# Jacobi Elliptic Function Solutions For Fractional Partial Differential Equations

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*Abstract*—**In this paper, we are concerned with seeking exact solutions expressed in the Jacobi elliptic functions for fractional partial differential equations, where the fractional derivative is defined in the sense of the modified Riemann-Liouville derivative. Based on a fractional complex transformation, certain fractional partial differential equation is converted into another ordinary differential equation of integer order, and the exact solutions of the latter are assumed to be expressed in a polynomial in the Jacobi elliptic functions including the Jacobi sine function, the Jacobi cosine function, and the Jacobi elliptic function of the third kind. The degree of the polynomial can be determined by the homogeneous balance principle. As for applications, we apply this method to seek Jacobi elliptic function solutions for the space-time fractional KP-BBM equation and the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System.**

*Index Terms*—**Fractional differential equation; Jacobi elliptic function; Exact solution; Fractional complex transformation**

## I. Introduction

It is well known that nonlinear partial differential equations are widely used to describe many complex phenomena in various fields including either the scientific work or engineering fields. During the past few decades, searching for explicit solutions of nonlinear partial differential equations by using various methods has been the main goal for many researchers, and many powerful methods for constructing exact solutions of nonlinear partial differential equations have been established and developed. Some of these methods include the homogeneous balance method [1,2], the tanh-method [3-5], the inverse scattering transform [6], the generalized Riccati equation method [7-9], the (G'/G) method [10-13], the Jacobi elliptic function method [14-15] and so on.

Fractional differential equations involving fractional derivatives are generalizations of classical differential equations of integer order, and are widely used as models to express many important physical phenomena such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and so on (See [16,17] for example). In order to illustrate better the described physical phenomena, one need to obtain their analytical solutions. So the research on how to extend those methods suitable for solving differential equations of integer order to be suitable for solving fractional differential equations has been paid an increasing attention. Recently, under the definition of the modified Riemann-Liouville derivative [18-21], many authors have extended some efficient methods from differential equations of integer order to fractional differential equations. For example, in [22], Zhang et al. generalized the traditional Riccati sub-equation method to be suitable for seeking exact solutions of partial differential equations in fractional case, and proposed a new fractional Riccati sub-equation method, where the sub-equation used is the fractional Riccati equation $D_\xi^\alpha \phi = \sigma + \phi^2$, and $D^\alpha$ denotes the modified Riemann-Liouville derivative of $\alpha-$ order. This method got improved in [23-26]. In [27-29], the authors extended the (G'/G) method to be suitable for solving fractional partial differential equations, while in [30], the simplest equation method is extended to seek exact solutions of fractional partial differential equations. The most important point in these methods lies that based on a certain fractional complex transformation or a traveling wave transformation, certain fractional differential equation can be converted into another differential equation in different form, which can be solved based on an auxiliary equation named sub-equation. With these methods, a variety of fractional differential equations arising in mathematical physics have been investigated, and analytical solutions in various forms for these equations were found. These obtained solutions have contributed much in understanding better the physical effects that the fractional differential equations demonstrate.

In this paper, we extend the traditional Jacobi elliptic function method to seek exact solutions for fractional partial differential equations in the sense of the modified Riemann-Liouville derivative. First by a fractional complex transformation, certain fractional partial differential equation is converted into another ordinary differential equation of integer order. Then the exact solutions of the converted ordinary differential equation are assumed to be expressed in a polynomial in the Jacobi elliptic functions, where the coefficients are unknown. By use of the concept of the sub-equation methods and the properties of the Jacobi elliptic functions, the coefficients can be determined with the aid of mathematical software.

For the definition and theoretic investigations of the modified Riemann-Liouville fractional derivative, we refer the reader to [31-34]. Some important properties for the modified Riemann-Liouville derivative are listed as follows [18,22-30]:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}. \tag{1}$$

$$D_t^\alpha (f(t)g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t). \tag{2}$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)] D_t^\alpha g(t). \tag{3}$$

$$D_t^\alpha f[g(t)] = D_g^\alpha f[g(t)](g'(t))^\alpha. \tag{4}$$

Manuscript received July 18, 2015; revised October 12, 2015.

