Some Inequalities for p-Geominimal Surface Area and Related Results

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***Abstract*—The concepts of p-affine and p-geominimal surface areas were introduced by Lutwak. In this paper, we establish some Brunn-Minkowski type inequalities of p-geominimal surface area combining L_p-polar curvature image with various combinations of convex bodies. Moreover, we discuss the equivalence of several inequalities, and also obtain some results similar to p-geominimal surface area for the p-affine surface area.**

***Index Terms*—convex bodies, p-affine surface area, p-geominimal surface area, Brunn-Minkowski type inequality.**

I. Introduction

LET $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in n-dimensional Euclidean space $\mathbf{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of convex bodies whose centroids lie at the origin and the set of origin-symmetric convex bodies in $\mathbf{R}^n$, we write $\mathcal{K}_o^n$, $\mathcal{K}_e^n$ and $\mathcal{K}_c^n$, respectively. $\mathcal{S}_o^n$ and $\mathcal{S}_c^n$ respectively denote the set of star bodies (about the origin) and the set of origin-symmetric star bodies in $\mathbf{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$, and let $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B_n in $\mathbf{R}^n$, we use $\omega_n = V(B_n)$ to denote its volume.

The study of affine surface area goes back to Blaschke [1] and is about one hundred years old. It was generalized to the p-affine surface area by Lutwak in [10]. Since then, considerable attention has been paid to the p-affine surface area, which is now at the core of the rapidly developing L_p-Brunn-Minkowski theory(see articles [4], [5], [6], [8], [11], [12], [13], [14], [17], [19], [24], [28] or books [7], [22]). In particular, affine isoperimetric inequalities related to the p-affine surface area can be found in [10], [29].

Another fundamental concept in convex geometry is geominimal surface area, introduced by Petty [19] more than three decades ago. As Petty explained in [19], the geominimal surface area connects the affine geometry, relative geometry and Minkowski geometry. Hence it receives a lot of attention (see [19], [20], [23]). The geominimal surface area was extended to p-geominimal surface area by Lutwak in his seminal paper [10]. The p-geominimal surface area shares many properties with the p-affine surface area. For instance, both are affine invariant and have the same degree of homogeneity. However, the p-geominimal surface area is different from the p-affine surface area. For instance, unlike the p-affine surface area, p-geominimal surface area has no nice integral expression. This leads to a big obstacle on extending the p-geominimal surface area. There are many papers on p-affine and p-geominimal surface areas, see e.g., [16], [18], [25], [26], [27], [28], [29], [30], [32].

Based on the notion of L_p-mixed volume, Lutwak introduced the concepts of p-affine and p-geominimal surface areas, respectively.

For $p \geq 1$ and $K \in \mathcal{K}_o^n$, the p-affine surface area, $\Omega_p(K)$, was defined in [10] by

$$n^{-\frac{p}{n}}\Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\}.$$

Here $V_p(K,Q^*)$ denotes the L_p-mixed volume of K and Q^* (see Section II. A) and Q^* denotes the polar of body Q (see Section II. C).

For $p \geq 1$, Lutwak in [10] defined the p-geominimal surface area, $G_p(K)$, of $K \in \mathcal{K}_o^n$ by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}. \quad (1)$$

Further, Lutwak obtained the following inequalities for the p-affine and the p-geominimal surface areas.

Lemma 1.1. (Theorem 4.8 in [10]) *Let $K \in \mathcal{K}_e^n$ and $p \geq 1$. Then*

$$\Omega_p(K)^{n+p} \leq n^{n+p}\omega_n^{2p}V(K)^{n-p}, \quad (2)$$

with equality if and only if K is an ellipsoid.

Lemma 1.2. (Theorem 3.12 in [10]) *Let $K \in \mathcal{K}_o^n$ and $p \geq 1$. Then*

$$G_p(K)^n \leq n^n\omega_n^pV(K)^{n-p}, \quad (3)$$

with equality if and only if K is an ellipsoid.

