IAENG International Journal of Applied Mathematics, 46:1, IJAM_46_1_15

The k-Path Vertex Cover in Product Graphs of Stars and Complete Graphs*

Liancui Zuo†, Bitao Zhang, and Shaoqiang Zhang

Abstract

For a graph G and a positive integer k, a subset S of vertices of G is called a k-path vertex cover if every path of order k in G contains at least one vertex from S. The cardinality of a minimum k-path vertex cover is denoted by $\psi_k(G)$. In this paper, we present the exact values of ψ_k in some product graphs of stars and complete graphs.

Keywords: k-path vertex cover; cartesian product; lexicographic product; strong product; direct product

1 Introduction

Let x be a real number, denoted by $\lfloor x \rfloor$ the maximum integer no more than x, and denoted by $\lceil x \rceil$ the minimum integer no less than x. Let G be a finite, simple and undirected graph, $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For a subset $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. The order of a path P_n is the number n of vertices while the length is the number $n-1$ of edges. For nonnegative integers a, b, let $[a, b] = \{a, a+1, \cdots, b\}$ if $a \leq b$, and $[a, b] = \emptyset$ if $a > b$.

In recent years, many parameters and classes of graphs were studied. For example, in [8], different properties of the intrinsic order graph are obtained, namely those dealing with its edges, chains, shadows, neighbors and degrees of its vertices, and some relevant subgraphs, as well as the natural isomorphisms between them. In [10], the n-dimensional cube-connected complete graph is studied. In [22], the multi-level distance number for a class of Lobster-like trees are researched. In [23, 24], the linear 4-arboricity of some complete bipartite graphs and the linear $(n-1)$-arboricity of some Cartesian product graphs are obtained.

For a graph G and a positive integer k, a subset S of the vertex set of G is called a k-path vertex cover if every path of order k in G contains at least one vertex from S. The set S is also called the set of covered vertices in a k-path vertex cover of G and we call $T = V(G) - S$ the set of uncovered vertices. The cardinality of a minimum k-path vertex cover is denoted by $\psi_k(G)$.

The motivation for the k-path vertex cover, which was introduced in [13], arises from secure communications in wireless sensor networks, as well as in traffic control. The topology of wireless sensor networks can be modeled as a graph, in which vertices represent sensor devices and edges represent communication channels between pairs of sensor devices. Traditional security techniques cannot be applied directly to wireless sensor networks since sensor devices are limited in their computation, energy, and communication capabilities. Furthermore, they are often deployed in accessible areas, where they can be captured by an attacker. Generally speaking, a standard sensor device is not taken into account as tamper-resistant and it is unnecessary to make all devices of a sensor network tamper-proof due to increasing cost. Hence, the design of wireless sensor networks safety contracts has become a challenge in security research. We focus on the Canvas scheme [6, 13, 14, 17] which should provide data integrity in a sensor network. The scheme combines the properties of cryptographic primitives and the network topology.

The model of communications in wireless sensor networks is just equivalent to the traffic control that is formulated in [19]. This problem also has its background in the real word. The increasing numbers of cars and buses lead to more and more traffic accidents, hence posing the installment of cameras to be in an urgent state. If every crossing is installed with several cameras, the cost would be enormous and unnecessary, since the installing fees can vary greatly because of different factors. Hence we need to install cameras at certain crossings which make sure that a driver will encounter at least one camera within n crossings. At the same time, we need to guarantee the lowest cost. This practical problem can, then, be turned into the k-path vertex cover problem.

The concept of k-path vertex cover is a generalization of the vertex cover. Clearly, $\psi_2(G)$ corresponds to the size of a minimum vertex cover, moreover

$$\psi_2(G) = |V(G)| - \alpha(G),$$

where $\alpha(G)$ stands for the independence number of graph G. This gives an interesting connection to the well studied independence number [9].

A subset of vertices in graph G is called a dissociation set if it induces a subgraph with maximum degree at most 1. The number of vertices in a maximum cardinality set in G is called the dissociation number of G and is denoted by $diss(G)$. The dissociation number problem is studied in several articles [1, 2, 5, 7], and a survey for this results is given in [15]. The value of $\psi_3(G)$ is in close relation to $diss(G)$ because it is easy to see that

$$\psi_3(G) = |V(G)| - diss(G).$$

*Manuscript received August 21,2015. This work was supported by NSFC for youth with code 61103073.

†Liancui Zuo, Bitao Zhang and Shaoqiang Zhang are with College of Mathematical Science, Tianjin Normal University, Tianjin, 300387, China. Email: lczuo@mail.tjnu.edu.cn;lczuo@163.com

(Advance online publication: 15 February 2016)

Some approximation algorithms for $\psi_3(G)$ are studied in [18, 19, 20]. In [12] an exact algorithm for computing $\psi_3(G)$ in running time $O(1.5171^n)$ for a graph of order n is presented.

The problem of computing $\psi_k(G)$ is in general NP-hard for any fixed integer $k \geq 2$, but for tree the problem can be solved in linear time, as shown in [3]. The authors also gave some upper bounds on the value of $\psi_k(G)$ and provide several estimations and the exact value of $\psi_k(G)$.

The concept of the k-path vertex cover was also studied in different graph products. The Cartesian product $G\Box H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u_1, v_1), (u_2, v_2)$ are adjacent whenever $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and $v_1 = v_2$.

