The Study on Almost Periodic Solutions for A Neutral Multi-species Logarithmic Population Model With Feedback Controls By Matrix's Spectral Theory

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***Abstract*—In this paper, a neutral multi-species logarithmic population model is investigated. By applying the matrix's spectral theory which is different from the methods employed in the literature, a set of sufficient conditions are obtained for the existence and uniqueness of almost periodic solution of the neutral multi-species logarithmic population model. The obtained sufficient conditions are given in terms of spectral radius of explicit matrices which are much different from those by the algebraic inequalities. An example is given to illustrate the feasibility and effectiveness of the obtained results. The results of this paper are completely new and generalize those of the previous studies.**

***Index Terms*—Nicholson-type system, positive solution, exponential stability, delay, Lyapunov method.**

I. Introduction

IN recent years, various multi-species logarithmic population models have been extensively investigated by many scholars due to their theoretical and practical significance in biology. Gopalsamy [1] and Kirlinger [2] proposed the following single species logarithmic model

$$\frac{dN(t)}{dt} = N(t)[a - b\ln N(t) - c\ln N(t-\tau)]. \quad (1)$$

In 1997, Li [3] generalized system (1) to the following non-autonomous form

$$\frac{dN(t)}{dt} = N(t)[a(t) - b(t)\ln N(t) - c(t)\ln N(t-\tau(t))]. \quad (2)$$

Applying the coincidence degree theory, Li [3] established some sufficient conditions for the existence of positive periodic solutions of system (2). In 2003, Chen et al. [4] generalized system (2) to the system with state dependent delays and investigated the existence of positive periodic solutions of the system. In [5], Liu proposed the following multispecies periodic logarithmic population model

$$\frac{dN_i(t)}{dt} = N_i(t)\Bigg[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)\ln N_j(t) - \sum_{j=1}^{n} b_{ij}(t)\ln N_j(t-\tau_{ij}(t))\Bigg]. \quad (3)$$

Applying the coincidence degree theory and constructing Lyapunov functional, a set of sufficient conditions which guarantee the existence, uniqueness and stability of the positive periodic solution of system (3) are established. In 2005, Chen [6] proposed the following multispecies logarithmic population model

$$\frac{dN_i(t)}{dt} = N_i(t)\Bigg[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)\ln N_j(t) - \sum_{j=1}^{n} b_{ij}(t)\ln N_j(t-\tau_{ij}(t)) + \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)\ln N_j(s)ds\Bigg]. \quad (4)$$

By using the fixed point theory and constructing a suitable Lyapunov functional, a set of easily applicable criteria are obtained for the existence, uniqueness and global attractivity of positive periodic solution (positive almost periodic solution) of the model (4). Gopalsamy [1] pointed out that in some case, the neutral delay population models are more realistic. Then Li [7] and Li et al. [8] considered the periodic solution or almost periodic solutions of the following two single species neutral Logarithmic models

$$\frac{dN(t)}{dt} = N(t)\left[r(t) - a(t)\ln N(t-\sigma) - b(t)\frac{d\ln N(t-\tau)}{dt}\right] \quad (5)$$

and

$$\frac{dN(t)}{dt} = N(t)\Bigg[r(t) - \sum_{j=1}^{n} a_j(t)\ln N(t-\sigma_j(t)) - \sum_{j=1}^{n} b_j(t)\frac{d\ln N(t-\tau_j(t))}{dt}\Bigg], \quad (6)$$

respectively. In 2003, Yang and Cao [9] addressed the existence of positive periodic solutions of the neutral logarithmic

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population model with multiple delays

$$\frac{dN(t)}{dt} = N(t)\Bigg[a(t) - \beta(t)N(t) - \sum_{i=1}^{n}(b_i(t) \times N(t-\tau_i(t))) - \sum_{i=1}^{n} c_i(t)\frac{d\ln N(t-\gamma_i(t))}{dt}\Bigg]. \quad (7)$$

In 2004, Lu and Ge [10] pointed out that the proof of Theorem 3.1 is incomplete and analyzed the existence of positive periodic solutions for neutral logarithmic population model with multiple delays

$$\frac{dN(t)}{dt} = N(t)\Bigg[r(t) - \sum_{j=1}^{n} a_i(t)\ln N(t-\sigma_i(t)) - \sum_{j=1}^{m} b_j(t)\frac{d\ln N(t-\tau_j(t))}{dt}\Bigg]. \quad (8)$$

With the help of an abstract continuous theorem of k-set contractive operator, authors obtained some sufficient conditions for the existence, global attractivity of positive periodic solution of (8). In 2009, Wang et al. [11] focused on the existence and uniqueness of positive periodic solutions for a following neutral logarithmic population model

$$\frac{dN(t)}{dt} = N(t)\Bigg[r(t) - a(t)\ln N(t) - \sum_{j=1}^{n} b_j(t)\ln N(t-\tau_j(t)) - \sum_{j=1}^{n} c_j(t)\int_{-\infty}^{t} k_j(t-s)\ln N(s)ds - \sum_{j=1}^{n} d_j(t)\frac{d\ln N(t-\eta_j(t))}{dt}\Bigg]. \quad (9)$$

Applying an abstract continuous theorem of k-set contractive operator, authors established some sufficient conditions for the existence, global attractivity of positive periodic solution of (9). In 2010 and 2011, Alzabut et al. [12-13] studied the almost periodic solutions for delay logarithmic population models. Recently, Chen [14] had investigated the periodic solution and almost periodic solutions of the following neutral multi-species logarithmic population model

$$\frac{dN_i(t)}{dt} = N_i(t)\Bigg[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)\ln N_j(t) - \sum_{j=1}^{n} b_{ij}(t)\ln N_j(t-\tau_{ij}(t)) + \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)\ln N_j(s)ds - \sum_{j=1}^{n} d_{ij}(t)\frac{d\ln N_j(t-\eta_{ij}(t)}{dt}\Bigg], \quad (10)$$

where $i = 1,2,\cdots,,n, a_{ij}(t), b_{ij}(t), c_{ij}(t), d_{ij}(t) \in C(R,(0,+\infty)), \tau_{ij}(t), \eta_{ij}(t) \in C(R,R^+)$ are all continuous functions. $\int_0^{+\infty} K_{ij}(s)ds = 1, \int_0^{+\infty} sK_{ij}(s)ds < +\infty$.