Lemma 1.3. ([10] p. 250) *Let $K \in \mathcal{F}_o^n$ and $p \geq 1$. Then*

$$\Omega_p(K)^{n+p} \leq (n\omega_n)^pG_p(K)^n, \quad (4)$$

with equality if and only if K is of p-elliptic type.

A convex body $K \in \mathcal{K}_o^n$ is said to have a L_p-curvature function (see [10]) $f_p(K,\cdot) : S^{n-1} \to \mathbf{R}$, if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{\mathrm{d}S_p(K,\cdot)}{\mathrm{d}S} = f_p(K,\cdot).$$

Let $\mathcal{F}_o^n, \mathcal{F}_c^n$ denote the set of all bodies in $\mathcal{K}_o^n, \mathcal{K}_c^n$ respectively, and both of them have a positive continuous curvature function.

If $K \in \mathcal{S}_c^n$, and $p \geq 1$, then define $\Lambda_p^\circ K \in \mathcal{F}_c^n$, the L_p-polar curvature image of K, by

$$f_p(\Lambda_p^\circ K,\cdot) = \frac{\omega_n}{V(K)}\rho(K,\cdot)^{n+p}. \quad (5)$$

When $p = 1$, we write $\Lambda_1^\circ K = \Lambda K$, it is just the classical curvature image (see [12], [14]); When $p > 1$, it was defined by Yuan, Zhu, Lv and Leng (see [15], [30], [31]).

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The following theorems are our main results: Combining L_p-polar curvature image with p-geominimal surface area, we establish several Brunn-Minkowski type inequalities of the p-geominimal surface area.

Theorem 1.4. *If $p \geq 1, K, L \in \mathcal{K}_c^n$, and $\lambda, \mu \geq 0$ (not both zero), then*

$$G_p\big(\Lambda_p^\circ(\lambda \cdot K +_p \mu \cdot L)\big) \geq \lambda G_p(\Lambda_p^\circ K) + \mu G_p(\Lambda_p^\circ L), \quad (6)$$

with equality for $p = 1$ if and only if K and L are homothetic, and for $p > 1$ if and only if K and L are dilates.

Here, $\lambda \cdot K +_p \mu \cdot L$ denotes the L_p-Firey combination of K and L (see (10)).

Theorem 1.5. *If $1 \leq p \leq n, K, L \in \mathcal{K}_c^n$, and $\lambda, \mu \geq 0$ (not both zero), then*

$$G_p\big(\Lambda_p^\circ(\lambda \circ K \widetilde{+}_p \mu \circ L)\big) \leq \lambda G_p(\Lambda_p^\circ K) + \mu G_p(\Lambda_p^\circ L). \quad (7)$$

The reverse inequality holds when $p > n$. Equality holds in every inequality when $p \neq n$ if and only if K is a dilate of L. Here, $\lambda \circ K \widetilde{+}_p \mu \circ L$ denotes the L_p-radial combination of K and L (see (13)).

Theorem 1.6. *If $p \geq 1, K, L \in \mathcal{K}_c^n$, and $\lambda, \mu \geq 0$ (not both zero), then*

$$G_p\big(\Lambda_p^\circ(\lambda * K \widehat{+}_{-p} \mu * L)\big)^{-1} \geq \lambda G_p(\Lambda_p^\circ K)^{-1} + \mu G_p(\Lambda_p^\circ L)^{-1}, \quad (8)$$

with equality if and only if K and L are dilates.

Here, $\lambda * K \widehat{+}_{-p} \mu * L$ denotes the L_p-harmonic radial combination of K and L (see (16)).

Theorem 1.7. *If $n \neq p \geq 1, K, L \in \mathcal{F}_c^n$, and $\lambda, \mu \geq 0$ (not both zero), then*

$$G_p\big(\lambda K \check{+}_p \mu L\big) \geq \lambda G_p(K) + \mu G_p(L), \quad (9)$$

with equality for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates.

Here, $\lambda K \check{+}_p \mu L$ denotes the Blaschke L_p-combination of K and L (see (23)).