The lexicographic product $G \circ H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u_1, v_1), (u_2, v_2)$ are adjacent whenever $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

The direct product $G \times H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u_1, v_1), (u_2, v_2)$ are adjacent whenever $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

The strong product $G \boxtimes H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u_1, v_1), (u_2, v_2)$ are adjacent whenever $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

The modular product $G \diamond H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u_1, v_1), (u_2, v_2)$ are adjacent whenever $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$, or $u_1u_2 \notin E(G)$ and $v_1v_2 \notin E(H)$.

Let G and H be arbitrary graphs, for a fixed vertex $v \in V(H)$, we refer to the set $V(G) \times \{v\}$ as a G-layer. Similarly $\{u\} \times V(H)$, for a fixed vertex $u \in V(G)$, is an H-layer. Whenever referring to a specific G- or H-layer, we denote them by G^v or ${}^uH$, respectively. Layers can also be regarded as the graphs induced on these sets. It is clear that in the Cartesian and lexicographic products, a G-layer or H-layer is isomorphic to G or H, respectively.

For the Cartesian product of two paths, an asymptotically tight bound and the exact value for ψ_3 are given in [4], and some bounds are improved in [11] and extended to the strong product of two paths. Also, an upper bound for ψ_3 and a lower bound of ψ_k of regular graphs are presented in [4]. For the lexicographic product of two arbitrary graphs, some results are also given in [11], and a good lower and an upper bounds for ψ_k, ψ_2 and ψ_3 are presented in [3].

2 Main results

Let S_m denote the star graph, whose vertex set $V(S_m) = \{u_1, u_2, \cdots, u_m\}$ and $d(u_1) = m - 1$ while $d(u_i) = 1$ for $2 \leq i \leq m$. Similarly, let K_n denote the complete graph, whose vertex set $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. In this paper, we present the exact values of $\psi_k(S_m\Box K_n)$, $\psi_k(S_m \circ K_n)$, $\psi_k(S_m \boxtimes K_n)$, $\psi_k(S_m \diamond K_n)$, and $\psi_k(S_m \times K_n)$, respectively.

Firstly, we give three lammas. It is obvious that the following result holds.

Lemma 2.1. *For any positive integers k and n with $2 \leq k \leq n$, we have*

$$\psi_k(P_n) = \lfloor \frac{n}{k} \rfloor,$$
$$\psi_k(C_n) = \lceil \frac{n}{k} \rceil,$$
$$\psi_k(K_n) = n - k + 1.$$

Lemma 2.2. *If H is a subgraph of G and k is a positive integer, then*

$$\psi_k(G) \geq \psi_k(H).$$

This is trivial since we can obtain one k-path vertex cover $S \cap V(H)$ of H from every k-path vertex cover S of G for every subgraph H of G.

Clearly, $\psi_1(G) = |V(G)|$ and $\psi_k(G) = 0$ for any graph G and each integer $k > |V(G)|$, so we always suppose that $2 \leq k \leq |V(G)|$ for $\psi_k(G)$ in the sequel.

Lemma 2.3. *[21] If $n \geq 2$ and $\lceil \frac{n}{2} \rceil + 1 \leq k \leq n + 1$, then $\psi_k(P_2\Box K_n) = n$.*

In the following, we provide the exact value of $\psi_k(S_m\Box K_n)$ at first.

Theorem 2.4. *For positive integers $m \geq 3$ and $n \geq 2$, the following results hold.*
(1)If $2 \leq k \leq \lceil \frac{n}{2} \rceil$, then $\psi_k(S_m\Box K_n) = m(n - k + 1)$.
(2)If $\lceil \frac{n}{2} \rceil + 1 \leq k \leq n + 1$, then

$$\psi_k(S_m\Box K_n) = n + (m - 2)(n - k + 1).$$

(3)If $m \leq n + 1$ and $k \in [n + 2, mn - n + m - 2]$, or $m \geq n + 2$ and $k \in [n + 2, n^2 + 2n]$, then

$$\psi_k(S_m\Box K_n) = n + 1 - \lfloor \frac{k}{n + 1} \rfloor.$$

(4)If $m \leq n + 1$ and $k \in [(m - 1)(n + 1), mn]$, then

$$\psi_k(S_m\Box K_n) = mn - k + 1.$$

(5)If $m \geq n + 2$ and $k \in [(n + 1)^2, mn]$, then

$$\psi_k(S_m\Box K_n) = 0.$$

Proof. (1)Let $S_1 = \{(u_1, v_j) \in V(S_m\Box K_n)|j \in [k, n]\}$ with $|S_1| = n-k+1$ and $S_i = \{(u_i, v_j) \in V(S_m\Box K_n)|1 \leq j \leq n - k + 1\}$ with $|S_i| = n - k + 1$, where $2 \leq i \leq m$. It is clear that $S = \cup_{i=1}^{m} S_i$ is a k-path vertex cover since the largest connected component induced by all vertices uncovered is isomorphic to K_{k-1}. Therefore, $\psi_k(S_m\Box K_n) \le |S| = m(n-k+1)$.