Many scholars [15-18,22-24] argue that ecosystem in the real world is continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. Of practical interest in ecology is the question of whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control variables, we call the disturbance functions as control variables. Motivated by the discussion above, we will investigate the neutral multi-species logarithmic population model with feedback controls as follows

$$\begin{cases} \dfrac{dN_i(t)}{dt} = N_i(t)\Bigg[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)\ln N_j(t) - \sum_{j=1}^{n} b_{ij}(t)\ln N_j(t-\tau_{ij}(t)) + \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)\ln N_j(s)ds - \sum_{j=1}^{n} d_{ij}(t)\dfrac{d\ln N_j(t-\eta_{ij}(t)}{dt} - e_i(t)u_i(t) - f_i(t)u_i(t-\sigma_i(t))\Bigg], \\ \dfrac{du_i(t)}{dt} = -\alpha_i(t)u_i(t) + \beta_i(t)\ln N_i(t) + \gamma_i(t)\ln N_i(t-\delta_i(t)), \end{cases} \quad (11)$$

where $i = 1,2,\cdots,n, u_i (i = 1,2,\cdots,n)$ denote indirect feedback control variables.

The main aim of this article is to establish some sufficient conditions for the existence and uniqueness of almost periodic solutions of (11). Our results are new and complement those of the previous studies in [8-14]. To the best of our knowledge, it is the first time to investigate the neutral multi-species logarithmic population model with feedback controls by applying the matrix's spectral theory. So far, there are very few paper that deal with the almost periodic solutions by applying the matrix's spectral theory.

The remainder of the paper is organized as follows. In Section II, we introduce some notations and assumptions, which can be used to check the existence and uniqueness of almost periodic solution of system (11). In Section III, we present some sufficient conditions for the existence and uniqueness of almost periodic solution of (11). An example is given to illustrate the effectiveness of the obtained results in Section V. A brief conclusion is drawn in Section VI.

II. Notations and Assumptions

In this section, we would like to introduce some notations and assumptions which are used in what follows. Let $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ denote a column vector, $D = (d_{ij})_{n\times n}$ be an $n \times n$ matrix, D^T be the transpose of D, and E_n be the identity matrix of size n. A matrix or vector $D > 0$ means that all entries of D are greater than zero, likewise for $D \geq 0$. For matrices or vectors D and E, $D > E (D \geq E)$ means that $D - E > 0 (D - E \geq 0)$. $\rho(D)$ denotes the spectral radius of the matrix D.

If $v = (v_1, v_2, \cdots, v_n)^T \in \mathbb{R}^n$, then we define the commonly used norms as follows

$$||v||_1 = \sum_{j=1}^{n} |v_i|, ||v||_2 = \left(\sum_{j=1}^{n} |v_i|^2 \right)^{\frac{1}{2}}, ||v||_\infty = \max_{1\le i\le n} |v_i|.$$

If $A = (a_{ij})_{n\times n}$, then we define the norm of the matrix $||A||$ as follows

$$||A|| = \sup_{v\in\mathbb{R}^n, v\neq 0} \frac{||Av||}{||v||} = \sup_{||v||=1} ||Av|| = \sup_{||v||\le 1} ||Av||.$$

In particular,

$$||A||_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|, ||A||_2 = [\lambda_{\max}(A^T A)]^{\frac{1}{2}},$$

$||A||_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$. Let

$$m(f) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T f(t)dt = 0,$$

where $f(t)$ is almost periodic function.

Throughout this paper, we make the following assumptions.

(H1) $r_i(t), a_{ij}(t), b_{ij}(t), c_{ij}(t), d_{ij}(t), \alpha_i(t), \beta_i(t), \gamma_i(t), e_i(t), f_i(t), i, j = 1, 2, \cdots, n$ are continuous real-valued nonnegative almost periodic functions on $\mathbb{R}$.

(H2) The kernels $K_{ij}(.), i, j = 1, 2, \cdots, n$ are nonnegative continuous functions defined on $[0, +\infty)$ satisfying $\int_0^{+\infty} K_{ij}(s)ds = 1$.

(H3) $\tau_{ij}(t), \sigma_i(t), \delta_i(t)$ and $\eta_{ij}(t)$ are nonnegative, continuously differentiable and almost periodic functions on $t \in \mathbb{R}$. Moreover, $\dot{\tau}_{ij}(t), \dot{\sigma}_i(t), \dot{\delta}_i(t)$ and $\dot{\eta}_{ij}(t)$ are all uniformly continuous on $\mathbb{R}$ with $\inf_{t\in\mathbb{R}}\{1 - \dot{\tau}_{ij}(t)\} > 0, \inf_{t\in\mathbb{R}}\{1 - \dot{\sigma}_i(t)\} > 0, \inf_{t\in\mathbb{R}}\{1 - \dot{\delta}_i(t)\} > 0, \inf_{t\in\mathbb{R}}\{1 - \dot{\eta}_{ij}(t)\} > 0$.

System (11) is supplemented with the initial value conditions

$$\begin{aligned} &N_i(s) = \varphi_{N_i}(s) \ge 0, \dot{N}_i(s) = \dot{\varphi}_{N_i}(s), s \in (-\infty, 0], \\ &\varphi_{N_i}(0) > 0, \sup_{s\in(-\infty,0]} \varphi_{N_i}(s) < +\infty, \sup_{s\in(-\infty,0]} \dot{\varphi}_{N_i}(s) \\ &< +\infty, u_i(s) = \varphi_{u_i}(s) \ge 0, \varphi_{u_i}(0) > 0, s \in (-\infty, 0]. \end{aligned} \tag{12}$$

It is easy to see that there exists a positive solution $y(t) = (N_1(t), N_2(t), \cdots, x_n(t), u_1(t), u_2(t), \cdots, u_n(t))$ of system (11) satisfying the initial value condition (12).

III. Existence and uniqueness of almost periodic solution

In this section, we will establish sufficient conditions on the existence and uniqueness of almost periodic solutions of (11). For convenience, we introduce some definitions and lemmas which will be used in what follows.