Please see the next section for above interrelated notations, definitions and their background materials. The proofs of Theorems 1.4-1.7 will be given in Section III of this paper. Moreover, we derive the equivalence of several inequalities in Section IV.

II. Preliminaries

A. L_p-Firey Combination and L_p-mixed Volume

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbf{R}^n \to (-\infty, \infty)$, is defined by (see [22]) $h(K, x) = \max\{x \cdot y : \ y \in K\}, \ \ x \in \mathbf{R}^n$, where $x \cdot y$ denotes the standard inner product of x and y.

For real $p \geq 1, K, L \in \mathcal{K}_o^n$, and $\alpha, \beta \geq 0$ (not both zero), the L_p-Firey combination, $\alpha \cdot K +_p \beta \cdot L$, is defined by (see [2])

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p. \quad (10)$$

For $p \geq 1$, the L_p-mixed volume, $V_p(K, L)$, of $K, L \in \mathcal{K}_o^n$, was defined in [9] by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(L)}{\varepsilon}.$$

It was shown in [9] that corresponding to each $K \in \mathcal{K}_o^n$ there is a positive Borel measure $S_p(K, \cdot)$ on S^{n-1} such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p \mathrm{d}S_p(K, u)$$

for all $Q \in \mathcal{K}_o^n$. It turns out that the L_p-surface area measure $S_p(K, \cdot)$ on S^{n-1} is absolutely continuous with respect to $S(K, \cdot)$, and has the Radon-Nikodym derivative

$$\frac{\mathrm{d}S_p(K, \cdot)}{\mathrm{d}S(K, \cdot)} = h^{1-p}(K, \cdot).$$

The L_p-Brunn-Minkowski inequality was given by Lutwak in [9]: If $K, L \in \mathcal{K}_o^n, \lambda, \mu > 0$, and $p \geq 1$, then

$$V(\lambda \cdot K +_p \mu \cdot L)^{p/n} \geq \lambda V(K)^{p/n} + \mu V(L)^{p/n}, \quad (11)$$

with equality for $p = 1$ if and only if K and L are homothetic, and for $p > 1$ if and only if K and L are dilates.

Taking $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (10), the L_p-difference body, $\Delta_p K$, of K was given by (see [9])

$$\Delta_p K = \frac{1}{2} \cdot K +_p \frac{1}{2} \cdot (-K). \quad (12)$$

B. L_p-radial Combination and L_p-dual Mixed Volume

If K is a compact star-shaped (about the origin) set in $\mathbf{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbf{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see [22]) $\rho(K, u) = \max\{\lambda \geq 0 : \ \lambda u \in K\}, \ \ u \in S^{n-1}$. If ρ_K is positive and continuous, then K will be called a star body (about the origin). Two star bodies K and L are said to be dilated of one another if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \geq 0$ (not both zero), then for $p > 0$, the L_p-radial combination, $\lambda \circ K \widetilde{+}_p \mu \circ L \in \mathcal{S}_o^n$, is defined by (see [3])

$$\rho(\lambda \circ K \widetilde{+}_p \mu \circ L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p. \quad (13)$$

For $p \geq 1$, and $K, L \in \mathcal{S}_o^n$, the L_p-dual mixed volume, $\widetilde{V}_p(K, L)$, was defined in [3] by

$$\frac{n}{p} \widetilde{V}_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_p \varepsilon \circ L) - V(K)}{\varepsilon}.$$

The following integral representation for the L_p-dual mixed volume was obtained in [3]: If $p \geq 1$, and $K, L \in \mathcal{S}_o^n$, then

$$\widetilde{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-p} \rho(L, u)^p \mathrm{d}S(u),$$

where S is the spherical Lebesgue measure on S^{n-1} (i.e., the $(n-1)$-dimensional Hausdorff measure).