On the other hand, each layer ${}^{u_i}K_n$ is isomorphic to K_n and $S_m\Box K_n$ has m such layers, where $1 \le i \le m$. So, we have $\psi_k(S_m\Box K_n) \ge m\psi_k(K_n) = m(n-k+1)$. Thus,

$$\psi_k(S_m\Box K_n) = m(n-k+1)$$

for $2 \le k \le \lceil \frac{n}{2} \rceil$.

(2)Let $S_1 = \{(u_1, v_j) \in V(S_m\Box K_n) | 1 \le j \le k-1\}$ with $|S_1| = k-1$ and $S_i = \{(u_i, v_j) \in V(S_m\Box K_n) | k \le j \le n\}$ with $|S_i| = n-k+1$ for $2 \le i \le m$. It is obvious that $S = \cup_{i=1}^{m} S_i$ is a k-path vertex cover since the largest connected subgraph of $S_m\Box K_n$ induced by all vertices uncovered is isomorphic to K_{k-1}. Therefore,

$$\begin{aligned}\psi_k(S_m\Box K_n) &\le |S| \\ &= k-1+(m-1)(n-k+1) \\ &= n+(m-2)(n-k+1).\end{aligned}$$

On the other hand, we delete all edges between the layers ${}^{u_1}K_n$ and ${}^{u_i}K_n$, where $3 \le i \le m$. The graph $S_m\Box K_n$ can be partitioned into a subgraph isomorphic to $P_2\Box K_n$ and $(m-2)$ subgraphs isomorphic to K_n. According to Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\psi_k(S_m\Box K_n) &\ge \psi_k(P_2\Box K_n) + (m-2)\psi_k(K_n) \\ &= n+(m-2)(n-k+1).\end{aligned}$$

Thus, (2) is proved.

(3)Firstly, we will construct a k-path vertex cover with $n+1-\lfloor \frac{k}{n+1} \rfloor$ vertices to prove that $\psi_k(S_m\Box K_n) \le n+1-\lfloor \frac{k}{n+1} \rfloor$. Let $G = S_m\Box K_n$ and $S = \{(u_1, v_j) \in V(S_m\Box K_n) | \lfloor \frac{k}{n+1} \rfloor \le j \le n\}$ with $|S| = n+1-\lfloor \frac{k}{n+1} \rfloor$. Clearly, every path P_{n+1} in $S_m\Box K_n$ contains at least one vertex that belongs to $V({}^{u_1}K_n)$, so graph $G[V(G)-S]$ contains $\lfloor \frac{k}{n+1} \rfloor - 1$ vertices which belong to $V({}^{u_1}K_n)$, thus the largest connected subgraph of $G[V(G)-S]$ has order at most

$$n+(n+1)(\lfloor \frac{k}{n+1} \rfloor - 1) \le n+(n+1)(\frac{k}{n+1} - 1) = k-1.$$

Therefore, S is a k-path vertex cover of $S_m\Box K_n$ and then $\psi_k(S_m\Box K_n) \le |S| = n+1-\lfloor \frac{k}{n+1} \rfloor$.

Secondly, we show that $\psi_k(S_m\Box K_n) \ge n+1-\lfloor \frac{k}{n+1} \rfloor$. Assume to the contrary that T is a k-path vertex cover of $\psi_k(S_m\Box K_n)$ with $|T| \le n-\lfloor \frac{k}{n+1} \rfloor$. Let $T_i = T \cap V({}^{u_i}K_n)$ and $n_i = |T_i|$, where $1 \le i \le m$. It is easy to see that $T = \cup_{i=1}^{m} T_i$ and $|T| = \sum_{i=1}^{m} n_i$. Since $n-n_i \ge n-|T| \ge a$ for $a = \lfloor \frac{k}{n+1} \rfloor$, there are at least a vertices which not belong to T in each layer ${}^{u_i}K_n$ for $1 \le i \le m$. By the symmetry of vertices $u_2, \cdots, u_m$ in graph S_m, we assume that $n_2 \ge n_3 \ge \cdots \ge n_l \ge 1$ and $n_j = 0$ for $j \in [l+1, m]$, where $2 \le l \le m$. If $m \le n$ and $n+2 \le k \le mn+m-n-2$, then $a+2 = \lfloor \frac{k}{n+1} \rfloor + 2 \le \lfloor \frac{(m-2)(n+1)+n}{n+1} \rfloor + 2 = m$. If $m \ge n+1$ and $n+2 \le k \le n^2+2n$, then $a+2 = \lfloor \frac{k}{n+1} \rfloor + 2 \le \lfloor \frac{n(n+1)+n}{n+1} \rfloor + 2 = n+2 \le m$. Therefore, there are at least $a+2$ K_n-layers in $S_m\Box K_n$ in both cases. We only need to show that all vertices which belong to vertex set $\cup_{i=1}^{a+2}(V({}^{u_i}K_n) - T_i)$ can form a path since

$$\begin{aligned}|\cup_{i=1}^{a+2}(V({}^{u_i}K_n) - T_i)| &= \textstyle\sum_{i=1}^{a+2}(n-|T_i|) \\ &\ge n(a+2) - \textstyle\sum_{i=1}^{m}|T_i| \\ &\ge n(a+2) - (n-a) \\ &= \lfloor \frac{k}{n+1} \rfloor (n+1) + n \\ &\ge \frac{k-n}{n+1}(n+1) + n \\ &= k.\end{aligned}$$

Clearly, if $a = 1$, then all vertices which belong to vertex set $\cup_{i=1}^{3}(V({}^{u_i}K_n) - T_i)$ can form a path. We assume that $a \ge 2$ and construct such a path P in two cases.