Definition 3.1 [19-20] *Let $f(t) : \mathbb{R} \to \mathbb{R}^n$ be continuous in t. $f(t)$ is said to almost periodic on $\mathbb{R}$, if for any $\varepsilon > 0$, the set $T(f, \varepsilon) = \{\delta : |f(t + \delta) - f(t)| < \varepsilon, \forall t \in \mathbb{R}\}$ is relatively dense, i.e., for $\forall\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|f(t + \delta) - f(t)| < \varepsilon$, for $\forall \in \mathbb{R}$.*

Definition 3.2 *Let $z \in \mathbb{R}^n$ and $Q(t)$ be a $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system*

$$\frac{dz}{dt} = Q(t)z(t) \tag{13}$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist constants $k, \lambda > 0$, projection P and the fundamental matrix $Z(t)$ of (13) satisfying

$$||Z(t)PZ^{-1}(s)|| \le ke^{-\lambda(t-s)}, for\ t \ge s,$$
$$||Z(t)(I - P)Z^{-1}(s)|| \le ke^{-\lambda(t-s)}, for\ t \le s.$$

Lemma 3.1 [20-21] *If the linear system (13) admits an exponential dichotomy, then almost periodic system*

$$\frac{dz}{dt} = Q(t)z(t) + g(t) \tag{14}$$

has a unique almost periodic solution $z(t)$ and

$$\begin{aligned} z(t) = &\int_{-\infty}^{t} Z(t)PZ^{-1}(s)g(s)ds \\ &- \int_{t}^{+\infty} Z(t)(I - P)Z^{-1}(s)g(s)ds. \end{aligned}$$

Lemma 3.2 [20-21] *Let $a_i(t)$ be an almost periodic function on $\mathbb{R}$ and $a_i(t) > 0$. Then the system*

$$\frac{dz}{dt} = diag(-a_1(t), -a_2(t), \cdots, -a_n(t))z(t) \tag{15}$$

admits an exponential dichotomy.

Remark 3.1 *It follows from Lemma 3.2 that system (15) has a unique almost periodic solution $z(t)$ which takes the form*

$$\begin{aligned} z(t) &= \int_{-\infty}^{t} Z(t)Z^{-1}(s)g(s)ds \\ &= \left(\int_{-\infty}^{t} e^{-\int_s^t a_1(u)du} g_1(s)ds, \cdots, \right. \\ &\left. \int_{-\infty}^{t} e^{-\int_s^t a_n(u)du} g_n(s)ds \right). \end{aligned}$$

Lemma 3.3 [14] *Assume that $v(t), \eta(t)$ are all continuously differentiable T-periodic functions, $a(t), b(t)$ are all nonnegative continuous T-periodic functions such that $\int_0^T a(t)dt > 0$, then*

$$\begin{aligned} &\int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau} b(s)v'(s)(s - \eta(s))ds = c(t)v(t - \eta(t)) \\ &- \int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau} (a(s)c(s) + c'(s))(v(s - \eta(s))ds, \end{aligned}$$

where $c(s) = \frac{b(s)}{1-\eta'(s)}$.

Lemma 3.4 *Let m be a positive integer and B be an Banach space. If the mapping $\Gamma : B \to B$ is a contraction mapping, then $\Gamma : B \to B$ has exactly one fixed point in B, where $\Gamma^m = \Gamma(\Gamma^{m-1})$.*

By (H1), $m(\alpha_i) > 0$. In view of Lemma 3.1, we have the following result.

Lemma 3.5 *$(N_1(t), N_2(t), \cdots, N_n(t), u_1(t), u_2(t), \cdots, u_n(t))^T$ is an almost periodic solution of system (11) if and only if it is an almost periodic solution of*

$$\begin{cases} \dfrac{dN_i(t)}{dt} = N_i(t)\Bigg[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)\ln N_j(t) \\ \quad - \sum_{j=1}^{n} b_{ij}(t)\ln N_j(t-\tau_{ij}(t)) \\ \quad + \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)\ln N_j(s)ds \\ \quad - \sum_{j=1}^{n} d_{ij}(t)\dfrac{d\ln N_j(t-\eta_{ij}(t)}{dt} \\ \quad - e_i(t)u_i(t) - f_i(t)u_i(t-\sigma_i(t))\Bigg], \\ u_i(t) = \int_{-\infty}^{t} e^{\int_s^t \alpha_i(\zeta)d\zeta}[\beta_i(s)\ln N_i(s) \\ \quad + \gamma_i(s)\ln N_i(s-\delta_i(s))]ds, \end{cases} \tag{16}$$

where $i = 1, 2, \cdots, n$. Obviously, (16) is equivalent to the following system

$$\begin{aligned} \frac{dN_i(t)}{dt} = \; & N_i(t)\Bigg[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)\ln N_j(t) \\ & - \sum_{j=1}^{n} b_{ij}(t)\ln N_j(t-\tau_{ij}(t)) \\ & + \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)\ln N_j(s)ds \\ & - \sum_{j=1}^{n} d_{ij}(t)\frac{d\ln N_j(t-\eta_{ij}(t)}{dt} \\ & -e_i(t)\int_{-\infty}^{t} e^{\int_s^t \alpha_i(\zeta)d\zeta}(\beta_i(s)\ln N_i(s) \\ & +\gamma_i(s)\ln N_i(s-\delta_i(s)))ds \\ & -f_i(t)\int_{-\infty}^{t-\sigma_i(t)} e^{\int_s^{t-\sigma_i(t)} \alpha_i(\zeta)d\zeta}(\beta_i(s)\ln N_i(s) \\ & +\gamma_i(s)\ln N_i(s-\delta_i(s)))ds\Bigg]. \end{aligned} \tag{17}$$

Now we are in a position to state our main results on the existence and uniqueness of almost periodic solution for system (11).