We shall need the following L_p-dual Brunn-Minkowski inequality (see [3]): If $K, L \in \mathcal{S}_o^n$ and $0 < p \leq n$, then

$$V(\lambda \circ K \widetilde{+}_p \mu \circ L)^{p/n} \leq \lambda V(K)^{p/n} + \mu V(L)^{p/n}. \quad (14)$$

The reverse inequality holds when $p > n$. Equality holds when $p \neq n$ if and only if K is a dilate of L.

Taking $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (13), the L_p-radial body, $\widetilde{\Delta}_p K$, of K is defined by

$$\widetilde{\Delta}_p K = \frac{1}{2} \circ K \widetilde{+}_p \frac{1}{2} \circ (-K). \quad (15)$$

C. L_p-harmonic Radial Combination and L_p-harmonic Mixed Volume

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination, $\lambda * K \hat{+}_{-p} \mu * L \in \mathcal{S}_o^n$, is defined by(see [10])

$$\rho(\lambda * K \hat{+}_{-p} \mu * L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}. \quad (16)$$

If $K \in \mathcal{K}_o^n$, the polar set, K^*, of K is defined by

$$K^* = \{x \in \mathbf{R}^n : x \cdot y \leq 1, \text{for all } y \in K\}. \quad (17)$$

From (17), we can easily have $(K^*)^* = K$, and

$$h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K} \quad (18)$$

for $K \in \mathcal{K}_o^n$.

By (10), (16) and (18), it follows that if $K, L \in \mathcal{K}_o^n$ and $\lambda, \mu \geq 0$ (not both zero), then

$$\lambda * K \hat{+}_{-p} \mu * L = (\lambda \cdot K^* +_p \mu \cdot L^*)^*.$$

Define the Santaló product of $K \in \mathcal{K}_o^n$ by $V(K)V(K^*)$. The Blaschke-Santaló inequality (see [22]) is one of the fundamental affine isoperimetric inequalities. It states that if $K \in \mathcal{K}_c^n$ then

$$V(K)V(K^*) \leq \omega_n^2,$$

with equality if and only if K is an ellipsoid.

For $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, the L_p-harmonic mixed volume, $\widetilde{V}_{-p}(K, L)$, is defined by (see [10])

$$-\frac{n}{p}\widetilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \hat{+}_{-p} \varepsilon * L) - V(K)}{\varepsilon}.$$

From the polar coordinate formula, the following integral representation was given in [10]: If $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, then

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} \mathrm{d}S(u).$$

The Minkowski's inequality for the L_p-harmonic mixed volume can be stated that (see [10]): If $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, then

$$\widetilde{V}_{-p}(K, L)^n \geq V(K)^{n+p} V(L)^{-p}, \quad (19)$$

with equality if and only if K and L are dilates.

The Brunn-Minkowski inequality for the L_p-harmonic radial combination can be stated that (see [10]): Suppose $K, L \in S_o^n, p \geq 1$ and $\lambda, \mu > 0$, then

$$V(\lambda * K \hat{+}_{-p} \mu * L)^{-p/n} \geq \lambda V(K)^{-p/n} + \mu V(L)^{-p/n}, \quad (20)$$

with equality if and only if K and L are dilates each other.

Taking $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (16), the L_p-harmonic radial body, $\widehat{\Delta}_p K$, of K is defined by

$$\widehat{\Delta}_p K = \frac{1}{2} * K \hat{+}_{-p} \frac{1}{2} * (-K). \quad (21)$$

D. L_p-affine Surface Area, L_p-curvature Image and Blaschke L_p-combination

In [10], Lutwak defined the L_p-affine surface area as follows: For $K \in \mathcal{F}_o^n$ and $p \geq 1$, the L_p-affine surface area, $\Omega_p(K)$, of K is defined by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K, u)^{\frac{n}{n+p}} \mathrm{d}S(u).$$

Further, Lutwak [10] showed the notion of L_p-curvature image as follows: For any $K \in \mathcal{F}_o^n$ and $p \geq 1$, define $\Lambda_p K \in \mathcal{S}_o^n$, the L_p-curvature image of K, by

$$\rho(\Lambda_p K, \cdot)^{n+p} = \frac{V(\Lambda_p K)}{\omega_n} f_p(K, \cdot). \quad (22)$$

Note that for $p = 1$, this definition is different from the classical curvature image (see [14]).