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The rest of this paper is organized as follows. In Section 2, we give the description of the Jacobi elliptic function method for solving fractional partial differential equations. Then in Section 3 and Section 4, we apply this method to seek exact solutions for the space-time fractional KP-BBM equation and the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System respectively. In Section 5, we present some concluding comments.

## II. Description of the Jacobi elliptic function method for solving fractional partial differential equations

In this section, we give the description of the Jacobi elliptic function method for solving fractional partial differential equations.

Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, ..., x_n$, is given by

$$P(u_1, ...u_k, D_t^\alpha u_1, ..., D_t^\alpha u_k, \frac{\partial u_1}{\partial x_1}, ..., \frac{\partial u_k}{\partial x_1}, D_{x_2}^\beta u_1, ...D_{x_2}^\beta u_1, ..., \frac{\partial u_1}{\partial x_{n-1}}, ..., \frac{\partial u_k}{\partial x_{n-1}}, D_{x_n}^\gamma u_1, ..., D_{x_n}^\gamma u_k, ...) = 0, \quad (5)$$

where $u_i = u_i(t, x_1, x_2, ..., x_n),\ i = 1, ..., k$ are unknown functions, $P$ is a polynomial in $u_i$ and their various partial derivatives including fractional derivatives in the sense of the modified Riemann-Liouville derivative.

Step 1. For Eq. (5), suppose that $u_i(t, x_1, x_2, ..., x_n) = U_i(\xi)$, and a fractional complex transformation for $\xi$ as follows:

$$\xi = \frac{ct^\alpha}{\Gamma(1+\alpha)} + k_1 x_1 + \frac{k_2 x_2^\beta}{\Gamma(1+\alpha)} + ...k_{n-1}x_{n-1} + \frac{k_n x_n^\gamma}{\Gamma(1+\gamma)} + \xi_0, \quad (6)$$

where $c,\ k_1, ...,\ k_{n-1},\ k_n,\ \xi_0$ are all nonzero constants. Based on the transformation above, for the terms in (5) containing fractional derivative, such as $D_t^\alpha u_1$, using (1) and (3) one can obtain that

$$D_t^\alpha u_1 = D_t^\alpha U_1(\xi) = U_1'(\xi) D_t^\alpha \xi = cU_1'(\xi).$$

For the terms in (5) containing derivative of integer order, such as $\frac{\partial u_1}{\partial x_1}$, one has

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial U_1}{\partial \xi}\xi'_{x_1} = k_1 U_1'(\xi).$$

So by this transformation for $\xi$, Eq. (5) can be turned into the following ordinary differential equation of integer order with respect to the variable $\xi$:

$$\widetilde{P}(U_1, ..., U_k,\ U_1', ..., U_k',\ U_1'', ..., U_k'',\ ...) = 0. \quad (7)$$

Step 2. Suppose that the solution of (7) can be expressed by a polynomial in the Jacobi elliptic functions as follows:

$$U_j(\xi) = a_j^{(0)} + \sum_{n+p+q=1}^{m_j} a_j^{(n,p,q)} sn^n(\xi) cn^p(\xi) dn^q(\xi), \quad j = 1, 2, ..., k, \quad (8)$$

where $n,\ p,\ q$ are nonnegative integers with $1 \le n + p + q \le m_j$, $a_j^{(0)},\ a_j^{(n,p,q)},\ j = 1, 2, ..., k$ are constants to be determined later, the positive integer $m_j$ can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (7), $sn(\xi),\ cn(\xi),\ dn(\xi)$ denote the Jacobi elliptic sine function, Jacobi elliptic cosine function, and the Jacobi elliptic function of the third kind respectively.

