$ (m^2/s). A further explanation of c_1 and c_2 is due here. Pollutant was assumed to be coming from 2 sources: 1) in the incoming water from the canal through the entrance gate located at the northeast of the upper pond, as shown in Fig. 3. , and 2) in the wastewater from the 5 nearby industrial plants, each plant releasing its wastewater at a wastewater release point. Concurrently, the outgoing water out of the lower pond into the

TABLE I. Pollutant concentration in discharged wastewater from five plants.

Plants	Pollutant concentration in wastewater (kg/m^3)
F_1	5.0000
F_2	5.1500
F_3	5.2000
F_4	5.6000
F_5	5.3400

canal through the exit gate, as shown in Fig. 3. , was also polluted at a certain concentration. c_1 and c_2 were the rate of change of incoming pollutant concentration with respect to x-coordinate ($\frac{\partial C}{\partial x} = c_1$ (kg/m^4)) and the rate of change of outgoing pollutant concentration with respect to y-coordinate ($\frac{\partial C}{\partial y} = -c_2$ (kg/m^4)), respectively. Another point of note is that, in our simulation, around 90 points were selected to be ‘observation points’, covering the 2 ponds evenly. The pollutant concentrations at these points were determined at completion of the simulation. Among the observation points, around 20 points were selected to be ‘scheme-comparing points’, shown in Fig. 4b. , the main purpose of these points being for comparing the pollutant concentrations calculated by the 3 different schemes described in the fourth section.

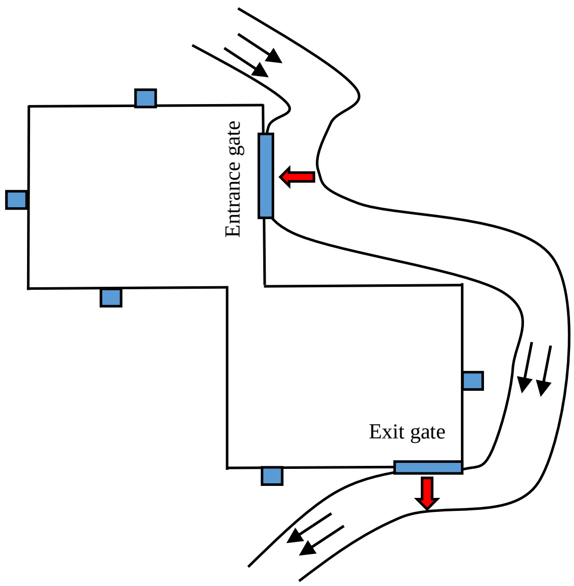


Fig. 3. The connected-pond reservoir with openings to a canal.