The definition of Blaschke L_p-combination for convex bodies may be stated that (see [9]) for $K, L \in \mathcal{K}_c^n, \lambda, \mu \geq 0$ (not both zero) and $n \neq p \geq 1$, the Blaschke L_p-combination, $\lambda K \check{+}_p \mu L \in \mathcal{K}_c^n$, of K and L is defined by

$$\mathrm{d}S_p(\lambda K \check{+}_p \mu L, \cdot) = \lambda \mathrm{d}S_p(K, \cdot) + \mu \mathrm{d}S_p(L, \cdot). \quad (23)$$

Taking $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (23), the Blaschke L_p-body, $\nabla_p K \in \mathcal{K}_c^n$, of K is defined by (see [9])

$$\nabla_p K = \frac{1}{2} K \check{+}_p \frac{1}{2}(-K). \quad (24)$$

From (22) and (23), Wang and Leng [26] proved the following L_p-Brunn-Minkowski inequality: If $K, L \in \mathcal{F}_c^n, \lambda, \mu > 0$ and $n \neq p \geq 1$, then

$$V(\Lambda_p(\lambda K \check{+}_p \mu L))^{p/n} \geq \lambda V(\Lambda_p K)^{p/n} + \mu V(\Lambda_p L)^{p/n}, \quad (25)$$

with equality for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates.

III. Proofs of Theorems

In this section, we prove Theorems 1.4-1.7. Taking $L = Q^*$ in Proposition 3.4 of [31], we immediately give:

Lemma 3.1. *If $p \geq 1$ and $K \in \mathcal{K}_c^n$, then for any $Q \in \mathcal{K}_o^n$,*

$$V_p(\Lambda_p^\circ K, Q) = \omega_n \widetilde{V}_{-p}(K, Q^*)/V(K). \quad (26)$$

Lemma 3.2. *If $p \geq 1$ and $K \in \mathcal{K}_c^n$, then*

$$G_p(\Lambda_p^\circ K) = n\omega_n^{\frac{n-p}{n}} V(K)^{\frac{p}{n}}. \quad (27)$$

Proof. By (1), (26) and (27), we have

$$\begin{aligned} & G_p(\Lambda_p^\circ K) \\ = \ & \omega_n^{-\frac{p}{n}} \inf\{nV_p(\Lambda_p^\circ K, Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\} \\ = \ & \omega_n^{-\frac{p}{n}} \inf\{n\omega_n \widetilde{V}_{-p}(K, Q^*)V(Q^*)^{\frac{p}{n}}/V(K) : Q \in \mathcal{K}_o^n\} \\ \geq \ & n\omega_n^{\frac{n-p}{n}} \inf\{V(K)^{\frac{n+p}{n}} V(Q^*)^{-\frac{p}{n}} V(Q^*)^{\frac{p}{n}}/V(K) \\ & : Q \in \mathcal{K}_o^n\} \\ = \ & n\omega_n^{\frac{n-p}{n}} V(K)^{\frac{p}{n}}. \end{aligned}$$

On the other hand, from (1) and (26), it follows that for any $Q \in \mathcal{K}_o^n$

$$\begin{aligned} G_p(\Lambda_p^\circ K) & \leq \omega_n^{-\frac{p}{n}} nV_p(\Lambda_p^\circ K, Q)V(Q^*)^{\frac{p}{n}} \\ & = n\omega_n^{\frac{n-p}{n}} \widetilde{V}_{-p}(K, Q^*)V(Q^*)^{\frac{p}{n}}/V(K). \end{aligned}$$

Since $K \in \mathcal{K}_c^n$, and taking $Q^* = K$, we obtian

$$G_p(\Lambda_p^\circ K) \le n\omega_n^{\frac{n-p}{n}} V(K)^{\frac{p}{n}}.$$

Above all, we yield equality (27).

