Restricted Connectivity of Cartesian Product Graphs

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Abstract—**For a connected graph $G = (V(G), E(G))$, a vertex set $S \subseteq V(G)$ is a k-restricted vertex-cut if $G - S$ is disconnected such that every component of $G - S$ has at least k vertices. The k-restricted connectivity $\kappa_k(G)$ of the graph G is the cardinality of a minimum k-restricted vertex-cut of G. In this paper, we give the 3-restricted connectivity and the 4-restricted connectivity of the Cartesian product graphs, and we proposed two conjectures for general cases of the k-restricted connectivity of the Cartesian product graphs.**

Keywords: restricted connectivity, Cartesian product, perfect matching.

1 Introduction

We follow [1] for graph-theoretical terminology and notation not defined here. A network can be modelled by an undirected graphs with no loops or multiple edges. The connectivity is a classic measure of network reliability. In [4], Harary proposed conditional connectivity, which is a more refined index than the connectivity. In this paper, we consider finite, undirected and simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and S be a non-empty subset of $V(G)$, then S is a *vertex-cut* if $G - S$ is disconnected, S is a *k-restricted vertex-cut* if $G - S$ is disconnected and every component of $G - S$ has at least k vertices, and S is a *cyclic vertex-cut* if $G - S$ is disconnected and has at least two components containing cycles. The *connectivity* $\kappa(G)$ is defined as the minimum cardinality over all vertex-cuts of G, the *k-restricted connectivity* $\kappa_k(G)$ is defined as the minimum cardinality over all k-restricted vertex-cuts of G, and the *cyclic connectivity* $\kappa_c(G)$ is defined as the minimum cardinality over all cyclic vertex-cuts of G. It should be pointed out that not all connected graphs have the k-restricted vertex-cut. A connected graph G is called *κ_k-connected* if $\kappa_k(G)$ exists. The *girth* $g(G)$ of the graph G is the length of its shortest cycle if G contains cycles. Results on the restricted connectivity are referred to [5, 8–11, 13].

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Let $u \in V(G)$ and G_1, G_2 be two subgraphs of G. Define $N(u) = \{v \in V(G)|v \text{ is adjacent to } u\}$, and $d(u) = |N(u)|$ be the *degree* of u in G, and $N_{G_1}(G_2) = \{v \in V(G_1) \setminus V(G_2) \mid v \text{ is adjacent to a vertex of } G_2\}$. The graph G is *k-regular* if $d(u) = k$ for any $u \in V(G)$. Let A be a subset of $V(G)$. We define $G[A]$ as a subgraph of G induced by A, $N(A) = \{u \in V(G) \setminus A \mid u \text{ is adjacent to a vertex of } A\}$, and $N[A] = A \cup N(A)$.

Throughout this paper, we present the same notations related to the Cartesian product graphs as in [6]. Assume $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 = \{x_1, x_2, \cdots, x_m\}$ and $V_2 = \{y_1, y_2, \cdots, y_n\}$. The *Cartesian product* of graphs G_1 and G_2, denoted by $G_1 \Box G_2$, is the graph with vertex set $V_1 \times V_2 = \{(x, y) \mid x \in V_1 \text{ and } y \in V_2\}$ such that two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$ with $y_1y_2 \in E_2$ or $y_1 = y_2$ with $x_1x_2 \in E_1$.

For convenience, we define two kinds of subgraphs G_{1y} and G_{2x} of $G_1 \Box G_2$ as follows: $V(G_{1y}) = \{(x, y) \mid x \in V_1\}$ and $E(G_{1y}) = \{(x_1, y)(x_2, y) \mid x_1x_2 \in E_1\}$ for any $y \in V_2$, $V(G_{2x}) = \{(x, y) \mid y \in V_2\}$ and $E(G_{2x}) = \{(x, y_1)(x, y_2) \mid y_1y_2 \in E_2\}$ for any $x \in V_1$. Obviously, G_{1y} is isomorphic to G_1 for any $y \in V_2$, and G_{2x} is isomorphic to G_2 for any $x \in V_1$. By definition, $V(G_{1y}) \cap V(G_{1y'}) = \emptyset$ for any $y \neq y'$, $V(G_{2x}) \cap V(G_{2x'}) = \emptyset$ for any $x \neq x'$, $V(G_{1y}) \cap V(G_{2x}) = \{(x, y)\}$ for any $x \in V_1$ and $y \in V_2$, and $V(G_1 \Box G_2) = \cup_{y \in V_2} V_{1y} = \cup_{x \in V_1} V_{2x}$. For some results on the connectedness of Cartesian product graphs, see [2, 3, 6–8].