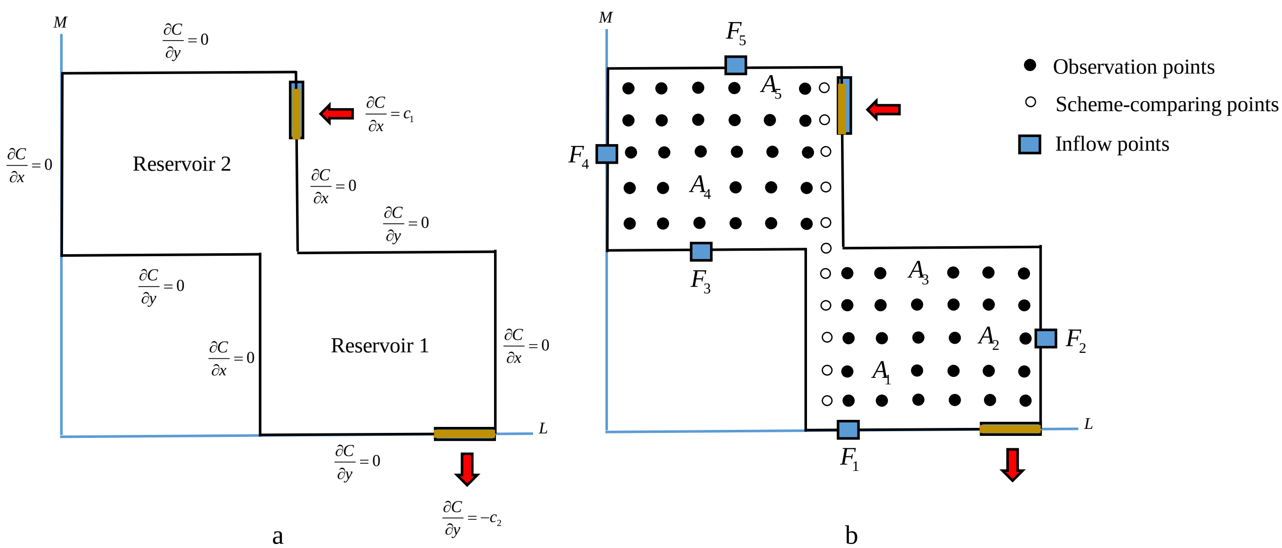
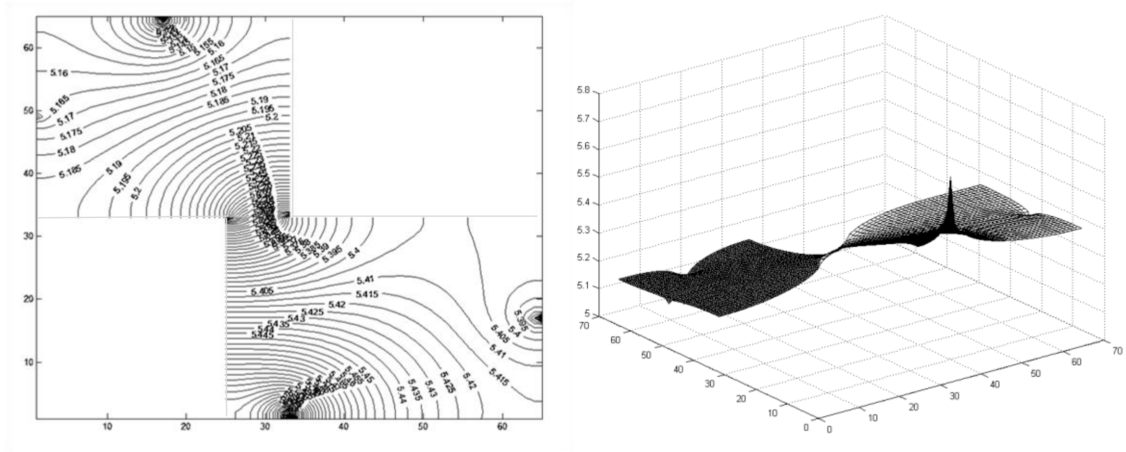
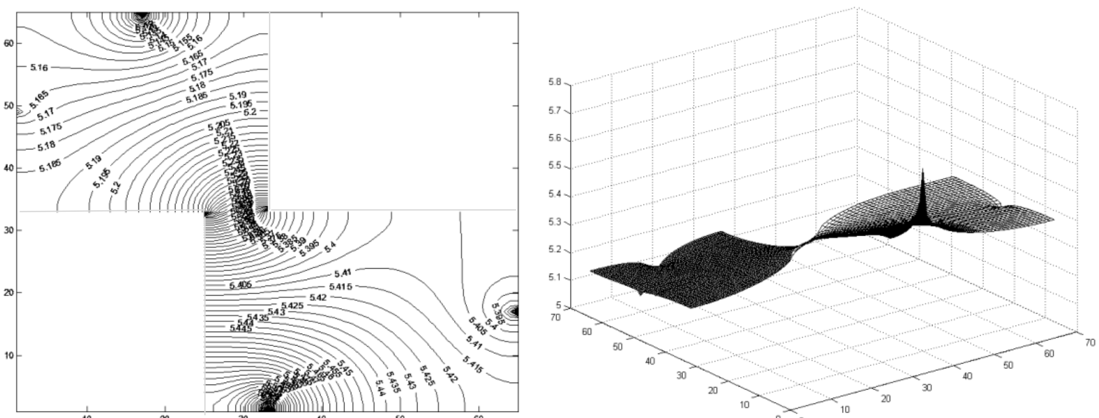


Fig. 4. The domain of the steady-state dispersion model (a); the observation points, the scheme-comparing points, and the wastewater release points (b).