Case 1. $l \ge a+2$.

Since $n_1+n_2+n_3 \le \sum_{i=1}^{m} n_i \le n-1$, there are three vertices (u_1, v_{y_2}), (u_2, v_{y_2}), $(u_3, v_{y_2}) \notin T$. Lying in the layer ${}^{u_2}K_n$, all the vertices which are not covered by T can form a path P_2 with terminate vertex (u_2, v_{y_2}), where $1 \le y_2 \le n$. Since $1+n_1+n_3+n_4 \le \sum_{i=1}^{m} n_i \le n-1$, there are three vertices (u_1, v_{y_3}), (u_3, v_{y_3}), $(u_4, v_{y_3}) \notin T$. Lying in the layer ${}^{u_3}K_n$, all vertices which are not covered by T can form a path P_3 with original vertex (u_3, v_{y_2}) and terminate vertex (u_3, v_{y_3}), where $1 \le y_3 \le n$ and $y_3 \ne y_2$. $\cdots$ Lying in the layer ${}^{u_{a+2}}K_n$, all vertices which are not covered by T can form a path P_{a+2} with the original vertex $(u_{a+2}, v_{y_{a+1}})$, where $1 \le y_{a+1} \le n$ and $y_{a+1} \ne y_i$ for $2 \le i \le a$. Let $V_1 = \{(u_1, v_{y_2}), (u_1, v_{y_3}), \cdots, (u_1, v_{y_{a+1}})\}$ and $V_2 = (V({}^{u_1}K_n) - T_1 - V_1) \cup \{(u_1, v_{y_a}), (u_1, v_{y_{a+1}})\}$. All vertices which belong to vertex set V_2 can form a path P_1 with the originate vertex (u_1, v_{y_a}) and the terminate vertex $(u_1, v_{y_{a+1}})$. Set

$$\begin{aligned}P = P_2 &+ (u_2, v_{y_2})(u_1, v_{y_2}) + (u_1, v_{y_2})(u_3, v_{y_2}) \\ &+ P_3 + (u_3, v_{y_3})(u_1, v_{y_3}) + \cdots + P_{a+1} \\ &+ (u_{a+1}, v_{y_a})(u_1, v_{y_a}) + P_1 \\ &+ (u_1, v_{y_{a+1}})(u_{a+2}, v_{y_{a+1}}) + P_{a+2}.\end{aligned}$$

We have a path P of order at least k with no vertex that belongs to T, a contradiction.

Case 2. $l < a+2$.

Since $n_1+n_2+n_3 \le \sum_{i=1}^{m} n_i \le n-1$, there are three vertices (u_1, v_{y_2}), (u_2, v_{y_2}), $(u_3, v_{y_2}) \notin T$. Lying in the layer ${}^{u_2}K_n$, all the vertices which are not covered by T can form a path P_2 with the terminate vertex (u_2, v_{y_2}), where $1 \le y_2 \le n$. Since $1+n_1+n_3+n_4 \le \sum_{i=1}^{m} n_i \le n-1$, there are three vertices (u_1, v_{y_3}), (u_3, v_{y_3}), $(u_4, v_{y_3}) \notin T$. Lying in the layer ${}^{u_3}K_n$, all vertices which are not covered by T can form a path P_3 with the original vertex (u_3, v_{y_2}) and the terminate vertex (u_3, v_{y_3}), where $1 \le y_3 \le n$ and $y_3 \ne y_2$. $\cdots$ Since

$$l-3+n_1+n_{l-1}+n_l \le \sum_{i=1}^{m} n_i \le n-1,$$

there are three vertices (u_1, v_{y_l}), (u_l, v_{y_l}), $(u_{l+1}, v_{y_l}) \notin T$. Lying in the layer ${}^{u_l}K_n$, all the vertices can form a path P_l with the original vertex $(u_l, v_{y_{l-1}})$ and the terminate vertex (u_l, v_{y_l}), where $1 \le y_l \le n$ and $y_l \ne y_i$ for $2 \le i \le$

$l-1$. Lying in the layer ${}^{u_{l+1}}K_n$, all vertices which are not covered by T can form a path P_{l+1} with the original vertex (u_{l+1}, v_{y_l}) and the terminate vertex $(u_{l+1}, v_{y_{l+1}})$, where $1 \le y_{l+1} \le n$, $(u_1, v_{y_{l+1}}) \notin T$ and $y_{l+1} \ne y_i$ for $2 \le i \le l$. $\cdots$ Lying in the layer ${}^{u_{a+2}}K_n$, all the vertices can form a path P_{a+2} with the original vertex $(u_{a+2}, v_{y_{a+1}})$, where $1 \le y_{a+1} \le n$, $(u_1, v_{y_{a+1}}) \notin T$ and $y_{a+1} \ne y_i$ for $2 \le i \le a$. Let

$$V_1 = \{(u_1, v_{y_2}), (u_1, v_{y_3}), \cdots, (u_1, v_{y_{a+1}})\}$$

and

$$V_2 = (V({}^{u_1}K_n) - T_1 - V_1) \cup \{(u_1, v_{y_a}), (u_1, v_{y_{a+1}})\}.$$