Theorem 3.1 *In addition to (H1)–(H3), if the following condition*
(H4) $\rho(\Lambda) < 1$, where $\Lambda = (\Lambda_{ij})_{n\times n}$ and

$$\begin{aligned} \Lambda_{ii} &= d_{ii}(t) + \int_{-\infty}^{t} e^{-\int_s^t a_{ii}(\zeta)d\zeta}\Theta_{ii}(s)ds, \\ \Lambda_{ij} &= d_{ij}(t) + \int_{-\infty}^{t} e^{-\int_s^t a_{ii}(\zeta)d\zeta}\Theta_{ij}(s)ds, i \neq j, \\ \Theta_{ii}(s) &= a_{ii}(s) + b_{ii}(s) + c_{ii}(s)\int_0^{\infty} K_{ii}(\theta)d\theta \\ &\quad + d_{ii}(s)(a_{ii}(s)d_{ii}(s) + |d_{ii}'(s)|) \\ &\quad +(e_i(s) + f_i(s))\int_{-\infty}^{s} e^{\int_\xi^s \alpha_i(\zeta)d\zeta}(\beta_i(\xi) + \gamma_i(\xi))d\xi, \\ \Theta_{ij}(s) &= (a_{ij}(s) + b_{ij}(s) + c_{ij}(s)\int_0^{\infty} K_{ij}(\theta)d\theta \\ &\quad +d_{ij}(s)(a_{ii}(s)d_{ij}(s) + |d_{ij}'(s)|), i \neq j. \end{aligned}$$

where $i = 1, 2, \cdots, n$. Then system (11) has a unique positive almost periodic solution.

Proof Let $N_i(t) = e^{x_i(t)}$, then (17) takes the form

$$\begin{aligned} \frac{dx_i(t)}{dt} &= r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) \\ & - \sum_{j=1}^{n} b_{ij}(t)x_j(t-\tau_{ij}(t)) \\ & + \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds \\ & - \sum_{j=1}^{n} d_{ij}(t)\dot{x}_j(t-\eta_{ij}(t)(1-\dot{\eta}_{ij}(t)) \\ & -e_i(t)\int_{-\infty}^{t} e^{\int_s^t \alpha_i(\zeta)d\zeta}(\beta_i(s)x_i(s) \\ & +\gamma_i(s)x_i(s-\delta_i(s)))ds \\ & -f_i(t)\int_{-\infty}^{t-\sigma_i(t)} e^{\int_s^{t-\sigma_i(t)} \alpha_i(\zeta)d\zeta}(\beta_i(s)x_i(s) \\ & +\gamma_i(s)x_i(s-\delta_i(s)))ds \\ & = -a_{ii}(t)x_i(t) - \sum_{j=1, j\neq i}^{n} a_{ij}(t)x_j(t) \\ & - \sum_{j=1}^{n} b_{ij}(t)x_j(t-\tau_{ij}(t)) \\ & + \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds \\ & - \sum_{j=1}^{n} d_{ij}(t)\dot{x}_j(t-\eta_{ij}(t)(1-\dot{\eta}_{ij}(t)) \\ & -e_i(t)\int_{-\infty}^{t} e^{\int_s^t \alpha_i(\zeta)d\zeta}(\beta_i(s)x_i(s) \\ & +\gamma_i(s)x_i(s-\delta_i(s)))ds \\ & -f_i(t)\int_{-\infty}^{t-\sigma_i(t)} e^{\int_s^{t-\sigma_i(t)} \alpha_i(\zeta)d\zeta}(\beta_i(s)x_i(s) \\ & +\gamma_i(s)x_i(s-\delta_i(s)))ds + r_i(t). \end{aligned} \tag{18}$$

Clearly, if system (18) has an almost periodic solution $(x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^T$, then $(N_1^*(t), N_2^*(t), \cdots, N_n^*(t))^T = (e^{x_1^*(t)}, e^{x_1^*(t)}, \cdots, e^{x_n^*(t)})^T$ is an almost periodic solution of (17). In view of Lemma 3.5, we can conclude that $(e^{x_1^*(t)}, e^{x_1^*(t)}, \cdots, e^{x_n^*(t)}, u_1^*(t), u_1^*(t), \cdots, u_n^*(t))^T$ is an almost periodic solution of (11), where

$$u_i^*(t) = \int_{-\infty}^{t} e^{\int_s^t \alpha_i(\zeta)d\zeta}[\beta_i(s)x_i^*(s) + \gamma_i(s)x_i^*(s-\delta_i(s))]ds,$$

where $i = 1, 2, \cdots, n$. Now we will show that (18) has a unique almost almost periodic solution. Firstly, we define $B = \{\psi(t) = (\psi_1(t), \psi_2(t), \cdots, \psi_n(t))^T | \psi(t)$ is a continuous almost periodic function$\}$. Obviously, B is a Banach space with the norm $||\psi|| = \max_{1\leq i\leq n}\sup_{t\in\mathbb{R}}|x_i(t)|$.

For any $\psi(t) = (\psi_1(t), \psi_2(t), \cdots, \psi_n(t))^T \in B$, we consider the following almost periodic system

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -a_{ii}(t)x_i(t) - \sum_{j=1,j\neq i}^{n} a_{ij}(t)\psi_j(t)\\ &\quad -\sum_{j=1}^{n} b_{ij}(t)\psi_j(t-\tau_{ij}(t))\\ &\quad +\sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)\psi_j(s)ds\\ &\quad -\sum_{j=1}^{n} d_{ij}(t)\dot{\psi}_j(t-\eta_{ij}(t)(1-\dot{\eta}_{ij}(t))\\ &\quad -e_i(t)\int_{-\infty}^{t} e^{\int_s^t \alpha_i(\zeta)d\zeta}(\beta_i(s)\psi_i(s)\\ &\quad +\gamma_i(s)\psi_i(s-\delta_i(s)))ds\\ &\quad -f_i(t)\int_{-\infty}^{t-\sigma_i(t)} e^{\int_s^{t-\sigma_i(t)} \alpha_i(\zeta)d\zeta}(\beta_i(s)\psi_i(s)\\ &\quad +\gamma_i(s)\psi_i(s-\delta_i(s)))ds + r_i(t). \qquad (19)\end{aligned}$$