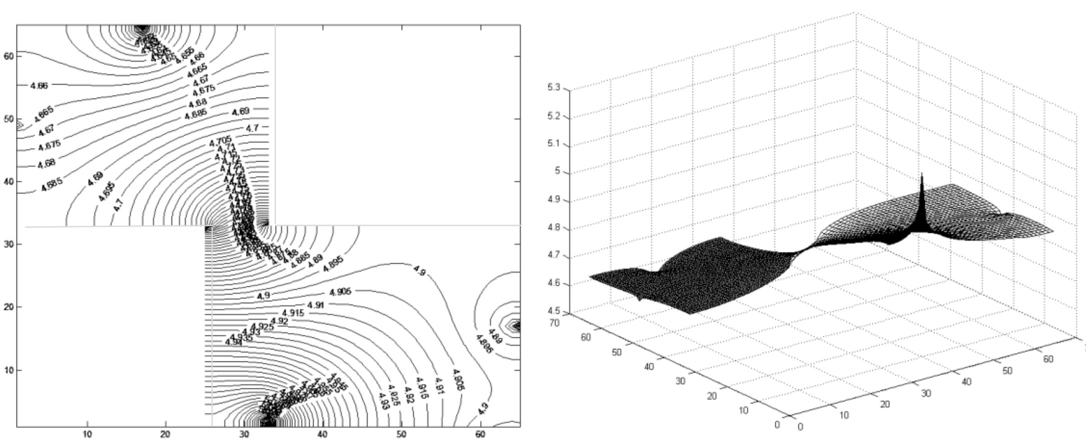
a b

Fig. 5. Pollutant concentration (kg/m^3) in the connected reservoir $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 1 in table 1) (a) Contour plot and (b) Surface plot.



a b

Fig. 6. Pollutant concentration (kg/m^3) in the connected reservoir $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 4 in table 1) (a) Contour plot and (b) Surface plot.



a b

Fig. 7. Pollutant concentration (kg/m^3) in the connected reservoir $\Delta x = 0.015625, \Delta y = 0.015625$ (Case 7 in table 1) (a) Contour plot and (b) Surface plot.

Table II shows the average pollutant concentrations in the upper pond (pond 1) and lower pond (pond 2) for 9 different combinations of c_1, c_2, F_1, F_2, F_3, F_4, and F_5.

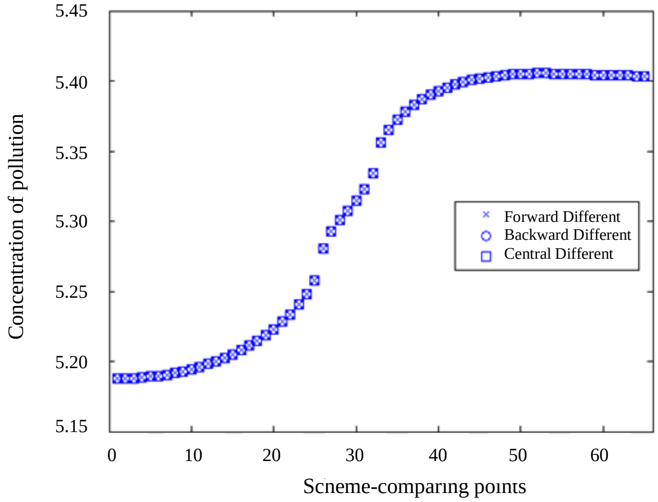


Fig. 8. Pollutant concentration at the scheme-comparing points calculated by the backward, forward, and central difference techniques.