All vertices which belong to vertex set V_2 can form a path P_1 with the originate vertex (u_1, v_{y_a}) and the terminate vertex $(u_1, v_{y_{a+1}})$. Set

$$\begin{aligned} P = & P_2 + (u_2, v_{y_2})(u_1, v_{y_2}) + (u_1, v_{y_2})(u_3, v_{y_2}) \\ & + P_3 + (u_3, v_{y_3})(u_1, v_{y_3}) + \cdots + P_{a+1} \\ & + (u_{a+1}, v_{y_a})(u_1, v_{y_a}) + P_1 \\ & + (u_1, v_{y_{a+1}})(u_{a+2}, v_{y_{a+1}}) + P_{a+2}. \end{aligned}$$

We have a path P of order at least k with no vertex that belong to T, a contradiction, too.

(4)Let S be a k-path vertex cover of $S_m \square K_n$ with $|S| = mn - k + 1$. Since $|V(S_m \square K_n)| - |S| = k - 1$, we have $\psi_k(S_m \square K_n) \le mn - k + 1$.

Next we prove that $\psi_k(S_m \square K_n) \ge mn - k + 1$. Assume to the contrary that T is a k-path vertex cover of $\psi_k(S_m \square K_n)$ with $|T| \le mn - k$. Let $T_i = T \cap V({}^{u_i}K_n)$ with $|T_i| = n_i$, where $1 \le i \le m$. It is easy to see that $T = \cup_{i=1}^{m} T_i$ and $|T| = \sum_{i=1}^{m} n_i$. Since $n - n_i \ge n - |T| \ge n - mn + k \ge n - mn + (m-1)(n+1) = m - 1$, there are at least $m-1$ vertices which not belong to T in each layer ${}^{u_i}K_n$ for $1 \le i \le n$. It is clear that $|V(S_m \square K_n) - T| \ge mn - (mn - k) = k$, and we can show that all vertices of $V(S_m \square K_n) - T$ can form a path as (3) similarly, a contradiction.

(5)Since every path in $S_m \square K_n$ with order $n+1$ has at least one vertex that belongs to $V({}^{u_1}K_n)$, every path of $S_m \square K_n$ has order at most

$$n + n(n+1) = n^2 + 2n \le k - 1.$$

Therefore, $\psi_k(S_m \square K_n) = 0$. □

In the next theorem, we study $S_m \circ K_n$, $S_m \boxtimes K_n$, and $S_m \diamond K_n$.

Theorem 2.5. *For positive integers $m \ge 3$ and $n \ge 2$, the following results hold.*
(1)If $2 \le k \le n+1$, then

$$\psi_k(S_m \circ K_n) = \psi_k(S_m \boxtimes K_n)$$
$$= \psi_k(S_m \diamond K_n) = n + (m-1)(n-k+1).$$

(2)If $m \le n+1$ and $n+2 \le k \le mn - n + m - 2$, or $m \ge n+2$ and $n+2 \le k \le n^2 + 2n$, then

$$\psi_k(S_m \circ K_n) = \psi_k(S_m \boxtimes K_n)$$
$$= \psi_k(S_m \diamond K_n) = n + 1 - \lfloor \frac{k}{n+1} \rfloor.$$

(3)If $m \le n+1$ and $(m-1)(n+1) \le k \le mn$, then

$$\psi_k(S_m \circ K_n) = \psi_k(S_m \boxtimes K_n)$$
$$= \psi_k(S_m \diamond K_n) = mn - k + 1.$$

(4)If $m \le n+2$ and $(n+1)^2 \le k \le mn$, then

$$\psi_k(S_m \circ K_n) = \psi_k(S_m \boxtimes K_n) = \psi_k(S_m \diamond K_n) = 0.$$

Proof. It is easy to see that both $S_m \boxtimes K_n$ and $S_m \diamond K_n$ are isomorphic to $S_m \circ K_n$, thus it is only need to show that results hold for $S_m \circ K_n$.

(1)Let

$$S_1 = \{(u_1, v_j) \in V(S_m \circ K_n) | 1 \le j \le n\}$$

with $|S_1| = n$ and

$$S_i = \{(u_i, v_j) \in V(S_m \circ K_n) | k \le j \le n\}$$

with $|S_i| = n - k + 1$, where $2 \le i \le m$. It is clear that $S = \cup_{i=1}^{m} S_i$ is a k-path vertex cover since the largest connected subgraph of $S_m \circ K_n$ induced by all vertices uncovered is isomorphic to K_{k-1}. Therefore,

$$\psi_k(S_m \circ K_n) \le |S| = n + (m-1)(n-k+1).$$

On the other hand, we delete all edges between layers ${}^{u_1}K_n$ and ${}^{u_i}K_n$, where $3 \le i \le m$. The graph $S_m \circ K_n$ can be partitioned into a subgraph isomorphic to K_{2n} and $(m-2)$ subgraphs isomorphic to K_n. According to Lemma 2.2, we have

$$\begin{aligned} \psi_k(S_m \circ K_n) & \ge \psi_k(K_{2n}) + (m-2)\psi_k(K_n) \\ & = 2n - k + 1 + (m-2)(n-k+1) \\ & = n + (m-1)(n-k+1). \end{aligned}$$