For the Jacobi elliptic functions, one has

$$sn'(\xi) = cn(\xi)dn(\xi),\ cn'(\xi) = -sn(\xi)dn(\xi),$$
$$dn'(\xi) = -m^2 sn(\xi)cn(\xi), \quad (9)$$

where $m$ is the modulus, and

$$cs(\xi) = \frac{cn(\xi)}{sn(\xi)},\ sd(\xi) = \frac{sn(\xi)}{dn(\xi)},\ dc(\xi) = \frac{dn(\xi)}{cn(\xi)},$$

$$sc(\xi) = \frac{1}{cs(\xi)},\ ds(\xi) = \frac{1}{sd(\xi)},$$

$$cd(\xi) = \frac{1}{dc(\xi)},\ nd(\xi) = \frac{1}{dn(\xi)},\ ns(\xi) = \frac{1}{sn(\xi)},\ nc(\xi) = \frac{1}{cn(\xi)}.$$

$$cn^2(\xi) = -sn^2(\xi) + 1,\ dn^2(\xi) = -m^2 sn^2(\xi) + 1,$$
$$dn^2(\xi) = m^2 cn^2(\xi) + 1 - m^2,\ ns^2(\xi) = cs^2(\xi) + 1,$$
$$ns^2(\xi) = ds^2(\xi) + m^2,\ ds^2(\xi) = cs^2(\xi) + 1 - m^2,$$
$$nc^2(\xi) = sc^2(\xi) + 1,\ dc^2(\xi) = (1 - m^2)nc^2(\xi) + m^2,$$
$$dc^2(\xi) = (1-m^2)sc^2(\xi) + 1,\ cd^2(\xi) = \frac{m^2-1}{m^2} nd^2(\xi) + \frac{1}{m^2},$$
$$cd^2(\xi) = (m^2 - 1)sd^2(\xi) + 1,\ nd^2(\xi) = m^2 sd^2(\xi) + 1.$$

Step 3. Substituting (8) into (7) and using (9), the left-hand side of (7) is converted into another polynomial in $sn^n(\xi)cn^p(\xi)dn^q(\xi)$. Collecting all coefficients of the same power and Equating them to zero, yields a set of algebraic equations for $a_j^{(0)},\ a_j^{(n,p,q)},\ j = 1, 2, ..., k$.

Step 4. Solving the equations system in Step 3, we can construct a variety of Jacobi elliptic function solutions for Eq. (5).

## III. Application of the Jacobi elliptic function method to the space-time fractional KP-BBM equation

In this section, we apply the Jacobi elliptic function method to seek exact solutions for the space-time fractional KP-BBM equation, which is denoted as follows:

$$D_x^\beta[D_t^\alpha u + D_x^\beta u - aD_x^\beta u^2 - bD_t^\alpha(D_x^{2\beta} u)] + eD_y^{2\gamma} u = 0, \quad (10)$$

where $0 < \alpha,\ \beta,\ \gamma \le 1$, $a,\ b,\ e$ are constants, $u = u(x, y, t)$ is unknown, and the concerned fractional derivative is defined by the modified Riemann-Liouville derivative. When $\alpha = \beta = \gamma = 1$, Eq. (10) becomes the following known KP-BBM equation of integer order:

$$(u_t + u_x - a(u^2)_x - bu_{xxt})_x + eu_{yy} = 0.$$

In order to apply the Jacobi elliptic function method to solve Eq. (10), we suppose $u(x, y, t) = U(\xi)$, where $\xi = \frac{c}{\Gamma(1+\alpha)} t^\alpha + \frac{k}{\Gamma(1+\beta)} x^\beta + \frac{k}{\Gamma(1+\gamma)} y^\gamma + \xi_0$, $c,\ k,\ \xi_0$ are all constants with $k,\ c\neq 0$. Then by use of (1) and (3) one can deduce that $D_t^{\alpha}\xi=c,\ D_x^{\beta}\xi=D_y^{\gamma}\xi=k$, and

$$\begin{cases} D_t^{\alpha}u=D_t^{\alpha}U(\xi)=U'(\xi)D_t^{\alpha}\xi=cU'(\xi), \\ D_x^{\beta}u=D_x^{\beta}U(\xi)=U'(\xi)D_x^{\beta}\xi=kU'(\xi), \\ D_y^{\gamma}u=D_y^{\gamma}U(\xi)=U'(\xi)D_y^{\gamma}\xi=kU'(\xi). \end{cases} \tag{11}$$