VII. Pollution Control and Cost Optimization

In this section, the method of calculation of optimal cost is described. A plant had to incur a cost to pretreat its wastewater before releasing it into the pond, in order to keep the pollutant level in its wastewater discharge to stay below a certain level. The more extensive the pre-treatment, the higher the treatment cost. The criterion for acceptable level of pollutant in the released wastewater was that the pollutant levels in the pond water at all of the observation points had to be lower than a specified standard level. The higher the pollutant level in the pond water than the standard level was, the more extensive pre-treatment was, and consequently, the higher the cost to the plant was. Let x_β be the observation points and r_α be the reduction of pollutant concentration in the wastewater at its release points. It follows that $C_\alpha - r_\alpha$ is the pollutant concentration after pre-treatment (or partial purification) at the release points. From Eq. (51), we have

$$[A]\{C\} = \{B\}, \tag{52}$$

which can be expressed as,

$$\{C\} = [A]^{-1}\{B\}. \tag{53}$$

Let

$$\{C\} = [D]\{B\} \tag{54}$$

where

$$[A]^{-1} = [D]. \tag{55}$$

Let $\tilde{C}_\beta$ be the pollutant concentration at an observation point x_β. Let r_α be the reduction of pollutant concentration at the wastewater release points; then, Eq. (54) becomes,

$$\tilde{C}_\beta = d_{\beta 1}b_1 + \ldots + d_{\beta\alpha}(b_\alpha - r_\alpha) + \ldots + d_{\beta N}b_N \tag{56}$$

or

$$\tilde{C}_\beta = \sum_{i=1}^{m} d_{\beta i}b_i + \sum_{j=1}^{n} d_{\beta\alpha_i}(b_{\alpha_j} - r_{\alpha_j}), \tag{57}$$

where m is the number of observation points and n is the number of wastewater release points $(N = m + n)$. Let C_{ST} be the standard allowable pollutant level in the pond water. $\tilde{C}_\beta$ must be at or below this level, i.e.,

$$\tilde{C}_\beta \le C_{ST}. \tag{58}$$

The objective function J is the cost of wastewater pre-treatment, so

$$J(x) = \sum_{j=1}^{m} \omega_j r_{\alpha_j}, \tag{59}$$

where ω_j is the cost of wastewater pre-treatment for the required reduction of pollutant concentration. The constraints are

$$\tilde{C}_\beta = \sum_{i=1}^{m} d_{\beta i}b_i + \sum_{j=1}^{n} d_{\beta\alpha_i}(b_{\alpha_j} - r_{\alpha_j}) \le C_{ST}. \tag{60}$$

The upper bound and lower bounds of the controls are,

$$l_{\alpha_j} \le r_{\alpha_j} \le u_{\alpha_j} \tag{61}$$

and the controls are non-negative

$$r_{\alpha_j} \ge 0, \tag{62}$$

where $l_{\alpha_j}, u_{\alpha_j}$ are the lower and upper bounds, respectively, of the reduction of pollutant level in the wastewater, specifying the minimum and maximum reduction of pollutant level in the wastewater that a plant can reduce by pre-treatment. This optimal control problem was solved by Simplex method. There were 5 water treatment plants that discharged wastewater into the connected ponds. Plant 1 to Plant 5 had the ability to purify its wastewater such that the maximum reductions in their pollutant concentration were 1.0, 1.0, 1.5, 1.5 and 1.0 (kg/m^3), respectively, while the minimum reduction specified by the law was 0.5 (kg/m^3). The physical parameters settings were the following: diffusion coefficient of $D = 10.0$ (m^2/s) and the same velocities in the X and Y directions of -0.025 (m/s). The standard allowable pollutant level in the pond water was 4.2 (kg/m^3). Therefore, all constraints were as follows:

$$C_{ST} = 4.2, \tag{63}$$
$$0.5 \le r_{\alpha_1} \le 1.0, \tag{64}$$
$$0.5 \le r_{\alpha_2} \le 1.0, \tag{65}$$
$$0.5 \le r_{\alpha_3} \le 1.5, \tag{66}$$
$$0.5 \le r_{\alpha_4} \le 1.5, \tag{67}$$
$$0.5 \le r_{\alpha_5} \le 1.0. \tag{68}$$

Column 3 in Table III shows pollutant concentrations at observation and wastewater release points before the wastewaters were pre-treated by the plants. They were higher than the standard, C_{ST}. After pre-treatment, the concentrations at these points were lower than C_{ST}, as shown in column 5.