(2)Firstly, we construct a k-path vertex cover with $n + 1 - \lfloor \frac{k}{n+1} \rfloor$ vertices to prove that

$$\psi_k(S_m \circ K_n) \le n + 1 - \lfloor \frac{k}{n+1} \rfloor.$$

Let $G = S_m \circ K_n$ and

$$S = \{(u_1, v_j) \in V(S_m \circ K_n) | \lfloor \frac{k}{n+1} \rfloor \le j \le n\}$$

with $|S| = n + 1 - \lfloor \frac{k}{n+1} \rfloor$. It is obvious that every path in $S_m \circ K_n$ with order $n+1$ contains at least one vertex that belongs to $V({}^{u_1}K_n)$, so graph $G[V(G) - S]$ contains $\lfloor \frac{k}{n+1} \rfloor - 1$ vertices which belong to $V({}^{u_1}K_n)$, thus the largest order of paths in $G[V(G) - S]$ is at most

$$n + (n+1)(\lfloor \frac{k}{n+1} \rfloor - 1) \le n + (n+1)(\frac{k}{n+1} - 1) = k - 1.$$

Therefore, S is a k-path vertex cover of $S_m \circ K_n$ and then

$$\psi_k(S_m \circ K_n) \le |S| = n + 1 - \lfloor \frac{k}{n+1} \rfloor.$$

Secondly, we can show that

$$\psi_k(S_m \circ K_n) \geq n+1-\lfloor\frac{k}{n+1}\rfloor$$

by Theorem 2.4 since $S_m \square K_n$ is a subgraph of $S_m \circ K_n$.

(3)Let S be a k-path vertex cover of $S_m \circ K_n$ with $|S| = mn-k+1$. Since $|V(S_m \circ K_n)|-|S| = k-1$, we have $\psi_k(S_m \circ K_n) \leq mn-k+1$.

We can obtain that $\psi_k(S_m \circ K_n) \geq mn-k+1$ as (2) similarly.

(4)Since every path in $S_m \circ K_n$ with order $n+1$ has at least one vertex that belongs to ${}^{u_1}K_n$, the largest order of paths in $S_m \circ K_n$ is at most

$$n+n(n+1) = n^2+2n \leq k-1.$$

Therefore, $\psi_k(S_m \circ K_n) = 0$. □

Finally, we present the exact value of $\psi_k(S_m \times K_n)$. Before giving the main result, we show the following lemma first.

Lemma 2.6. *If $n \geq 3$ and $4 \leq k \leq 2n-1$, then*

$$\psi_k(P_2 \times K_n) \geq n+1-\lfloor\frac{k}{2}\rfloor.$$

Proof. Assume to the contrary that T is a k-path vertex cover of the graph $P_2 \times K_n$ with $|T| = n-\lfloor\frac{k}{2}\rfloor$. Let $T_i = T \cap V({}^{u_i}K_n)$ with $n_i = |T_i|$ for $i = 1, 2$. Because of the symmetry of two layers in graph $P_2 \times K_n$, we may assume that $n_1 \geq n_2$. Since $n_1+n_2 = |T| = n-\lfloor\frac{k}{2}\rfloor$, we have

$$n_2 \leq \lfloor\frac{n-\lfloor\frac{k}{2}\rfloor}{2}\rfloor \leq \lfloor\frac{n-2}{2}\rfloor \leq \lfloor\frac{2n-5}{2}\rfloor = n-3.$$

Let $a = n-|T_1|$ and $b = n-|T_2|$. It is easy to see that

$$n \geq a = n-|T_1| \geq n-|T| = \lfloor\frac{k}{2}\rfloor.$$

According to the symmetry of vertices in each layer of graph $P_2 \times K_n$, we may assume that $V({}^{u_1}K_n)-T_1 = \{(u_1, v_1), (u_1, v_2), \cdots, (u_1, v_a)\}$. Next we discuss on a in two cases.

Case 1. $a = \lfloor\frac{k}{2}\rfloor$.

Then $T_1 = T$ and $T_2 = \emptyset$.

Let

$$P = (u_2, v_2)(u_1, v_1)(u_2, v_3)(u_1, v_2)(u_2, v_4)$$
$$(u_1, v_3)(u_2, v_5)\cdots(u_1, v_a)(u_2, v_{a+2})$$

with $a+2 = \lfloor\frac{k}{2}\rfloor+2 \leq \lfloor\frac{2n-1}{2}\rfloor+2 = n+1$, where indices are taken modulo n. Since $|V(P)| = 2a+1 = 2\lfloor\frac{k}{2}\rfloor+1 \geq k$, we have a path P of order at least k with no vertex that belongs to T, a contradiction.

Case 2. $\lfloor\frac{k}{2}\rfloor+1 \leq a \leq n$.