By the definition of the Cartesian product, the graph $G = G_1 \Box G_2$ can be viewed as formed from m disjoint copies of G_2, denoted by $G_{2x_1}, G_{2x_2}, \cdots, G_{2x_m}$, respectively, by connecting vertex (x, y_i) of G_{2x} with vertex (x', y_i) of $G_{2x'}$ for any $y_i \in V_2$ whenever $xx' \in E_1$. These new edges are called cross edges. That is, there exists a perfect matching between two copies G_{2x} and $G_{2x'}$ for any $xx' \in E_1$. Similarly, G can also be viewed as formed from n disjoint copies of G_1, denoted by $G_{1y_1}, G_{1y_2}, \cdots, G_{1y_n}$, respectively, by connecting vertex (x_i, y) of G_{1y} with vertex (x_i, y') of $G_{1y'}$ for any $x_i \in V_1$ whenever $yy' \in E_2$. Thus there is also a perfect matching between two copies G_{1y} and $G_{1y'}$ for any $yy' \in E_2$.

For $S \subseteq V(G_1 \Box G_2)$, let $G'_{1y} = G_{1y} - S$ for any $y \in V_2$,

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and $G'_{2x} = G_{2x} - S$ for any $x \in V_1$. It is clear that $V(G'_{1y}) = V(G_{1y}) \setminus S$ for any $y \in V_2$, $V(G'_{2x}) = V(G_{2x}) \setminus S$ for any $x \in V_1$, and $V(G_1 \Box G_2 - S) = \cup_{y \in V_2} V(G'_{1y}) = \cup_{x \in V_1} V(G'_{2x})$.

In the graph theory, Menger's theorem and Whitney Criterion are well-known [12]:

Theorem 1. *(Menger's theorem) Let $G = (V, E)$ be a connected graph with $x, y \in V$. Then the minimum number of vertices separating vertex x from vertex y in G is equal to the maximum number of internally disjoint (x, y)-paths in G if $xy \notin E$.*

Theorem 2. *(Whitney Criterion) If G is a graph with order at least $k + 1 (k \geq 1)$, then $\kappa(G) \geq k$ if and only if there are at least k internally disjoint (x, y)-paths in G for any $x, y \in V$.*

2 Main Results

In this section, we always assume that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have m and n vertices, respectively. For the 2-restricted connectivity of the Cartesian product graphs, we can find the following theorems in [6] and [8].

Theorem 3. *([6]) $\kappa_2(K_m \Box K_n) = min\{m+2n-4, 2m+n-4\}$ for $m + n \geq 6$.*

Theorem 4. *([8]) Let G_2 be a $k_2(\geq 2)$-regular and maximally connected graph. If $g(G_2) \geq 4$, then $\kappa_2(K_m \Box G_2) = 2k_2 + m - 2$ for $m \geq 2$.*

Theorem 5. *([8]) Let G_i be a $k_i(\geq 2)$-regular and maximally connected graph with $g(G_i) \geq 4$ for $i \in \{1, 2\}$. Then $\kappa_2(G_1 \Box G_2) = 2k_1 + 2k_2 - 2$.*

In the following, we consider the k-restricted connectivity of the Cartesian product graphs for $k \geq 3$.

By the exercise 2.1.9 in [1], we have $|V(G)| \geq k^2 + 1$ for $g(G) \geq 5$, where k is the regularity of G, thus we have the following lemma.

Lemma 2.1. *Let G be a k-regular graph with girth $g(G) \geq 5$. Then $|V(G)| \geq 3k - 1$ for $k \geq 2$.*

Theorem 6. *(i) $\kappa_3(K_m \Box K_n) = min\{3m + n - 6, m + 3n - 6\}$ for $m + n \geq 8$ with $m \geq n + 2$ or $n \geq m + 2$;*
(ii) $\kappa_3(K_m \Box K_n) = 2m + 2n - 8$ for $m + n \geq 8$ with $m = n$, or $n = m + 1$, or $m = n + 1$.

Proof. Denote $G = K_m \Box K_n$. Assume $G[A]$ is a connected subgraph of G such that $|A| \geq 3$.

(i)If A is contained in one copy G_{2x} or G_{1y} for $x \in V_1$ and $y \in V_2$, then since $|A| \geq 3$, $G_{2x} \cong K_n$, and $G_{1y} \cong K_m$, there exist cycles in $G[A]$. Without loss of generality, assume $A \subseteq G_{2x}$. When $|A| = 3$, $|N(A)|$ is minimum, there are at least three vertices in each component of $G - N(A)$, and there exist cycles in $G \setminus N[A]$. Thus $N(A)$ is a 3-restricted vertex-cut, and also is a cyclic vertex-cut. Hence, $\kappa_3(G) = \kappa_c(G) =min\{3m + n - 6, m + 3n - 6\}$ by the theorem 2.2(i) in [2].