In Table IV, column 3 and column 6 show the extents of pollutant reduction in terms of concentration. The Non-optimal reduction was for achieving 4.2 kg/m^3 pollutant level, C_{ST}, in the wastewater at the release points of all 5 plants, while the optimal reduction was for achieving a C_{ST} pollutant level or lower in the pond water. These reductions multiplied by the corresponding cost of treatment to each plant (column 4) gave the Non-optimal cost of reduction (column 7) and the optimal cost of reduction (column 5). It can be seen that the total cost of optimal wastewater treatment was 150,915.50 USD, significantly lower than 162,400.0 0 USD of the Non-optimal treatment. Fig. 9 shows the comparison of the cost among five plants in case of optimal cost and non-optimal cost.

TABLE II. Average pollutant concentration pond 1 and pond 2 for 9 combinations of parameter settings.

Case	c_1 (kg/m^4)	c_2 (kg/m^4)	Concentration in discharge (kg/m^3) from plant F_1	F_2	F_3	F_4	F_5	Average concentration in pond (kg/m^3) Pond 1	Pond 2
1	0.0100	0.0100	5.0000	5.1500	5.2000	5.6000	5.3400	5.4222	5.1850
2	0.0100	0.0500	5.0000	5.1500	5.2000	5.6000	5.3400	5.4191	5.1844
3	0.0100	0.1000	5.0000	5.1500	5.2000	5.6000	5.3400	5.4152	5.1837
4	0.0100	0.0050	5.0000	5.1500	5.2000	5.6000	5.3400	5.4226	5.1851
5	0.0100	0.0100	4.5000	4.6500	4.7000	5.1000	4.8400	4.9222	4.6850
6	0.1000	0.0500	4.5000	4.6500	4.7000	5.1000	4.8400	4.9191	4.6844
7	0.0100	0.1000	4.5000	4.6500	4.7000	5.1000	4.8400	4.9152	4.6837
8	0.0100	0.0050	4.5000	4.6500	4.7000	5.1000	4.8400	4.9226	4.6851
9	0.0100	1.0000	4.5000	4.6500	4.7000	5.1000	4.8400	4.8454	4.6706

TABLE III. Pollutant concentration at 5 observation points and 5 wastewater release points.

Points	Pollutant concentration Untreated Inflow	Observations	Pre-treated Inflow	Observations
A_1		5.4236		4.1160
A_2		5.3689		4.1528
A_3		5.3944		4.1276
A_4		5.1723		4.0759
A_5		5.1465		4.1825
F_1	5.0000	5.0000	4.5000	4.5000
F_2	5.1500	5.1500	4.2179	4.2179
F_3	5.2000	5.2000	3.7150	3.7150
F_4	5.6000	5.6000	4.1000	4.1000
F_5	5.3400	5.3400	4.3400	4.3400

TABLE IV. Optimal cost of wastewater treatment.