Then Eq. (10) can be turned into the following form with respect to the new variable $\xi$:

$$ckU''(\xi)+k^2U''(\xi)-2ak^2[(U'(\xi))^2+U(\xi)U''(\xi)] \\ -bck^3U^{(4)}(\xi)+ek^2U''(\xi)=0. \tag{12}$$

Suppose that the solution of Eq. (12) can be expressed by a polynomial in the Jacobi elliptic functions as follows:

$$U(\xi)=a^{(0)}+\sum_{n+p+q=1}^{m}a^{(n,p,q)}sn^n(\xi)cn^p(\xi)dn^q(\xi). \tag{13}$$

By balancing the order of $U^{(4)}(\xi)$ and $U(\xi)U''(\xi)$ in (12) one can obtain $m=2$. So

$$U(\xi)=a^{(0)}+a^{(1,0,0)}sn(\xi)+a^{(0,1,0)}cn(\xi)+a^{(0,0,1)}dn(\xi) \\ +a^{(2,0,0)}sn^2(\xi)+a^{(1,1,0)}sn(\xi)cn(\xi)+a^{(0,1,1)}cn(\xi)dn(\xi) \\ +a^{(0,2,0)}cn^2(\xi)+a^{(0,0,2)}dn^2(\xi)+a^{(1,0,1)}sn(\xi)dn(\xi). \tag{14}$$

Substituting (14) into (12), using (9) and collecting all the terms with the same power of $sn^n(\xi)cn^p(\xi)dn^q(\xi)$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations with the aid of mathematical software such as Maple, yields the following values, where $i$ denotes the unit of the imaginary numbers.

Case 1:

$$a^{(0)}=\frac{k+bck^2m^2+4bck^2+ek+c}{2ak},\ a^{(1,0,0)}=0,$$
$$a^{(0,1,0)}=0,\ a^{(0,0,1)}=0,\ a^{(2,0,0)}=-\frac{3m^2kcb}{a},$$
$$a^{(1,1,0)}=\pm\frac{3cbkm^2}{a}i,\ a^{(0,1,1)}=0,\ a^{(1,0,1)}=0,$$
$$a^{(0,2,0)}=a^{(0,0,2)}=0.$$

Case 2:

$$a^{(0)}=\frac{k+4bck^2m^2+4bck^2+ek+c}{2ak},\ a^{(1,0,0)}=0,$$
$$a^{(0,1,0)}=0,\ a^{(0,0,1)}=0,\ a^{(2,0,0)}=-\frac{6m^2kcb}{a},$$
$$a^{(1,1,0)}=0,\ a^{(0,1,1)}=0,\ a^{(1,0,1)}=0,$$
$$a^{(0,2,0)}=a^{(0,0,2)}=0.$$

Case 3:

$$a^{(0)}=\frac{k+4bck^2m^2+bck^2+ek+c}{2ak},\ a^{(1,0,0)}=0,$$
$$a^{(0,1,0)}=0,\ a^{(0,0,1)}=0,\ a^{(2,0,0)}=-\frac{3m^2kcb}{a},$$
$$a^{(1,1,0)}=0,\ a^{(0,1,1)}=0,\ a^{(1,0,1)}=\pm\frac{3cbkm}{a}i,$$
$$a^{(0,2,0)}=a^{(0,0,2)}=0.$$

Case 4:

$$a^{(0)}=\frac{k+bck^2m^2+bck^2+ek+c}{2ak},\ a^{(1,0,0)}=0,$$
$$a^{(0,1,0)}=0,\ a^{(0,0,1)}=0,\ a^{(2,0,0)}=-\frac{3m^2kcb}{a},$$
$$a^{(1,1,0)}=0,\ a^{(0,1,1)}=\pm\frac{3mkcb}{a},\ a^{(1,0,1)}=0,$$
$$a^{(0,2,0)}=a^{(0,0,2)}=0.$$