(ii)If A is not contained in one copy G_{2x} or G_{1y} for any $x \in V_1$ and $y \in V_2$. When there is no cycles in $G[A]$, $G[A] = G[A_1] \cup G[A_2]$ is a path of length $k_1 + k_2$ for $G_{2x} \cong K_n$ and $G_{1y} \cong K_m$, where $|A_i| = k_i \geq 2$ for $i \in \{1, 2\}$. Assume $G[A_1] \subseteq G_{2x}$, $G[A_2] \subseteq G_{1y}$ for $x \in V_1$ and $y \in V_2$, then $|N(A)| = (m - 1)k_1 + (n - 1)k_2 - k_1k_2 + 1$. When $k_1 = k_2 = 2$, $|N(A)|$ is minimum, and $|G \setminus N[A]| > 3$. Let $B = (G_{2x_1} \cup G_{2x_2}) \cap (G_{1y_1} \cup G_{1y_2})$ for $x_1, x_2 \in V_1$ and $y_1, y_2 \in V_2$, then $|N(B)| = 2m + 2n - 8$, $|G \setminus N[B]| > 3$, and there are cycles in $G \setminus N[B]$, thus $N(B)$ is a 3-restricted vertex-cut, and $|N(A)| > |N(B)|$, a contradiction. We have $A = B$, and $N(A)$ is a 3-restricted vertex-cut, and also is a cyclic vertex-cut of G. Thus $\kappa_3(G) = \kappa_c(G) = 2m + 2n - 8$ by the theorem 2.2(ii) in [2]. ☐

In the following, we consider the graph $G = G_1 \Box G_2$ with $G_1 \ncong K_m$ or $G_2 \ncong K_n$. For $G = K_m \Box G_2$, when $m = 2$, we easily obtain $\kappa_3(K_2 \Box G_2) = 3k_2 - 1$ for $k_2 \geq 2$, where G_2 is a k_2-regular and maximally connected graph with $g(G_2) \geq 5$.

Lemma 2.2. *Let G_2 be a $k_2(\geq 2)$-regular connected graph. If $g(G_2) \geq 5$, then $\kappa_3(K_m \Box G_2) \leq 3k_2 + m - 3$ for $m \geq 3$.*

Proof. Denote $G = K_m \Box G_2$ and $G_1 = K_m$. The graph G can be viewed as formed from m disjoint copies of G_2, denoted by G_{2x_1}, G_{2x_2}, $\cdots$, G_{2x_m}, respectively, by connecting vertex (x, y_j) of G_{2x} with vertex (x', y_j) of $G_{2x'}$ for any $y_j \in V_2$ and $x \neq x'$.

When $m = 3$. Let $P = u_1u_2u_3$ be a path of G, where $u_1 = (x_1, y_1)$, $u_2 = (x_1, y_2)$, and $u_3 = (x_2, y_2)$. Since G_{2x_i} is a k_2-regular graph for $i \in \{1, 2\}$, $G_{1y_j} \cong K_m$ for $j \in \{1, 2\}$, and $g(G_2) \geq 5$, we have

$$\begin{aligned} |N(\{u_1, u_2, u_3\})| &= |N_{G_{2x_1}}(\{u_1, u_2\})| + |N_{G_{2x_2}}(u_3)| \\ &+ |N_{G_{1y_1}}(u_1)| + |N_{G_{1y_2}}(\{u_2, u_3\})| \\ &- 1 \\ &= 2k_2 - 2 + k_2 + m - 2 + m - 1 - 1 \\ &= 3k_2 + m - 3. \end{aligned}$$

Since $g(G_2) \geq 5$ and $G_1 \cong K_m$, $G - N[\{u_1, u_2, u_3\}]$ is connected, and $|V(G) \setminus N[\{u_1, u_2, u_3\}]| \geq 3$. Hence $N(\{u_1, u_2, u_3\})$ is a 3-restricted vertex-cut.

When $m > 3$. Let $P = u_1u_2u_3$ be a path of G_{1y_1} for $y_1 \in V_2$. Suppose $u_i = (x_i, y_1)$ for $i \in \{1, 2, 3\}$. Since $G_{1y_1} \cong K_m$, G_{2x_i} is a k_2-regular graph for $i \in \{1, 2, 3\}$, and $g(G_2) \geq 5$, we have

$$\begin{aligned} |N(\{u_1,u_2,u_3\})| &= |N_{G_{1y_1}}(\{u_1,u_2,u_3\})| \\ &+ \Sigma_{i=1}^{3}|N_{G_{2x_i}}(\{u_i\})| \\ &= 3k_2+m-3. \end{aligned}$$

Let $G'_{2x} = G_{2x} - \{(x,y)\}$ for some $x \in \{x_1,x_2,x_3\}$ and $y \in V_2 \setminus \{y_1\}$. Since there is a perfect matching between G_{2x} and $G_{2x'}$ for any $x \neq x'$, each vertex $v \in V(G) \setminus N[\{u_1,u_2,u_3\}]$ is adjacent to some vertex in G'_{2x}, that is, $G - N[\{u_1,u_2,u_3\}]$ is connected, and $|V(G) \setminus N[\{u_1,u_2,u_3\}]| \geq 3$. Hence $N(\{u_1,u_2,u_3\})$ is a 3-restricted vertex-cut.