By (H1), we know that $m(a_{ii} > 0$. In view of Lemma 3.2, the linear system

$$\frac{dx_i(t)}{dt} = -a_{ii}(t)x_i(t), i = 1, 2, \cdots, n, \qquad (20)$$

admits an exponential dichotomy on $\mathbb{T}$. Then system (20) has exactly one almost periodic solution as follows

$$\begin{aligned}x_i^{\psi}(t) &= (x_1^{\psi}(t), x_2^{\psi}(t), \cdots, x_n^{\psi}(t))^T\\ &= \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_s^t a_{11}(\zeta)d\zeta} h_1^{\psi}(s)ds \\ \int_{-\infty}^{t} e^{-\int_s^t a_{22}(\zeta)d\zeta} h_2^{\psi}(s)ds \\ \cdots \\ \int_{-\infty}^{t} e^{-\int_s^t a_{nn}(\zeta)d\zeta} h_n^{\psi}(s)ds \end{pmatrix}, \qquad (21)\end{aligned}$$

where

$$\begin{aligned}h_i^{\psi}(s) &= -\sum_{j=1,j\neq i}^{n} a_{ij}(s)\psi_j(s) - \sum_{j=1}^{n} b_{ij}(s)\psi_j(s-\tau_{ij}(s))\\ &\quad +\sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} K_{ij}(s-\theta)\psi_j(\theta)d\theta\\ &\quad -\sum_{j=1}^{n} d_{ij}(s)\dot{\psi}_j(s-\eta_{ij}(s)(1-\dot{\eta}_{ij}(s))\\ &\quad -e_i(s)\int_{-\infty}^{s} e^{\int_\xi^s \alpha_i(\zeta)d\zeta}(\beta_i(\xi)\psi_i(\xi)\\ &\quad +\gamma_i(\xi)\psi_i(\xi-\delta_i(\xi)))d\xi\\ &\quad -f_i(s)\int_{-\infty}^{s-\sigma_i(s)} e^{\int_\xi^{s-\sigma_i(s)} \alpha_i(\zeta)d\zeta}(\beta_i(\xi)\psi_i(\xi)\\ &\quad +\gamma_i(\xi)\psi_i(\xi-\delta_i(\xi)))ds + r_i(s). \qquad (22)\end{aligned}$$

In view of Lemma 3.2, $x_i^{\psi}(t)$ can be expressed as

$$\begin{aligned}&x_i^{\psi}(t) = (x_1^{\psi}(t), x_2^{\psi}(t), \cdots, x_n^{\psi}(t))^T\\ &= (A_1, A_2, \cdots, A_n)^T, \qquad (23)\end{aligned}$$

where

$$A_1 = \sum_{j=1}^{n} d_{1j}(t)\psi_j(t-\eta_{1j}(t)) + \int_{-\infty}^{t} e^{-\int_s^t a_{11}(\zeta)d\zeta} l_1^{\psi}(s)ds,$$

$$A_2 = \sum_{j=1}^{n} d_{2j}(t)\psi_j(t-\eta_{2j}(t)) + \int_{-\infty}^{t} e^{-\int_s^t a_{22}(\zeta)d\zeta} l_2^{\psi}(s)ds,$$

$$A_n \sum_{j=1}^{n} d_{nj}(t)\psi_j(t-\eta_{nj}(t)) + \int_{-\infty}^{t} e^{-\int_s^t a_{nn}(\zeta)d\zeta} l_n^{\psi}(s)ds$$

and

$$\begin{aligned}l_i^{\psi}(s) &= -\sum_{j=1,j\neq i}^{n} a_{ij}(s)\psi_j(s)\\ &\quad -\sum_{j=1}^{n} b_{ij}(s)\psi_j(s-\tau_{ij}(s))\\ &\quad +\sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} K_{ij}(s-\theta)\psi_j(\theta)d\theta\\ &\quad +\sum_{j=1}^{n} d_{ij}(s)(a_{ii}(s)d_{ij}(s) + d_{ij}'(s))\psi_j(s-\eta_{ij}(s))\\ &\quad -e_i(s)\int_{-\infty}^{s} e^{\int_\xi^s \alpha_i(\zeta)d\zeta}(\beta_i(\xi)\psi_i(\xi)\\ &\quad +\gamma_i(\xi)\psi_i(\xi-\delta_i(\xi)))d\xi\\ &\quad -f_i(s)\int_{-\infty}^{s-\sigma_i(s)} e^{\int_\xi^{s-\sigma_i(s)} \alpha_i(\zeta)d\zeta}(\beta_i(\xi)\psi_i(\xi)\\ &\quad +\gamma_i(\xi)\psi_i(\xi-\delta_i(\xi)))d\xi + r_i(s). \qquad (24)\end{aligned}$$

Define a mapping $F : B \to B$ as follows

$$F\psi(t) = Z^{\psi}(t), \text{ for any } \psi \in B. \qquad (25)$$

For any $\phi, \psi \in B$, we have

$$\begin{aligned}&|(F(\phi) - F(\psi))|\\ &= (|(F(\phi(t)) - F(\psi(t)))_1|, |(F(\phi(t)) - F(\psi(t)))_2|,\\ &\cdots, |(F(\phi(t)) - F(\psi(t)))_n|)^T \le\\ &\begin{pmatrix} \sum_{j=1}^{n} d_{1j}(t)|\phi_j(t-\eta_{1j}(t)) - \psi_j(t-\eta_{1j}(t))| \\ +\int_{-\infty}^{t} e^{-\int_s^t a_{11}(\zeta)d\zeta}|l_1^{\phi}(s) - l_1^{\psi}(s)|ds \\ \sum_{j=1}^{n} d_{2j}(t)|\phi_j(t-\eta_{2j}(t)) - \psi_j(t-\eta_{2j}(t))| \\ +\int_{-\infty}^{t} e^{-\int_s^t a_{22}(\zeta)d\zeta}|l_2^{\phi}(s) - l_2^{\psi}(s)|ds \\ \cdots \\ \sum_{j=1}^{n} d_{nj}(t)|\phi_j(t-\eta_{nj}(t)) - \psi_j(t-\eta_{nj}(t))| \\ +\int_{-\infty}^{t} e^{-\int_s^t a_{nn}(\zeta)d\zeta}|l_n^{\phi}(s) - l_n^{\psi}(s)|ds \end{pmatrix}. \qquad (26)\end{aligned}$$