Case 5:

$$a^{(0)}=\frac{k+bck^2m^2+bck^2+ek+c}{2ak},\ a^{(1,0,0)}=0,$$
$$a^{(0,1,0)}=0,\ a^{(0,0,1)}=0,\ a^{(2,0,0)}=-\frac{3m^2kcb}{2a},$$
$$a^{(1,1,0)}=\pm\frac{3cbkm^2}{2a}i,\ a^{(0,1,1)}=\pm\frac{3mkcb}{2a},$$
$$a^{(1,0,1)}=\pm\frac{3cbkm}{2a}i,\ a^{(0,2,0)}=a^{(0,0,2)}=0.$$

Substituting the results above into Eq. (14) we can obtain the following exact solutions in the forms of the Jacobi elliptic functions for Eq. (10), where $\xi=\frac{c}{\Gamma(1+\alpha)}t^{\alpha}+\frac{k}{\Gamma(1+\beta)}x^{\beta}+\frac{k}{\Gamma(1+\gamma)}y^{\gamma}+\xi_0$.

Family 1:

$$u_1(x,y,t)=\frac{k+bck^2m^2+4bck^2+ek+c}{2ak} \\ -\frac{3m^2kcb}{a}sn^2(\xi)\pm\frac{3cbkm^2i}{a}sn(\xi)cn(\xi). \tag{15}$$

Family 2:

$$u_2(x,y,t)=\frac{k+4bck^2m^2+4bck^2+ek+c}{2ak}-\frac{6m^2kcb}{a}sn^2(\xi). \tag{16}$$

Family 3:

$$u_3(x,y,t)=\frac{k+4bck^2m^2+bck^2+ek+c}{2ak} \\ -\frac{3m^2kcb}{a}sn^2(\xi)\pm\frac{3cbkm}{a}isn(\xi)dn(\xi). \tag{17}$$

Family 4:

$$u_4(x,y,t)=\frac{k+bck^2m^2+bck^2+ek+c}{2ak} \\ -\frac{3m^2kcb}{a}sn^2(\xi)\pm\frac{3mkcb}{a}cn(\xi)dn(\xi). \tag{18}$$

Family 5:

$$u_5(x,y,t)=\frac{k+bck^2m^2+bck^2+ek+c}{2ak} \\ -\frac{3m^2kcb}{2a}sn^2(\xi)\pm\frac{3cbkm^2i}{2a}sn(\xi)cn(\xi) \\ \pm\frac{3mkcb}{2a}cn(\xi)dn(\xi)\pm\frac{3cbkmi}{2a}sn(\xi)dn(\xi). \tag{19}$$

**Remark 1**. We note that the Jacobi elliptic function solutions established in (15)-(19) for the space-time fractional KP-BBM equation (10) are new exact solutions so far in the literature.

## IV. Application of the Jacobi elliptic function method to the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System

Consider the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System [28]

$$\begin{cases} D_t^{\alpha}u + aD_x^{3\beta}u + bD_y^{3\gamma}u + cD_x^{\beta}u + dD_y^{\gamma}u \\ \qquad = 3aD_x^{\beta}(uv) + 3bD_y^{\gamma}(uw), \\ D_x^{\beta}u = D_y^{\gamma}v, \\ D_y^{\gamma}u = D_x^{\beta}w, \end{cases} ,0<\alpha,\ \beta,\ \gamma \le 1. \tag{20}$$