In conclusion,

$$\kappa_3(G) \leq |N(\{u_1,u_2,u_3\})| = 3k_2+m-3. \qquad \square$$

Theorem 7. *Let G_2 be a $k_2(\geq 2)$-regular and maximally connected graph. If $g(G_2) \geq 5$, then $\kappa_3(K_m \Box G_2) = 3k_2+m-3$ for $m \geq 3$.*

Proof. Denote $G = K_m \Box G_2$ and $G_1 = K_m$. By lemma 2.2, we have $\kappa_3(G) \leq 3k_2+m-3$. By contraction, $\kappa_3(G) < 3k_2+m-3$. Assume S is a 3-restricted vertex-cut of G with $|S| = \kappa_3(G)$. Let $S_x = S \cap V_{2x}$ for any $x \in V_1$. Since $G_1 = K_m$, there exists a perfect matching between two copies G_{2x} and $G_{2x'}$ for any $x \neq x'$. If $S_{\overline{x}} = \emptyset$ for some $\overline{x} \in V_1$, then all the vertices of G'_{2x} are connected to $G'_{2\overline{x}}(= G_{2\overline{x}})$ by cross edges for any $x \in V_1 \setminus \{\overline{x}\}$, which implies $G-S$ is connected, contradicting that S is a 3-restricted vertex-cut of G. Therefore, $S_x \neq \emptyset$ for any $x \in V_1$.

Let r be the number of copies of G_2 in G which are disconnected in $G-S$. If $r \geq 3$, then since G_2 is maximally connected and $S_x \neq \emptyset$ for any $x \in V_1$, we have $|S| = \Sigma_{x \in V_1}|S_x| \geq 3k_2+m-3$. We suppose $r \leq 2$ in the following.

Case 1. $r = 0$.

Without loss of generality, assume $G'_{2x_1} \subseteq C$ and $G'_{2x_2} \subseteq C'$ for $x_1, x_2 \in V_1$, where C and C' are distinct components of $G-S$. Since there is a perfect matching between G_{2x_1} and G_{2x_2}, we have $|S_{x_1}| \geq |V(G'_{2x_2})| = n - |S_{x_2}|$, that is $|S_{x_1}| + |S_{x_2}| \geq n$. Since $S_x \neq \emptyset$ for any $x \in V_1$, we have

$$\begin{aligned} |S| &= |S_{x_1}| + |S_{x_2}| + \Sigma_{x \in V_1 \setminus \{x_1,x_2\}}|S_x| \\ &\geq n+m-2 \\ &\geq 3k_2+m-3 \end{aligned}$$

for $g(G_2) \geq 5$, a contradiction. Hence $G-S$ is connected, a contradiction.

Case 2. $r = 1$.

Assume G'_{2x_1} is disconnected in $G-S$ for $x_1 \in V_1$. Then G'_{2x} is connected in $G-S$ for any $x \in V_1 \setminus \{x_1\}$. Suppose G'_{2x_2} is contained in a component C of $G-S$ for $x_2 \in V_1 \setminus \{x_1\}$. If there exists a copy $G'_{2x_3} \nsubseteq C$ for $x_3 \in V_1 \setminus \{x_1,x_2\}$, then since there is a perfect matching between G_{2x_2} and G_{2x_3}, we have $|S_{x_2}| + |S_{x_3}| \geq n$, and

$$\begin{aligned} |S| &= |S_{x_2}| + |S_{x_3}| + \Sigma_{x \in V_1 \setminus \{x_2,x_3\}}|S_x| \\ &\geq n+m-2 \\ &\geq 3k_2+m-3 \end{aligned}$$

for $g(G_2) \geq 5$, a contradiction. Hence $G'_{2x} \subseteq C$ for any $x \in V_1 \setminus \{x_1\}$.