Let $c = \lfloor\frac{k}{2}\rfloor+1$ and

$$V({}^{u_2}K_n)-T_2 = \{(u_2, v_{x_i})|1 \leq i \leq b,$$
$$1 \leq x_i \leq n, x_p < x_q \ for\ 1 \leq p < q \leq b\}.$$

Let $U_x = \{x_i|(u_2, v_{x_i}) \in (V({}^{u_2}K_n)-T_2)\}$ and x_l be the smallest index of x_i not less than 3, then we have $3 \leq x_l \leq n_2+3$, where $1 \leq l \leq b$ and $x_l \in U_x$. For any $d \geq 1$, we have $3+d \leq x_{l+d} \leq n_2+d+3$, where indices are taken modulo n. Since, for any integer $1 \leq j \leq c-1$, we have $x_{l+j-1} \geq j+2$ and $x_{l+j-1} \leq n_2+j+2 \leq n+j-1$, $(u_2, v_{x_{l+j-1}})$ is incident with (u_1, v_j) and (u_1, v_{j+1}). Let

$$P = (u_1, v_1)(u_2, v_{x_l})(u_1, v_2)(u_2, v_{x_{l+1}})$$
$$\cdots(u_1, v_{c-1})(u_2, v_{x_{c+l-2}})(u_1, v_c),$$

where $x_{c+l-2} \leq n_2+c+1 \leq n_1+a+1 = n+1$ and indices are taken modulo n. Since

$$|V(P)| = 2c-1 = 2\lfloor\frac{k}{2}\rfloor+1 \geq k,$$

we have a path P of order at least k with no vertex that belongs to T, a contradiction, too. □

By the definition, we have

$$E(S_m \times K_n) = \{(u_1, v_j)(u_i, v_l)|2 \leq i \leq m, 1 \leq j \neq l \leq n\}$$

and $|E(S_m \times K_n)| = n(n-1)(m-1)$. It is easy to see that $S_m \times K_n$ is a bipartite graph with a partition (X, Y), where $X = V({}^{u_1}K_n)$ and $Y = V(S_m \times K_n)-X$.

Theorem 2.7. *For any positive integers $m \geq 3$ and $n \geq 2$, we have*

$$\psi_k(S_m \times K_n) = \begin{cases} 2, \ if\ n = 2\ and\ 2 \leq k \leq 3, \\ 0, \ if\ n = 2\ and\ 4 \leq k \leq mn, \\ n+1-\lfloor\frac{k}{2}\rfloor, \ if\ n \geq 3\ and \\ \qquad 2 \leq k \leq 2n+1, \\ 0, \ if\ n \geq 3\ and\ 2n+2 \leq k \leq mn. \end{cases}$$

Proof. (1)If $n = 2$, then graph $S_m \times K_n$ consists of two vertex-disjoint isomorphic stars with order m. Therefore, we have $\psi_k(S_m) = 1$ and

$$\psi_k(S_m \times K_n) = 2\psi_k(S_m) = 2$$

for $2 \leq k \leq 3$. Since every path in graph $S_m \times K_n$ contains at most three vertices in this case, we have $\psi_k(S_m \times K_n) = 0$ for $4 \leq k \leq mn$.

(2)Assume that $n \geq 3$ and $2 \leq k \leq 2n+1$.

Firstly, we construct a k-path vertex cover with $n+1-\lfloor\frac{k}{2}\rfloor$ vertices to prove that

$$\psi_k(S_m \times K_n) \leq n+1-\lfloor\frac{k}{2}\rfloor.$$

Let $G = S_m \times K_n$ and

$$S = \{(u_1, v_j) \in V(S_m \times K_n)|\lfloor\frac{k}{2}\rfloor \leq j \leq n\}$$

with $|S| = n+1-\lfloor\frac{k}{2}\rfloor$. Since every edge in $S_m \times K_n$ contains one vertex that belongs to $V({}^{u_1}K_n)$, graph

$G[V(G)-S]$ contains $\lfloor\frac{k}{2}\rfloor - 1$ vertices which belong to $V({}^{u_1}K_n)$, thus the largest order of paths in $G[V(G)-S]$ is at most

$$1+2(\lfloor\frac{k}{2}\rfloor-1) \le 1+2(\frac{k}{2}-1) = k-1.$$

Therefore, S is a k-path vertex cover of $S_m \times K_n$ and then

$$\psi_k(S_m \times K_n) \le |S| = n+1-\lfloor\frac{k}{2}\rfloor.$$

Secondly we show that

$$\psi_k(S_m \times K_n) \ge n+1-\lfloor\frac{k}{2}\rfloor$$

in two cases.

Case 1. $2 \le k \le 2n-1$.

If $2 \le k \le 3$, then set $P_j = (u_2, v_j)(u_1, v_{j+1})(u_3, v_j)$ for $1 \le j \le n$, where indices are taken modulo n. Since graph $S_m \times K_n$ contains n vertex-disjoint paths P_j of order three, according to Lemma 2.2, we have

$$\psi_k(S_m \times K_n) \ge n\psi_k(P_3) \ge n = n+1-\lfloor\frac{k}{2}\rfloor.$$

Assume that $4 \le k \le 2n-1$, then we have

$$\psi_k(S_m \times K_n) \ge \psi_k(P_2 \times K_n) \ge n+1-\lfloor\frac{k}{2}\rfloor$$

according to Lemmas 2.2 and 2.6 since $P_2 \times K_n$ is a subgraph of $S_m \times K_n$.

Case 2. $2n \le k \le 2n+1$.