Treatment Factory	Location	Optimal Reduction of Pollutant (kg/m^3) Concentration	Cost of Treatment for Reduction by 1 (kg/m^3)	Optimal Cost of Reduction (USD)	Non-optimal Reduction of Pollutant (kg/m^3) Concentration	Non-optimal Cost of Reduction (USD)
Factory1	F_1	0.5000	40,000	20,000.00	0.8000	32,000.00
Factory2	F_2	0.9321	20,000	18,642.00	0.9500	19,000.00
Factory3	F_3	1.4849	15,000	22,273.50	1.0000	15,000.00
Factory4	F_4	1.5000	20,000	30,000.00	1.4000	28,000.00
Factory5	F_5	1.0000	60,000	60,000.00	1.1400	68,400.00
Total Cost				150,915.50		162,400.00

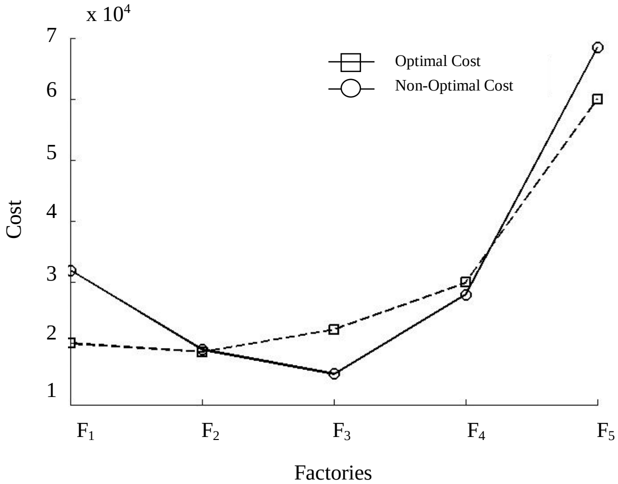


Fig. 9. Compare costs of water treatment of five plants between the optimal cost and non-optimal cost.

VIII. Discussion

In this study, mathematical models were proposed for determining pollution levels in the water of a connected-pond reservoir, with openings to a canal, polluted by wastewater discharged from nearby industrial plants. In the equations of the models, the parameters affecting the dispersion of pollution were water velocity and diffusion coefficient, but they did not have a significant effect in this study. More influential was the initial pollutant level in the wastewater discharged from industrial plants, as shown in Table 2. In Case 3, its c_2 was 10 times higher than that of Case 1, thus, making the average pollutant level in the pond water slightly but significantly lower. In Case 1 and Case 7, their c_2 were the same, but the initial pollutant level in Case 7 was reduced by 0.5 kg/m^3. It can be seen that the average pollutant level in the pond water in this case was 5 times lower than the reduced level in Case 3. Fig. 5 -7 show contour and surface plots of pollutant level versus two spatial coordinates of Case (1), (4), and (7). The contour plots show different patterns of pollutant dispersion of the three cases, while the surface plots better show the different pollutant levels at various locations. Fig. 8. shows plots of pollutant levels at scheme-comparing points calculated by 3 different finite difference schemes—backward, forward, and central. The curves of the 3 plots matched perfectly, indicating that all 3 schemes were equally valid.

Concerning the costs to the industrial plants, they were optimized with respect to the pollutant level in their wastewater discharge under the condition that the pollutant level in the pond water not exceeds the acceptable standard. If every plant can control the pollutant level in its wastewater discharge to match its corresponding optimal level shown in Table 4, they can save 11,484.50 USD of their wastewater pre-treatment cost in a year.

IX. Conclusion

In this paper, we proposed a numerical simulation of water-quality control using a couple of an optimization method and an implicit finite difference technique in the two ponds with an entrance and an exit gate to open water of a canal. The simulation results of these models showed that the maximum pollutant level in the two ponds could be effectively controlled at a minimum cost to each plant by optimally limiting the pollutant level in the wastewater it released. The mathematical model in this paper could also be used in irregular boundary with open-connected reservoir. In order to have an effective calculation of pollutant concentration, it is advised that the bottom topography function in hydrodynamic model must be improved realistic by changing flat bottom topography function to anisotropic bottom topography function, retrieving from the interpolation of field data, resulting in smooth anisotropic bottom topography function, helping to calculate the velocities, precisely, used as the input data for calculating the pollutant concentration in steady state of dispersion model. When decreasing the pollutant concentration, releasing less waste water from the factories is ideal or treating the waste water in some part of the reservoir as to make the pollutant concentration in the reservoir even less, leading to less cost of treatment but better water quality.

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