Denote the components of G'_{2x_1} by $H_1, H_2, \cdots, H_l (l \geq 2)$. When $|V(H_i)| \leq 2$ for some $i \in \{1,2,\cdots,l\}$. Since S is a 3-restricted vertex-cut of G, we have $H_i \subseteq C$. When $|V(H_i)| \geq 3k_2$ for some $i \in \{1,2,\cdots,l\}$ and $H_i \nsubseteq C$, $V_{G_{2x}}(H_i) \subseteq S_x$ for any $x \in V_1 \setminus \{x_1\}$, and we have

$$\begin{aligned} |S| &= \Sigma_{x \in V_1}|S_x| \\ &\geq \kappa(G_2) + (m-1)|V(H_i)| \\ &\geq k_2+m-1+|V(H_i)|-1 \\ &> 3k_2+m-3, \end{aligned}$$

a contradiction. When $3 \leq |V(H_i)| \leq 3k_2-1$ for some $i \in \{1,2,\cdots,l\}$ and $H_i \nsubseteq C$, $V_{G_{2x}}(H_i) \subseteq S_x$ for any $x \in V_1 \setminus \{x_1\}$, and

$$\begin{aligned} |S_{x_1}| &\geq |N_{G_{2x_1}}[\{u_1,u_2,u_3\}]| - |V(H_i)| \\ &= 3k_2-1-|V(H_i)| \end{aligned}$$

for $g(G_2) \geq 5$, where $P = u_1u_2u_3$ is a path of H_i. We have

$$\begin{aligned} |S| &= \Sigma_{x \in V_1}|S_x| \\ &\geq 3k_2-1-|V(H_i)|+(m-1)|V(H_i)| \\ &\geq 3k_2-1-|V(H_i)|+m-1+|V(H_i)|-1 \\ &= 3k_2+m-3, \end{aligned}$$

a contradiction. Hence $H_i \subseteq C$ for any $i \in \{1,2,\cdots,l\} (l \geq 2)$, and $G-S$ is connected, a contradiction.

Case 3. $r = 2$.

Assume G'_{2x_1} and G'_{2x_2} are disconnected in $G-S$. Then G'_{2x} is connected in $G-S$ for any $x \in V_1 \setminus \{x_1,x_2\}$, with the similar manner as the case 2, G'_{2x} is contained in a component C of $G-S$, and if there are at least three vertices in component H of G'_{2x_1} or G'_{2x_2}, then $H \subseteq C$. If $H \subseteq G'_{2x_1} \cup G'_{2x_2}$ and $|H| \leq 2$, then $H \subseteq C$ by S being a 3-restricted vertex-cut of G. Thus if $H \nsubseteq C$ for $H \subseteq G'_{2x_1} \cup G'_{2x_2}$, then H is one of graphs in figure 1. Since $|N(F_1)| = 3k_2+2m-6$, $|N(F_2)| = 4k_2+2m-8$, and $|N(F_3)| = 4k_2+3m-10$, we have $|N(F_i)| \geq 3k_2+m-3$ for any $i \in \{1,2,3\}$, a contradiction. Hence $H \subseteq C$, and $G-S$ is connected, a contradiction.

Since all possible cases lead to a contradiction, we have

$$\kappa_3(G) = 3k_2+m-3. \qquad \square$$

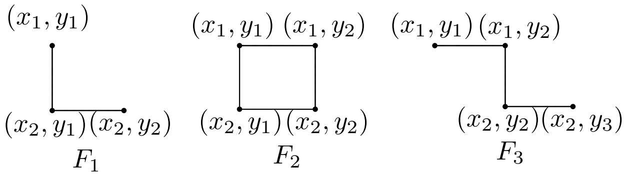

Figure 1. The subgraphs are not contained in C of $G - S$.

Lemma 2.3. *Let G_i be a $k_i(\geq 2)$-regular connected graph with $g(G_i) \geq 5$ for $i \in \{1, 2\}$. Then $\kappa_3(G_1 \square G_2) \leq 3k_1 + 3k_2 - 5$.*

Proof. Denote $G = G_1 \square G_2$. Let $P = u_1u_2u_3$ be a path of G, where $u_1 = (x_1, y_1)$, $u_2 = (x_2, y_1)$, and $u_3 = (x_2, y_2)$ for $x_1, x_2 \in V_1$ and $y_1, y_2 \in V_2$. Since G_{1y_j} is a k_1-regular graph for $j \in \{1, 2\}$ and G_{2x_i} is a k_2-regular graph for $i \in \{1, 2\}$, we have

$$\begin{aligned}|N(\{u_1, u_2, u_3\})| &= |N_{G_{2x_1}}(\{u_1\})| + |N_{G_{2x_2}}(\{u_2, u_3\})| \\ &+ |N_{G_{1y_1}}(\{u_1, u_2\})| + |N_{G_{1y_2}}(\{u_3\})| \\ &- 1 \\ &= 2k_1 - 2 + k_1 + k_2 + 2k_2 - 2 - 1 \\ &= 3k_1 + 3k_2 - 5\end{aligned}$$

for $g(G_1) \geq 5$ and $g(G_2) \geq 5$.