Proof of Theorem 1.4. From (27) and (11), it follows that

$$\begin{aligned} & G_p(\Lambda_p^\circ(\lambda \cdot K +_p \mu \cdot L)) \\ = \;& n\omega_n^{\frac{n-p}{n}} V(\lambda \cdot K +_p \mu \cdot L))^{\frac{p}{n}} \\ \ge \;& \lambda n\omega_n^{\frac{n-p}{n}} V(K)^{\frac{p}{n}} + \mu n\omega_n^{\frac{n-p}{n}} V(L)^{\frac{p}{n}} \\ = \;& \lambda G_p(\Lambda_p^\circ K) + \mu G_p(\Lambda_p^\circ L). \end{aligned}$$

From the equality condition of inequality (11), we know that equality holds in (6) for $p = 1$ if and only if K and L are homothetic, and for $p > 1$ if and only if K and L are dilates.

According to (6) and (12), we easily get that if $K \in \mathcal{K}_c^n$ and $p \ge 1$, then

$$G_p(\Lambda_p^\circ(\Delta_p K)) = G_p(\Lambda_p^\circ K).$$

Proof of Theorem 1.5. It follows from (27) and (14) that for $1 \le p \le n$,

$$\begin{aligned} & G_p(\Lambda_p^\circ(\lambda \circ K \widetilde{+}_p \mu \circ L)) \\ = \;& n\omega_n^{\frac{n-p}{n}} V(\lambda \circ K \widetilde{+}_p \mu \circ L))^{\frac{p}{n}} \\ \le \;& \lambda n\omega_n^{\frac{n-p}{n}} V(K)^{\frac{p}{n}} + \mu n\omega_n^{\frac{n-p}{n}} V(L)^{\frac{p}{n}} \\ = \;& \lambda G_p(\Lambda_p^\circ K) + \mu G_p(\Lambda_p^\circ L). \end{aligned}$$

The reverse inequality holds when $p > n$. From the equality condition of inequality (14), we know that equality holds in (7) when $p \ne n$ if and only if K is a dilate of L.

Together (7) with (15), we easily get that if $K \in \mathcal{K}_c^n$ and $p \ne n$, then

$$G_p(\Lambda_p^\circ(\widetilde{\Delta}_p K)) = G_p(\Lambda_p^\circ K).$$

Proof of Theorem 1.6. By (27) and (20), we have

$$\begin{aligned} & G_p(\Lambda_p^\circ(\lambda * K \widehat{+}_p \mu * L))^{-1} \\ = \;& \big(n\omega_n^{\frac{n-p}{n}}\big)^{-1} V(\lambda * K \widehat{+}_p \mu * L))^{-\frac{p}{n}} \\ \ge \;& \lambda\big(n\omega_n^{\frac{n-p}{n}}\big)^{-1} V(K)^{-\frac{p}{n}} + \mu\big(n\omega_n^{\frac{n-p}{n}}\big)^{-1} V(L)^{-\frac{p}{n}} \\ = \;& \lambda G_p(\Lambda_p^\circ K)^{-1} + \mu G_p(\Lambda_p^\circ L)^{-1}. \end{aligned}$$

From the equality condition of inequality (20), we know that equality holds in (8) if and only if K and L are dilates.

An immediate consequence of Theorem 1.6 is:

Corollary 3.3. *With the same assumptions of Theorem I, if $\lambda, \mu > 0$, then*

$$4G_p(\Lambda_p^\circ(\lambda * K \widehat{+}_p \mu * L)) \le \frac{1}{\lambda} G_p(\Lambda_p^\circ K) + \frac{1}{\mu} G_p(\Lambda_p^\circ L), \quad (28)$$

with equality if and only if K and L are dilates each other.