In [28], the author solved Eqs. (20) by use of the (G'/G) method, and obtained some exact solutions including hyperbolic function solutions, trigonometric function solutions, and rational function functions for it. Now we apply the Jacobi function method method to solve it. Suppose $u(x,y,t) = U(\xi),\ v(x,y,t) = V(\xi),\ w(x,y,t) = W(\xi)$, where $\xi = \frac{m}{\Gamma(1+\alpha)}t^{\alpha} + \frac{k}{\Gamma(1+\beta)}x^{\beta} + \frac{l}{\Gamma(1+\gamma)}y^{\gamma} + \xi_0$, $m,\ k,\ l,\ \xi_0$ are all constants with $k,\ l,\ m \neq 0$. By use of (1) and (3), we obtain

$$\begin{cases} D_t^{\alpha}u = D_t^{\alpha}U(\xi) = U'(\xi)D_t^{\alpha}\xi = mU'(\xi), \\ D_x^{\beta}u = D_x^{\beta}U(\xi) = U'(\xi)D_x^{\beta}\xi = kU'(\xi), \\ D_y^{\gamma}u = D_y^{\gamma}U(\xi) = U'(\xi)D_y^{\gamma}\xi = lU'(\xi), \end{cases}$$

and then Eqs. (17) can be turned into the following forms

$$\begin{cases} mU' + ak^3U''' + bl^3U''' + ckU' + dlU' \\ \qquad = 3ak(UV)' + 3bl(UW)', \\ kU' = lV', \\ lU' = kW'. \end{cases} \tag{21}$$

Suppose that the solution of Eqs. (21) can be expressed by a polynomial in the Jacobi elliptic functions as follows:

$$\begin{cases} U(\xi) = a^{(0)} + \sum\limits_{n+p+q=1}^{m_1} a^{(n,p,q)} sn^n(\xi)cn^p(\xi)dn^q(\xi), \\ V(\xi) = b^{(0)} + \sum\limits_{n+p+q=1}^{m_2} b^{(n,p,q)} sn^n(\xi)cn^p(\xi)dn^q(\xi), \\ W(\xi) = c^{(0)} + \sum\limits_{n+p+q=1}^{m_3} c^{(n,p,q)} sn^n(\xi)cn^p(\xi)dn^q(\xi). \end{cases} \tag{22}$$

Balancing the order of $U'''$ and $(UV)'$, the order of $U'$ and $V'$, the order of $U'$ and $W'$ in (21), we can obtain $m_1 = m_2 = m_3 = 2$. So we have

$$\begin{cases} U(\xi) = a^{(0)} + a^{(1,0,0)}sn(\xi) + a^{(0,1,0)}cn(\xi) + a^{(0,0,1)}dn(\xi) \\ \quad +a^{(2,0,0)}sn^2(\xi) + a^{(1,1,0)}sn(\xi)cn(\xi) \\ \quad +a^{(0,1,1)}cn(\xi)dn(\xi) + a^{(1,0,1)}sn(\xi)dn(\xi) \\ \quad +a^{(0,2,0)}cn^2(\xi) + a^{(0,0,2)}dn^2(\xi), \\ V(\xi) = b^{(0)} + b^{(1,0,0)}sn(\xi) + b^{(0,1,0)}cn(\xi) + b^{(0,0,1)}dn(\xi) \\ \quad +b^{(2,0,0)}sn^2(\xi) + b^{(1,1,0)}sn(\xi)cn(\xi) \\ \quad +b^{(0,1,1)}cn(\xi)dn(\xi) + b^{(1,0,1)}sn(\xi)dn(\xi) \\ \quad +b^{(0,2,0)}cn^2(\xi) + b^{(0,0,2)}dn^2(\xi), \\ W(\xi) = c^{(0)} + c^{(1,0,0)}sn(\xi) + c^{(0,1,0)}cn(\xi) + c^{(0,0,1)}dn(\xi) \\ \quad +c^{(2,0,0)}sn^2(\xi) + c^{(1,1,0)}sn(\xi)cn(\xi) \\ \quad +c^{(0,1,1)}cn(\xi)dn(\xi) + c^{(1,0,1)}sn(\xi)dn(\xi) \\ \quad +c^{(0,2,0)}cn^2(\xi) + c^{(0,0,2)}dn^2(\xi). \end{cases} \tag{23}$$

Substituting (23) into (21), using (9) and collecting all the terms with the same power of $sn^n(\xi)cn^p(\xi)dn^q(\xi)$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations with the aid of mathematical software, yields the following values.