If there exists a isolated vertex (x_0, y_1) in $G_{1y_1} - N_{G_{1y_1}}(\{u_1, u_2\})$, then $G'_{2x_0} = G_{2x_0}$. If there exists a isolated vertex (x_2, y_0) in $G_{2x_2} - N_{G_{2x_2}}(\{u_2, u_3\})$, then $G'_{1y_0} = G_{1y_0}$. Thus there are at least three vertices in each component of $G - N(\{u_1, u_2, u_3\})$, that is, $N(\{u_1, u_2, u_3\})$ is a 3-restricted vertex-cut of G. Hence

$$\kappa_3(G) \leq |N(\{u_1, u_2, u_3\})| = 3k_1 + 3k_2 - 5. \qquad \square$$

Theorem 8. *Let G_i be a $k_i(\geq 2)$-regular and maximally connected graph with $g(G_i) \geq 5$ for $i \in \{1, 2\}$. Then $\kappa_3(G_1 \square G_2) = 3k_1 + 3k_2 - 5$.*

Proof. Denote $G = G_1 \square G_2$. By lemma 2.3, we have $\kappa_3(G) \leq 3k_1 + 3k_2 - 5$. By contraction, $\kappa_3(G) < 3k_1 + 3k_2 - 5$. Assume S is a 3-restricted vertex-cut of G with $|S| = \kappa_3(G)$. Let $S_x = S \cap V_{2x}$ for any $x \in V_1$, we have $S_x \neq \emptyset$ for any $x \in V_1$. Otherwise, assume $S_{x_1} = \emptyset$ for some $x_1 \in V_1$, and $G'_{2x_1}(= G_{2x_1})$ is contained in a component C of $G - S$, then $G'_{2x} \subseteq C$ for any $x \in N_{G_1}(x_1)$. For any $x' \notin N_{G_1}[x_1]$, we denote these components of $G'_{2x'}$ by $H_1, H_2, \cdots, H_l$. Since there exist at least $\kappa(G_1)$ internally disjoint paths between x_1 and x' in G_1, there are $\kappa(G_1)|V(H_i)|$ internally disjoint paths between G'_{2x_1} and H_i in $G - S_{x_1} - S_{x'}$ for $i \in \{1, 2, \cdots, l\}$. If $|V(H_i)| \geq 3k_2$, then $H_i \subseteq C$, otherwise

$$\begin{aligned}|S| &\geq \kappa(G_1)|V(H_i)| \\ &\geq 3k_1k_2 \\ &\geq 3(k_1 + k_2 - 1) \\ &> 3k_1 + 3k_2 - 5,\end{aligned}$$

a contradiction. If $3 \leq |V(H_i)| \leq 3k_2 - 1$, and $H_i \nsubseteq C$, then since $N_{G_{2x'}}(H_i) \subseteq S_{x'}$ and $g(G_{2x'}) \geq 5$, we have

$$\begin{aligned}|S_{x'}| &\geq |N_{G_{2x'}}[\{u_1, u_2, u_3\}]| - |V(H_i)| \\ &= 3k_2 - 1 - |V(H_i)|,\end{aligned}$$

where $P = u_1u_2u_3$ is a path of H_i. Hence

$$\begin{aligned}|S| &\geq \kappa(G_1)|V(H_i)| + 3k_2 - 1 - |V(H_i)| \\ &= |V(H_i)|(k_1 - 1) + 3k_2 - 1 \\ &> 3k_1 + 3k_2 - 5,\end{aligned}$$

a contradiction.

If $|V(H_i)| \leq 2$ for some $i \in \{1, 2, \cdots, l\}$, then since S is a 3-restricted vertex-cut of G, there exists a copy G_{2x_2} of G_2 such that some vertex of H_i is adjacent to a vertex of a component H of G'_{2x_2} for $x_2 \in V_1 \setminus \{x'\}$, and $|V(H_i \cup H)| \geq 3$. When $x_2 \in N_{G_1}[x_1]$ or $|H| \geq 3$, we have $H_i \subseteq C$ using the similar above manner. In the following, we assume $x_2 \notin N_{G_1}[x_1]$ and $|H| \leq 2$, then $F = H_i \cup H$ is one of graphs in figure 1. If $F \nsubseteq C$, then $|N(F_1)| = 3k_1 + 3k_2 - 5$, and $|N(F_2)| = |N(F_3)| = 4k_1 + 4k_2 - 8$, we have $|N(F_i)| \geq 3k_1 + 3k_2 - 5$ for any $i \in \{1, 2, 3\}$, a contradiction. Thus $F \subseteq C$, and $G - S$ is connected, a contradiction. Hence $S_x \neq \emptyset$ for any $x \in V_1$.