On the other hand, by (24), we get

$$\begin{aligned}&|l_i^{\phi}(s) - l_i^{\psi}(s)|\\ &= \sum_{j=1,j\neq i}^{n} a_{ij}(s)|\phi_j(s) - \psi_j(s)|\\ &\quad +\sum_{j=1}^{n} b_{ij}(s)|\phi_j(s-\tau_{ij}(s)) - \psi_j(s-\tau_{ij}(s))|\\ &\quad +\sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} K_{ij}(s-\theta)|\phi_j(\theta) - \psi_j(\theta)|d\theta\\ &\quad +\sum_{j=1}^{n} d_{ij}(s)(a_{ii}(s)d_{ij}(s) + |d_{ij}'(s)|)|\phi_j(s-\eta_{ij}(s))\end{aligned}$$

$$
\begin{aligned}
&-\psi_j(s-\eta_{ij}(s))| \\
&+e_i(s)\int_{-\infty}^{s} e^{\int_\xi^s \alpha_i(\zeta)d\zeta}(\beta_i(\xi)|\phi_i(\xi)-\psi_i(\xi)| \\
&+\gamma_i(\xi)|\phi_i(\xi-\delta_i(\xi))-\psi_i(\xi-\delta_i(\xi))|)d\xi \\
&+f_i(s)\int_{-\infty}^{s-\sigma_i(s)} e^{\int_\xi^{s-\sigma_i(s)} \alpha_i(\zeta)d\zeta}(\beta_i(\xi)|\phi_i(\xi)-\psi_i(\xi)| \\
&+\gamma_i(\xi)|\phi_i(\xi-\delta_i(\xi))-\psi_i(\xi-\delta_i(\xi))|)d\xi \\
&\le \sum_{j=1,j\neq i}^{n} a_{ij}(s)\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)| \\
&+\sum_{j=1}^{n} b_{ij}(s)\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)| \\
&+\sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} K_{ij}(s-\theta)\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)|d\theta \\
&+\sum_{j=1}^{n} d_{ij}(s)(a_{ii}(s)d_{ij}(s)+|d'_{ij}(s)|) \\
&\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)|+e_i(s)\int_{-\infty}^{s} e^{\int_\xi^s \alpha_i(\zeta)d\zeta} \\
&\times(\beta_i(\xi)\sup_{t\in\mathbb{R}}|\phi_i(t)-\psi_i(t)| \\
&+\gamma_i(\xi)\sup_{t\in\mathbb{R}}|\phi_i(t)-\psi_i(t)|)d\xi \\
&+f_i(s)\int_{-\infty}^{s-\sigma_i(s)} e^{\int_\xi^{s-\sigma_i(s)} \alpha_i(\zeta)d\zeta}(\beta_i(\xi) \\
&\sup_{t\in\mathbb{R}}|\phi_i(t)-\psi_i(t)|+\gamma_i(\xi)\sup_{t\in\mathbb{R}}|\phi_i(t)-\psi_i(t)|)d\xi \\
&=\Bigg[\sum_{j=1,j\neq i}^{n} a_{ij}(s)+\sum_{j=1}^{n} b_{ij}(s) \\
&+\sum_{j=1}^{n} c_{ij}(s)\int_0^\infty K_{ij}(\theta)d\theta \\
&+\sum_{j=1}^{n} d_{ij}(s)(a_{ii}(s)d_{ij}(s)+|d'_{ij}(s)|)\Bigg] \\
&\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)| \\
&+\left[(e_i(s)+f_i(s))\int_{-\infty}^{s} e^{\int_\xi^s \alpha_i(\zeta)d\zeta}(\beta_i(\xi)+\gamma_i(\xi))d\xi\right] \\
&\times\sup_{t\in\mathbb{R}}|\phi_i(t)-\psi_i(t)| \\
&=\Bigg[a_{ii}(s)+b_{ii}(s)+c_{ii}(s)\int_0^\infty K_{ii}(\theta)d\theta \\
&+d_{ii}(s)(a_{ii}(s)d_{ii}(s)+|d'_{ii}(s)|) \\
&+(e_i(s)+f_i(s))\int_{-\infty}^{s} e^{\int_\xi^s \alpha_i(\zeta)d\zeta}(\beta_i(\xi)+\gamma_i(\xi))d\xi\Bigg] \\
&\times\sup_{t\in\mathbb{R}}|\phi_i(t)-\psi_i(t)| \\
&+\Bigg\{\sum_{j=1,j\neq i}^{n}\Bigg[(a_{ij}(s)+b_{ij}(s)+c_{ij}(s)\int_0^\infty K_{ij}(\theta)d\theta \\
&+d_{ij}(s)(a_{ii}(s)d_{ij}(s)+|d'_{ij}(s)|)\Bigg]\Bigg\} \\
&\times\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)|. \qquad (27)
\end{aligned}
$$

Let

$$
\begin{aligned}
&\Theta_{ii}(s)=a_{ii}(s)+b_{ii}(s)+c_{ii}(s)\int_0^\infty K_{ii}(\theta)d\theta \\
&+d_{ii}(s)(a_{ii}(s)d_{ii}(s)+|d'_{ii}(s)|) \\
&+(e_i(s)+f_i(s))\int_{-\infty}^{s} e^{\int_\xi^s \alpha_i(\zeta)d\zeta}(\beta_i(\xi)+\gamma_i(\xi))d\xi, \\
&\Theta_{ij}(s)=(a_{ij}(s)+b_{ij}(s)+c_{ij}(s)\int_0^\infty K_{ij}(\theta)d\theta \\
&+d_{ij}(s)(a_{ii}(s)d_{ij}(s)+|d'_{ij}(s)|), i\neq j.
\end{aligned}
$$

where $i=1,2,\cdots,n$. It follows from (27) that

$$
\begin{aligned}
&|l_i^\phi(s)-l_i^\psi(s)|\le\Theta_{ii}(s)\sup_{t\in\mathbb{R}}|\phi_i(t)-\psi_i(t)| \\
&+\sum_{j=1,j\neq i}^{n}\Theta_{ij}(s)\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)|, \qquad (28)
\end{aligned}
$$