Let

$$P = (u_2, v_2)(u_1, v_1)(u_2, v_3)(u_1, v_2)\cdots(u_2, v_n)$$
$$(u_1, v_{n-1})(u_2, v_1)(u_1, v_n)(u_3, v_1)$$

with $|P| = 2n+1 \ge k$. Since $n \ge 3$ and $S_m \times K_n$ contains a path P with order at least k, we have

$$\psi_k(S_m \times K_n) \ge 1 = n+1-\lfloor\frac{k}{2}\rfloor.$$

(3) Assume that $n \ge 3$ and $2n+2 \le k \le mn$.

Since every edge in $S_m \times K_n$ has one vertex that belongs to $V({}^{u_1}K_n)$, the largest order of paths in $S_m \times K_n$ is at most $2n+1 \le k-1$ in this case. Therefore, $\psi_k(S_m \times K_n) = 0$. □

References

[1] V. E. Alekseev, R. Boliac, D. V. Korobitsyn, V. V. Lozin, NP-hard graph problems and boundary classes of graphs, Theoret. Comput. Sci. 389(1-2)(2007)219-236.

[2] R. Boliac, K. Cameron, V. V. Lozin, On computing the dissociation number and the induced matching number of bipartite graphs, Ars Combin. 72(2004)241-253.

[3] B. Brešar, F. Kardos, J. Katrenič, G. Semanišin, Minimum k-path cover, Discrete Appl. Math.159(12)(2011)1189-1195.

[4] B. Brešar, M. Jakovac, J. Katrenič, G. Semanišin, A. Taranenko, On the vertex k-path cover, Discrete Appl. Math. 161(2013)1943-1949.

[5] K. Cameron, P. Hell, Indepedent packings in structured graphs, Math Program. 105(2-3)(2006)201-213.

[6] D. Gollmann, Protocol analysis for concrete environments, in: EUROCAST 2005, in: LNCS, vol. 3643, Springer, Heidelberg, 2005, pp. 365-372.

[7] F. Göring, J. Harant, D. Rautenbach, I. Schiermeyer, On F-independence in graphs, Discuss. Math. Graph Theory 29(2)(2009)377-383.

[8] L. González, Edges, chains, shadows, neighbors and subgraphs in the intrinsic order graph, IAENG International Journal of Applied Mathematics, vol. 42, no. 1, pp 66-73, 2012.

[9] J. Harant, M.A. Henning, D. Rautenbach, I. Schiermeyer, The Indepence Number in Graphs of Maximum Degree Three, Discrete Math. 308(2008)5829-5833.

[10] J. Liu and X. Zhang, Cube-connected complete graphs, IAENG International Journal of Applied Mathematics, vol. 44, no. 3, pp 134-136, 2014.

[11] M. Jakovac, A. Taranenko, On the k-path vertex cover of some graph products, Discrete Math. 313(1)(2013)94-100.

[12] F. Kardoš, J. Katrenič, I. Schiermeyer, On computing the minimum 3-path vertex cover and dissociation number of graphs, Theoret. Comput. Sci. 412(2011)7009-7017.

[13] M. Ncvotný, in: P. Samarati, M. Tunstall, J. Posegga, K. Markantonakis, D. Sauveron (Eds.), Design and Analysis of a Generalized Canvas Protocol, in: information Security Theory and Practices, Security and Privacy of Pervasive Systems and Smart Devices, Springer, Berlin/Heidelberg, 2010, pp. 106-121.

[14] M. Novotný, Formal analysis of security protocols for wireless sensor networks, Tatra Mt. Math. Publ. 47 (2010) 81-97.

[15] Y. Orlovich, A. Dolguib, G. Finkec, V. Gordond, F. wernere, The complexity of dissociation set problems in graphs, Discrete Appl. Math. 159(13)(2011)1352-1366.

[16] Z. Shao, R.S. Oba, L(2,1)-labelings on the modular product of two graphs, Theoret. Comput. Sci. 487(2013)74-81.

[17] H. Vogt, Integrity preservation for communication in sensor networks. Technical Report 434, Institute for Pervasive Computing, ETH Zürich, 2004.

[18] J. Tu, F. Yang, The vertex cover P_3 problem in cubic graphs, Inform. Process. Lett. 113(13)(2013)481-485.

[19] J. Tu, W. Zhou, A factor 2 approximation algorithm for the vertex cover P3 problem, Inform. Process. Lett. 111(2011)683-686.

[20] J. Tu, W. Zhou, A primal-dual approximation algorithm for the vertex cover P_3 problem, Theoret. Comput. Sci. 412(50)(2011)7044-7048.

[21] B. Zhang and L. Zuo, The k-path vertex cover of some product graphs, WSEAS Transactions on Mathematics, accepted.

[22] Liancui Zuo, Chunhong Shang, and Hongfang Guo, The Multi-level Distance Number for Symmetric Lobster-like Trees about the Weight Center, IAENG International Journal of Applied Mathematics, vol. 45, no. 1, pp 41-50, 2015.

[23] Liancui Zuo, Shengjie He, and Ranqun Wang, The linear 4-arboricity of balanced complete bipartite graphs, IAENG International Journal of Applied Mathematics, vol. 45, no. 1, pp 23-30, 2015.

[24] Liancui Zuo, Shengjie He, and Bing Xue, The linear $(n-1)$-arboricity of Cartesian product graphs, Applicable Analysis and Discrete Mathematics, 9(2015), 13-28.