Let r be the number of copies of G_2 in G which are disconnected in $G - S$. If $r \geq 3$, then

$$\begin{aligned}|S| &= \Sigma_{x \in V_1}|S_x| \\ &\geq 3k_2 + m - 3 \\ &> 3k_1 + 3k_2 - 5\end{aligned}$$

for $g(G_1) \geq 5$. We suppose $r \leq 2$ in the following.

Case 1. $r = 0$.

Without loss of generality, assume G'_{2x_1} is contained in one component C of $G - S$ for $x_1 \in V_1$. By the minimality of S, there exists one vertex $v \in N(u) \cap V(C')$ for any $u \in S_{x_1}$, where C' is a component of $G - S$ different from C. Assume $v \in G'_{2x_2}$ for $x_2 \in V_1 \setminus \{x_1\}$, then $G'_{2x_2} \subseteq C'$, and there exists a perfect matching between the copies G_{2x_1} and G_{2x_2}. Thus $|S_{x_1}| + |S_{x_2}| \geq n$, and

$$\begin{aligned}|S| &= |S_{x_1}| + |S_{x_2}| + \Sigma_{x \in V_1 \setminus \{x_1, x_2\}}|S_x| \\ &\geq n + m - 2 \\ &> 3k_1 + 3k_2 - 5\end{aligned}$$

by $g(G_i) \geq 5$ for $i \in \{1, 2\}$, a contradiction.

Case 2. $r = 1$.

Assume G'_{2x_1} is disconnected in $G - S$ for $x_1 \in V_1$. G'_{2x} is connected in $G - S$ for any $x \in V_1 \setminus \{x_1\}$. There is at least one copy G_{2x_2} of G_2 such that $|S_{x_2}| < \kappa(G_2)$ for

$x_2 \in V_1 \setminus \{x_1\}$. Otherwise

$$\begin{aligned} |S| &\geq m\kappa(G_2) \\ &\geq (3k_1-1)k_2 \\ &= 3(k_1-1)k_2+2k_2 \\ &\geq 3(k_1-1+k_2-1)+2k_2 \\ &> 3k_1+3k_2-5, \end{aligned}$$

a contradiction. Let G'_{2x_2} be contained in a component C of $G-S$. By the minimality of S, there exists one vertex $v \in N(u) \cap V(C')$ for any $u \in S_{x_2}$, where C' is a component of $G-S$ different from C. Assume $v \in G'_{2x_3}$ for $x_3 \in V_1 \setminus \{x_2\}$. If $x_3 \neq x_1$, then $G'_{2x_3} \subseteq C'$, and there is a perfect matching between the copies G_{2x_2} and G_{2x_3}. Thus $|S_{x_2}|+|S_{x_3}| \geq n$, and

$$\begin{aligned} |S| &= |S_{x_2}|+|S_{x_3}|+\Sigma_{x\in V_1\setminus\{x_2,x_3\}}|S_x| \\ &\geq n+m-2 \\ &> 3k_1+3k_2-5 \end{aligned}$$

by $g(G_i) \geq 5$ for $i \in \{1,2\}$, a contradiction. Hence $x_3 = x_1$.

Let v belong to a component H of G'_{2x_1}. Since $G'_{2x_2} \subseteq C$ and $v \in C'$, we have $N_{G_{2x_2}}(H) \subseteq S_{x_2}$. When $|V(H)| \geq 3k_2$,

$$\begin{aligned} |S| &= |S_{x_2}|+\Sigma_{x\in V_1\setminus\{x_2\}}|S_x| \\ &\geq 3k_2+m-1 \\ &> 3k_1+3k_2-5 \end{aligned}$$

for $g(G_1) \geq 5$, a contradiction. When $3 \leq |V(H)| \leq 3k_2-1$,

$$\begin{aligned} |N_{G_{2x_1}}(H)| &\geq |N_{G_{2x_1}}[\{u_1,u_2,u_3\}]|-|V(H)| \\ &\geq 3k_2-1-|V(H)|, \end{aligned}$$

where $P = u_1u_2u_3$ is a path of H. Since $N_{G_{2x_1}}(H) \subseteq S_{x_1}$ and $N_{G_{2x_2}}(H) \subseteq S_{x_2}$, we have

$$\begin{aligned} |S| &= |S_{x_1}|+|S_{x_2}|+\Sigma_{x\in V_1\setminus\{x_1,x_2\}}|S_x| \\ &\geq 3k_2-1-|V(H)|+|S_{x_2}|+m-2 \\ &> 3k_1+3k_2-5 \end{aligned}$$

for $g(G_1) \geq 5$, a contradiction. When $|V(H)| \leq 2$. Since S is a 3-restricted vertex-cut of G, v is adjacent to another vertex w in one copy G'_{2x_4} of $G-S$ for $x_4 \in V_1 \setminus \{x_1, x_2\}$. If $|V(G'_{2x_4})| \leq |S_{x_2}|+2$, then