Proof. Using Cauchy's inequality and the arithmetic mean-harmonic mean inequality in (8), we have

$$\begin{aligned} & G_p(\Lambda_p^\circ(\lambda * K \widehat{+}_p \mu * L)) \\ \le \;& \frac{1}{\lambda G_p(\Lambda_p^\circ K)^{-1} + \mu G_p(\Lambda_p^\circ L)^{-1}} \\ \le \;& \frac{1}{4\lambda} G_p(\Lambda_p^\circ K) + \frac{1}{4\lambda} G_p(\Lambda_p^\circ L). \end{aligned}$$

This yields the desired inequality.

Combining (8) with (21), we easily get that if $K \in \mathcal{K}_c^n$ and $p \ge 1$, then

$$G_p(\Lambda_p^\circ(\widehat{\triangle}_p K)) = G_p(\Lambda_p^\circ K).$$

Lemma 3.4. *For $n \ne p \ge 1$, the mapping $\Lambda_p : \mathcal{F}_c^n \to \mathcal{S}_c^n$ is bijective.*

Proof. For the case $p = 1$, since $\Lambda = \Lambda_1^\circ$ is the classical curvature image and $\Lambda : \mathcal{S}_c^n \to \mathcal{F}_c^n$ is a bijection (see [14], p.50), Λ_1° is a bijection. For $n \ne p > 1$, $\Lambda_p^\circ : \mathcal{S}_c^n \to \mathcal{F}_c^n$ was proved in Proposition 3.6 of [31] that it is also a bijection. Thus for $n \ne p \ge 1$, $\Lambda_p^\circ : \mathcal{S}_c^n \to \mathcal{F}_c^n$ is bijective. From the definition of the L_p-polar curvature image Λ_p°, we know that it is the inverse of the L_p-curvature image Λ_p. This implies that Λ_p is a bijection on the class of origin-symmetric bodies for $n \ne p \ge 1$.

Proof of Theorem 1.7. It follows from (5) that $\Lambda_p^\circ = \Lambda_p^{-1}$ is the inverse image of Λ_p. By Lemma 3.4, equation (27) and inequality (25), we have

$$\begin{aligned} & G_p(\lambda K \check{+}_p \mu L) \\ = \;& G_p(\Lambda_p^\circ \Lambda_p(\lambda K \check{+}_p \mu L) \\ = \;& n\omega_n^{\frac{n-p}{n}} V(\Lambda_p(\lambda K \check{+}_p \mu L))^{\frac{p}{n}} \\ \ge \;& \lambda n\omega_n^{\frac{n-p}{n}} V(\Lambda_p K)^{\frac{p}{n}} + \mu n\omega_n^{\frac{n-p}{n}} V(\Lambda_p L)^{\frac{p}{n}} \\ = \;& \lambda G_p(K) + \mu G_p(L). \end{aligned}$$

From the equality condition of (25), we know that equality holds in (9) for $p = 1$ if and only if K and L are homothetic, and for $p > 1$ if and only if K and L are dilates.

By (9) and (24), we easily get that if $K \in \mathcal{F}_c^n$ and $n \ne p \ge 1$, then

$$G_p(\nabla_p K) = G_p(K).$$

IV. The Equivalence of Several Inequalities

Define

$$\begin{aligned} \mathcal{M}_p^n = \{K \in \mathcal{F}_o^n : & \text{ there exists a } Q \in \mathcal{K}_o^n \\ & \text{with } f_p(K, \cdot) = h(Q, \cdot)^{-(n+p)}\}, \end{aligned}$$

and call it the p-elliptic type if $K \in \mathcal{M}_p^n$ (see [10]).

The following lemma is a direct consequence of Lemma 1.3.