where $i=1,2,\cdots,n$. Then

$$
\begin{aligned}
&\sum_{j=1}^{n} d_{ij}(t)|\phi_j(t-\eta_{ij}(t))-\psi_j(t-\eta_{ij}(t))| \\
&+\int_{-\infty}^{t} e^{-\int_s^t a_{ii}(\zeta)d\zeta}|l_i^\phi(s)-l_i^\psi(s)|ds \\
&\le\sum_{j=1}^{n} d_{ij}(t)\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)| \\
&+\int_{-\infty}^{t} e^{-\int_s^t a_{ii}(\zeta)d\zeta}\Theta_{ii}(s)ds\sup_{t\in\mathbb{R}}|\phi_i(t)-\psi_i(t)| \\
&+\sum_{j=1,j\neq i}^{n}\int_{-\infty}^{t} e^{-\int_s^t a_{ii}(\zeta)d\zeta}\Theta_{ij}(s)ds \\
&\times\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)| \\
&=\Lambda_{ii}\sup_{t\in\mathbb{R}}|\phi_i(t)-\psi_i(t)| \\
&+\sum_{j=1,j\neq i}^{n}\Lambda_{ij}\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)|, \qquad (29)
\end{aligned}
$$

where

$$
\begin{aligned}
&\Lambda_{ii}=d_{ii}(t)+\int_{-\infty}^{t} e^{-\int_s^t a_{ii}(\zeta)d\zeta}\Theta_{ii}(s)ds, \\
&\Lambda_{ij}=d_{ij}(t)+\int_{-\infty}^{t} e^{-\int_s^t a_{ii}(\zeta)d\zeta}\Theta_{ij}(s)ds, i\neq j,
\end{aligned}
$$

where $i=1,2,\cdots,n$. It follows from (26) and (29) that

$$
\begin{aligned}
&|(F(\phi)-F(\psi))| \\
&=(|(F(\phi(t))-F(\psi(t)))_1|,|(F(\phi(t))-F(\psi(t)))_2|, \\
&\cdots,|(F(\phi(t))-F(\psi(t)))_n|)^T
\end{aligned} \qquad (30)
$$

$$\leq \begin{pmatrix} \Lambda_{11}\sup_{t\in\mathbb{R}}|\phi_1(t)-\psi_1(t)| \\ +\sum_{j=1,j\neq 1}^{n}\Lambda_{1j}\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)| \\ \Lambda_{22}\sup_{t\in\mathbb{R}}|\phi_2(t)-\psi_2(t)| \\ +\sum_{j=1,j\neq 2}^{n}\Lambda_{2j}\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)| \\ \cdots \\ \Lambda_{nn}\sup_{t\in\mathbb{R}}|\phi_n(t)-\psi_n(t)| \\ +\sum_{j=1,j\neq n}^{n}\Lambda_{nj}\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)| \end{pmatrix}$$
$$= \begin{pmatrix} \Lambda_{11} & \Lambda_{11} & \cdots & \Lambda_{1n} \\ \Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \Lambda_{21} & \Lambda_{n2} & \cdots & \Lambda_{nn} \end{pmatrix}_{n\times n}$$
$$\times \begin{pmatrix} \sup_{t\in\mathbb{R}}|\phi_1(t)-\psi_1(t)| \\ \sup_{t\in\mathbb{R}}|\phi_2(t)-\psi_2(t)| \\ \cdots \\ \sup_{t\in\mathbb{R}}|\phi_n(t)-\psi_n(t)| \end{pmatrix}_{n\times 1}$$
$$= \Lambda(\sup_{t\in\mathbb{R}}|\phi_1(t)-\psi_1(t)|, \sup_{t\in\mathbb{R}}|\phi_2(t)-\psi_2(t)|, \cdots, \sup_{t\in\mathbb{R}}|\phi_n(t)-\psi_n(t)|)^T$$
$$= \Lambda(\sup_{t\in\mathbb{R}}|(\phi(t)-\psi(t)_1|, \sup_{t\in\mathbb{R}}|(\phi(t)-\psi(t))_2|, \cdots, \sup_{t\in\mathbb{R}}|(\phi(t)-\psi_n(t))_n|)^T. \quad (31)$$

Then we get

$$\begin{pmatrix} \sup_{t\in\mathbb{R}}|(F(\phi(t))-F(\psi(t)))_1| \\ \sup_{t\in\mathbb{R}}|(F(\phi(t))-F(\psi(t)))_2| \\ \cdots \\ \sup_{t\in\mathbb{R}}|(F(\phi(t))-F(\psi(t)))_n| \end{pmatrix} \leq \begin{pmatrix} \sup_{t\in\mathbb{R}}|(\phi(t)-\psi(t))_1| \\ \sup_{t\in\mathbb{R}}|(\phi(t)-\psi(t))_2| \\ \cdots \\ \sup_{t\in\mathbb{R}}|(\phi(t)-\psi(t))_n| \end{pmatrix}. \quad (32)$$

For any positive integer m, by (31), we have

$$\begin{pmatrix} \sup_{t\in\mathbb{R}}|(F^m(\phi(t))-F^m(\psi(t)))_1| \\ \sup_{t\in\mathbb{R}}|(F^m(\phi(t))-F^m(\psi(t)))_2| \\ \cdots \\ \sup_{t\in\mathbb{R}}|(F^m(\phi(t))-F^m(\psi(t)))_n| \end{pmatrix} = \begin{pmatrix} \sup_{t\in\mathbb{R}}|(F(F^{m-1}(\phi(t)))-F(F^{m-1}(\psi(t))))_1| \\ \sup_{t\in\mathbb{R}}|(F(F^{m-1}(\phi(t)))-F(F^{m-1}(\psi(t))))_2| \\ \cdots \\ \sup_{t\in\mathbb{R}}|(F(F^{m-1}(\phi(t)))-F(F^{m-1}(\psi(t))))_n| \end{pmatrix}$$
$$\leq \Lambda \begin{pmatrix} \sup_{t\in\mathbb{R}}|(F^{m-1}(\phi(t))-F^{m-1}(\psi(t)))_1| \\ \sup_{t\in\mathbb{R}}|(F^{m-1}(\phi(t))-F^{m-1}(\psi(t)))_2| \\ \cdots \\ \sup_{t\in\mathbb{R}}|(F^{m-1}(\phi(t))-F(F^{m-1}(\psi(t))_n| \end{pmatrix}$$
$$\leq \cdots$$
$$\leq \Lambda^m \begin{pmatrix} \sup_{t\in\mathbb{R}}|(\phi(t)-\psi(t))_1| \\ \sup_{t\in\mathbb{R}}|(\phi(t)-\psi(t))_2| \\ \cdots \\ \sup_{t\in\mathbb{R}}|(\phi(t)-\psi(t))_n| \end{pmatrix} \quad (33)$$