$$\begin{aligned} |S| &= |S_{x_1}|+|S_{x_2}|+|S_{x_4}|+\Sigma_{x\in V_1\setminus\{x_1,x_2,x_4\}}|S_x| \\ &\geq \kappa(G_2)+|S_{x_2}|+n-|V(G'_{2x_4})|+m-3 \\ &\geq 3k_1+3k_2-5 \end{aligned}$$

by $k_2 \geq 2$ and $g(G_i) \geq 5$ for $i \in \{1,2\}$, a contradiction. If $|V(G'_{2x_4})| \geq |S_{x_2}|+3$, then since there are $(|V(G'_{2x_4})| - |S_{x_2}|)\kappa(G_1)$ disjoint paths between G'_{2x_2} and G'_{2x_4} in $G-S_{x_2}-S_{x_4}$, we have

$$\begin{aligned} |S| &\geq |S_{x_2}|+|S_{x_4}|+(|V(G'_{2x_4})|-|S_{x_2}|)\kappa(G_1) \\ &= |S_{x_2}|+n-|V(G'_{2x_4})|+(|V(G'_{2x_4})| \\ &- |S_{x_2}|)\kappa(G_1) \\ &= n+(|V(G'_{2x_4})|-|S_{x_2}|)(k_1-1) \\ &> 3k_1+3k_2-5, \end{aligned}$$

a contradiction.

Case 3. $r = 2$.

Assume G'_{2x_1} and G'_{2x_2} are disconnected in $G-S$ for $x_1, x_2 \in V_1$, then G'_{2x} is connected for any $x \in V_1 \setminus \{x_1, x_2\}$. Let G'_{2x_3} be contained in a component C of $G-S$ for $x_3 \in V_1 \setminus \{x_1,x_2\}$. By the minimality of S, there exists one vertex $v \in N(u) \cap V(C')$ for any $u \in S_{x_3}$, where C' is a component of $G-S$ different from C. Using the similar manner as the case 2, if there are at least three vertices in a component H of G'_{2x_1} or G'_{2x_2}, then we obtain $H \subseteq C$. If $H \nsubseteq C$ for $H \subseteq G'_{2x_1} \cup G'_{2x_2}$, then H is one of graphs in figure 1. Since $|N(F_1)| = 3k_1+3k_2-5$, and $|N(F_2)| = |N(F_3)| = 4k_1+4k_2-8$, we have $|N(F_i)| \geq 3k_1+3k_2-5$ for any $i \in \{1,2,3\}$, a contradiction.

Since all possible cases lead to a contradiction, we have

$$\kappa_3(G) = 3k_1+3k_2-5. \qquad \square$$

Using the similar above manner, we can obtain the similar results for the 4-restricted connectivity of the Cartesian product graphs.

Theorem 9. *(i)* $\kappa_4(K_m \square K_n)$ $=min\{4m+n-8, m+4n-8\}$ *for* $m+n \geq 10$ *with* $n \geq 2m$ *or* $m \geq 2n$;
(ii) $\kappa_4(K_m \square K_n) = 2m+2n-8$ *for* $4 \leq m \leq 2n-1$ *and* $4 \leq n \leq 2m-1$.

Theorem 10. *Let* G_2 *be a* $k_2(\geq 2)$*-regular and maximally connected graph. If* $g(G_2) \geq 6$*, then* $\kappa_4(K_m \square G_2) = 4k_2+m-4$ *for* $m \geq 4$.

Theorem 11. *Let* G_i *be a* $k_i(\geq 2)$*-regular and maximally connected graph with* $g(G_i) \geq 6$ *for* $i \in \{1,2\}$*. Then* $\kappa_4(G_1 \square G_2) = 4k_1+4k_2-8$.

Let G_i be a $k_i(\geq 2)$-regular and maximally connected graph. If $g(G_i) \geq k+2 (k \geq 3)$, then using the similar manner as the lemmas 2.2 and 2.3, we have $\kappa_k(K_m \square G_2) \leq kk_2+m-k$ for $m \geq k (k \leq 10)$, and $\kappa_k(G_1 \square G_2) \leq kk_1+kk_2-3k+4 (k \leq 10)$. Hence, we have the following conjectures.

Conjecture 1. *Let* G_2 *be a* $k_2(\geq 2)$*-regular and maximally connected graph. If* $g(G_2) \geq k+2$*, then* $\kappa_k(K_m \square G_2) = kk_2+m-k$ *for* $m \geq k (k \geq 2)$.

Conjecture 2. *Let G_i be a $k_i(\geq 2)$-regular and maximally connected graph with $g(G_i) \geq k+2$ for $i \in \{1,2\}$. Then $\kappa_k(G_1 \Box G_2) = kk_1 + kk_2 - 3k + 4 (k \geq 2)$.*

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