Lemma 4.1. *Suppose $K \in \mathcal{M}_p^n$ and $p \ge 1$, then*

$$\Omega_p(K)^{n+p} = (n\omega_n)^p G_p(K)^n. \quad (29)$$

Let $\mathcal{F}_e^n$ denote the set of all bodies in $\mathcal{K}_e^n$ which has a positive continuous curvature function. Combining inequality (2) with inequality (3), it follows from Lemma 4.1 that

Theorem 4.2. *Suppose $K \in \mathcal{F}_e^n$ and $p \ge 1$. If $K \in \mathcal{M}_p^n$, then inequality (3) is equivalent to inequality (2).*

Lutwak [10] proved the following Blaschke-Santaló type inequality for p-affine surface area (Theorem 4.10 in [10]): If $p \ge 1$ and $K \in \mathcal{K}_e^n$, then

$$\Omega_p(K)\Omega_p(K^*) \le (n\omega_n)^2, \quad (30)$$

with equality if and only if K is an ellipsoid.

From (29) and (30), we get the following Blaschke-Santaló type inequality for p-geominimal surface area.

Theorem 4.3. *For $p \geq 1$ and $K \in \mathcal{K}_e^n$, if $K \in \mathcal{M}_p^n$, then*

$$G_p(K)G_p(K^*) \leq (n\omega_n)^2, \tag{31}$$

with equality if and only if K is an ellipsoid.

If $p \geq 1$ and $K \in \mathcal{K}_o^n$, then there exists a unique body $T_pK \in \mathcal{K}_o^n$ such that(see see Proposition 3.3 in [10])

$$G_p(K) = nV_p(K, T_pK) \quad \text{and} \quad V(T_p^*K) = \omega_n.$$

A body in $\mathcal{K}_o^n$ will be called p-selfminimal if T_pK and K are dilates of each other.

For $K \in \mathcal{K}_o^n$, Lutwak [10] defined the p-geominimal area ratio of K by

$$\left(\frac{G_p(K)^n}{n^nV(K)^{n-p}}\right)^{1/p},$$

and proved that the p-geominimal area ratios are monotone non-decreasing in p (see Theorem 6.3 in [10]): If $K \in \mathcal{K}_o^n$, and $1 \leq p \leq q$, then

$$\left(\frac{G_p(K)^n}{n^nV(K)^{n-p}}\right)^{1/p} \leq \left(\frac{G_q(K)^n}{n^nV(K)^{n-q}}\right)^{1/q}, \tag{32}$$

with equality if and only if K is p-selfminimal.

For $K \in \mathcal{K}_o^n$, Lutwak [10] defined the p-affine area ratio of K by

$$\left(\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{1/p},$$

and also obtained that the p-affine area ratios are monotone non-decreasing in p (see Proposition 5.13 in [10]): If $K \in \mathcal{F}_o^n$, and $1 \leq p \leq q$, then

$$\left(\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{1/p} \leq \left(\frac{\Omega_q(K)^{n+q}}{n^{n+q}V(K)^{n-q}}\right)^{1/q}, \tag{33}$$

with equality if and only if K^* and Λ_pK are dilates .

The equation (29) implies that if $K \in \mathcal{M}_p^n$ and $p \geq 1$, then

$$\left(\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{1/p} = \omega_n\left(\frac{G_p(K)^n}{n^nV(K)^{n-p}}\right)^{1/p}. \tag{34}$$

It is clear from (34) that for $K \in \mathcal{M}_p^n$ inequality (32) and inequality (33) are equivalent.

Lutwak proved the following inequalities (35) and (36) for the p-affine area ratio of K and the p-geominimal area ratio of K. Obviously, they are also equivalent for $K \in \mathcal{M}_p^n$.

If $K \in \mathcal{F}_o^n$, and $p \geq 1$, then (see Proposition 4.7 in [10])

$$\left(\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{1/p} \leq V(K)V(K^*), \tag{35}$$

with equality if and only if K^* and Λ_pK are dilates .

If $K \in \mathcal{K}_o^n$, and $p \geq 1$, then (see Proposition 6.2 in [10])

$$\left(\frac{G_p(K)^n}{n^nV(K)^{n-p}}\right)^{1/p} \leq V(K)V(K^*)/\omega_n, \tag{36}$$

with equality if and only if K is p-selfminimal.

We note that due to equality (29), Theorems 1.4-1.7 have obvious analogs for the p-affine surface area.

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