By (H4), we get

$$\lim_{m\to+\infty}\Lambda^m = 0, \quad (34)$$

which implies that there exists a positive integer N^* and a positive constant $\mu_0 < 1$ such that

$$\Lambda^{N^*} = (\kappa_{ij})_{n\times n} \text{ and } \sum_{j=1}^{n}\kappa_{ij} \leq \mu_0, i = 1,2,\cdots,n. \quad (35)$$

It follows form (32) and (34) that

$$\begin{aligned} &|(F^{N^*}(\phi)-F^{N^*}(\psi))_i| \\ &\leq \sum_{j=1}^{n}\kappa_{ij}\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)| \\ &\leq \max_{1\leq i\leq n}\sup_{t\in\mathbb{R}}|\phi(t)-\psi(t)|\sum_{j+1}^{n}\kappa_{ij} \\ &\leq \mu_0||\phi-\psi||, i = 1,2,\cdots,n. \end{aligned} \quad (36)$$

Thus

$$\begin{aligned} &||F^{N^*}(\phi)-F^{N^*}(\psi)|| = \\ &\max_{1\leq i\leq n}|(F^{N^*}(\phi)-F^{N^*}(\psi))_i| \leq \mu_0||\phi-\psi||, \end{aligned} \quad (37)$$

which implies that the mapping $F^{N^*}: B \to B$ is a contraction mapping. In view of Lemma 3.4, F has a unique a fixed point $x^*(t)$ in B. Thus system (18) has a unique almost periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^T$, then $(N_1^*(t), N_2^*(t), \cdots, N_n^*(t))^T = (e^{x_1^*(t)}, e^{x_2^*(t)}, \cdots, e^{x_n^*(t)})^T$ is the unique almost periodic solution of (17). Thus, by Lemma 3.5, $(e^{x_1^*(t)}, e^{x_2^*(t)}, \cdots, e^{x_n^*(t)}, u_1^*(t), u_2^*(t), \cdots, u_n^*(t))^T$ is the unique almost periodic solution of (11). The proof of Theorem 3.1 is completed.

IV. Numerical example

In this section, we will give an example to illustrate the feasibility and effectiveness of our main results obtained in previous sections. Considering the following neutral multi-species logarithmic population model with feedback controls

$$\begin{cases} \dfrac{dN_1(t)}{dt} = N_1(t)\Bigg[r_1(t) - \sum_{j=1}^{2}a_{1j}(t)\ln N_j(t) \\ \qquad -\sum_{j=1}^{2}b_{1j}(t)\ln N_j(t-\tau_{1j}(t)) \\ \qquad +\sum_{j=1}^{2}c_{1j}(t)\int_{-\infty}^{t}K_{1j}(t-s)\ln N_j(s)ds \\ \qquad -\sum_{j=1}^{2}d_{1j}(t)\dfrac{d\ln N_j(t-\eta_{1j}(t)}{dt} \\ \qquad - e_1(t)u_1(t) - f_1(t)u_1(t-\sigma_1(t))\Bigg], \\ \dfrac{du_1(t)}{dt} = -\alpha_1(t)u_1(t) + \beta_1(t)\ln N_1(t) \\ \qquad + \gamma_1(t)\ln N_1(t-\delta_1(t)), \end{cases} \quad (38)$$

where $k_{ij} = e^{-s}, r_1(t) = 1+\sin t, a_{11} = 1+\sin t, a_{12} = 1+\cos t, b_{11} = 0.3+\sin t, b_{12} = 0.2+\cos t, c_{11} = 0.1+\sin t, c_{12} = 0.3+\cos t, d_{11} = 0.4+\sin t, d_{12} = 0.5+\cos t, \tau_{11} = 0.2+0.4\sin t, \tau_{12} = 0.3+0.1\cos t, \eta_{11} = 0.3+0.2\sin t, \eta_{12} = 0.2+0.1\cos t, e_1(t) = 0.2+\sin t, f_1(t) = 0.2+\cos t, \sigma_1(t) = 0.4+0.2\sin t, \delta_1(t) = 0.3+0.3\sin t, \alpha_1(t) = 0.4+\cos t, \beta_1(t) = 0.3+\sin t, \gamma_1(t) = 0.5+\cos t$. Then by Matlab software, we have $\int_0^\infty K_{ij}(s)ds = 1, \rho(\Lambda) \approx 0.3472 < 1$. Thus all assumptions in Theorems 3.1 are fulfilled. Thus we can conclude that (37) has a unique positive periodic solution. The results are verified by the numerical simulations in Fig. 1.

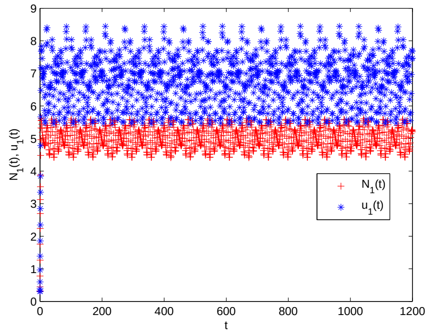

Fig. 1. Time response of state variables $N_1(t)$ and $u_1(t)$.

V. Conclusions

In this paper, we study a neutral multi-species logarithmic population model. Applying the matrix's spectral theory, we establish some sufficient conditions for the existence and uniqueness of almost periodic solution of the neutral multi-species logarithmic population model. The obtained sufficient conditions are given in terms of spectral radius of explicit matrices which are much different from those by the algebraic inequalities. An example is given to illustrate the feasibility and effectiveness of the obtained results. The results of this paper are completely new and generalize those of the previous studies in [8-14]. Recently, the almost periodic solution of discrete neutral multi-species logarithmic population models has also paid more attention by numerous researchers. However, there are very few results on the almost periodic solutions of discrete neutral multi-species logarithmic population models, which might be our future research topic.

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