Case 1:

$$a^{(0)} = a^{(0)},\ a^{(1,0,0)} = 0,\ a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,$$

$$a^{(2,0,0)} = \frac{m^2l(bl^3+ak^3)}{k(bl+ak)},\ a^{(1,1,0)} = 0,\ a^{(0,1,1)} = 0,$$

$$a^{(1,0,1)} = \pm\frac{ml(bl^3+ak^3)}{k(bl+ak)}i,\ a^{(0,2,0)} = a^{(0,0,2)} =$$

$$b^{(0,2,0)} = b^{(0,0,2)} = c^{(0,2,0)} = c^{(0,0,2)} = 0$$

$$b^{(0)} = -\frac{4ak^3m^2l + ak^3l + 3a^{(0)}k^2a + 3a^{(0)}kbl}{3l(bl+ak)}$$

$$-\frac{-ckl - dl^2 + bl^4 + 4bl^4m^2 - ml}{3l(bl+ak)},$$

$$b^{(1,0,0)} = 0,\ b^{(0,1,0)} = 0,\ b^{(0,0,1)} = 0,$$

$$b^{(2,0,0)} = \frac{m^2(bl^3+ak^3)}{bl+ak},\ b^{(1,1,0)} = 0,\ b^{(0,1,1)} = 0,$$

$$b^{(1,0,1)} = \pm\frac{m(bl^3+ak^3)}{bl+ak}i,\ c^{(0)} = c^{(0)},\ c^{(1,0,0)} = 0,$$

$$c^{(0,1,0)} = 0,\ c^{(0,0,1)} = 0,\ c^{(2,0,0)} = \frac{l^2m^2(bl^3+ak^3)}{k^2(bl+ak)},$$

$$c^{(1,1,0)} = 0,\ c^{(0,1,1)} = 0,\ c^{(1,0,1)} = \pm\frac{l^2(bl^3+ak^3)m}{k^2(bl+ak)}i.$$

Case 2:

$$a^{(0)} = a^{(0)},\ a^{(1,0,0)} = 0,\ a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,$$

$$a^{(2,0,0)} = \frac{m^2l(bl^3+ak^3)}{k(bl+ak)},\ a^{(1,1,0)} = \pm\frac{m^2l(bl^3+ak^3)}{k(bl+ak)}i,$$

$$a^{(0,1,1)} = 0,\ a^{(1,0,1)} = 0,$$

$$b^{(0)} = -\frac{ak^3m^2l + 4ak^3l + 3a^{(0)}k^2a + 3a^{(0)}kbl}{3l(bl+ak)}$$

$$-\frac{-ckl + bl^4m^2 - ml - dl^2 + 4bl^4}{3l(bl+ak)},$$

$$b^{(1,0,0)} = 0,\ b^{(0,1,0)} = 0,\ b^{(0,0,1)} = 0,\ b^{(2,0,0)} = \frac{m^2(bl^3+ak^3)}{bl+ak},$$

$$b^{(1,1,0)} = \pm\frac{m^2(bl^3+ak^3)}{bl+ak}i,\ b^{(0,1,1)} = 0,\ b^{(1,0,1)} = 0,$$

$$c^{(0)} = c^{(0)},\ c^{(1,0,0)} = 0,\ c^{(0,1,0)} = 0,\ c^{(0,0,1)} = 0,$$

$$c^{(2,0,0)} = \frac{l^2m^2(bl^3+ak^3)}{k^2(bl+ak)},\ c^{(1,1,0)} = \pm\frac{l^2(bl^3+ak^3)m^2}{k^2(bl+ak)}i,$$

$$c^{(0,1,1)} = 0,\ c^{(1,0,1)} = 0,\ a^{(0,2,0)} = a^{(0,0,2)} =$$

$$b^{(0,2,0)} = b^{(0,0,2)} = c^{(0,2,0)} = c^{(0,0,2)} = 0.$$

Case 3:

$$a^{(0)} = a^{(0)},\ a^{(1,0,0)} = 0,\ a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,$$

$$a^{(2,0,0)} = \frac{2m^2l(bl^3+ak^3)}{k(bl+ak)},\ a^{(1,1,0)} = 0,$$

$$a^{(0,1,1)} = 0,\ a^{(1,0,1)} = 0,$$

$$b^{(0)} = -\frac{4ak^3l + 4ak^3m^2l + 3a^{(0)}k^2a + 3a^{(0)}kbl}{3l(bl+ak)}$$

$$-\frac{-ckl - dl^2 + 4bl^4m^2 + 4bl^4 - ml}{3l(bl+ak)},$$

$$b^{(1,0,0)} = 0,\ b^{(0,1,0)} = 0,\ b^{(0,0,1)} = 0,$$

$$b^{(2,0,0)} = \frac{2m^2(bl^3+ak^3)}{bl+ak},\ b^{(1,1,0)} = 0,\ b^{(0,1,1)} = 0,\ b^{(1,0,1)} = 0,$$

$$c^{(0)} = c^{(0)},\ c^{(1,0,0)} = 0,\ c^{(0,1,0)} = 0,\ c^{(0,0,1)} = 0,$$

$$c^{(2,0,0)} = \frac{2l^2m^2(bl^3+ak^3)}{k^2(bl+ak)},\ c^{(1,1,0)} = 0,$$

$$c^{(0,1,1)} = 0,\ c^{(1,0,1)} = 0,\ a^{(0,2,0)} = a^{(0,0,2)} = b^{(0,2,0)} = b^{(0,0,2)} = c^{(0,2,0)} = c^{(0,0,2)} = 0.$$

Case 4:

$$a^{(0)} = -\frac{(ak^3+bl^3m^2-ck+3blb^{(0)}+ak^3m^2)l}{3k(bl+ak)} - \frac{(-m+3ab^{(0)}k-dl+bl^3)l}{3k(bl+ak)},$$

$$a^{(1,0,0)} = 0,\ a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,$$

$$a^{(2,0,0)} = \frac{m^2l(bl^3+ak^3)}{k(bl+ak)},\ a^{(1,1,0)} = 0,$$

$$a^{(0,1,1)} = \pm\frac{ml(bl^3+ak^3)}{k(bl+ak)},\ a^{(1,0,1)} = 0,$$

$$b^{(0)} = b^{(0)},\ b^{(1,0,0)} = 0,\ b^{(0,1,0)} = 0,\ b^{(0,0,1)} = 0,$$

$$b^{(2,0,0)} = \frac{m^2(bl^3+ak^3)}{bl+ak},\ b^{(1,1,0)} = 0,$$

$$b^{(0,1,1)} = \pm\frac{m(bl^3+ak^3)}{bl+ak},\ b^{(1,0,1)} = 0,$$

$$c^{(0)} = c^{(0)},\ c^{(1,0,0)} = 0,\ c^{(0,1,0)} = 0,\ c^{(0,0,1)} = 0,$$

$$c^{(2,0,0)} = \frac{l^2m^2(bl^3+ak^3)}{k^2(bl+ak)},\ c^{(1,1,0)} = 0,$$

$$c^{(0,1,1)} = \pm\frac{l^2m(bl^3+ak^3)}{k^2(bl+ak)},\ c^{(1,0,1)} = 0,\ a^{(0,2,0)} = a^{(0,0,2)} = b^{(0,2,0)} = b^{(0,0,2)} = c^{(0,2,0)} = c^{(0,0,2)} = 0.$$

Case 5:

$$a^{(0)} = -\frac{(ak^3+bl^3m^2-ck+3blb^{(0)}+ak^3m^2)l}{3k(bl+ak)} - \frac{(-m+3ab^{(0)}k-dl+bl^3)l}{3k(bl+ak)},$$

$$a^{(1,0,0)} = 0,\ a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,\ a^{(0,2,0)} = a^{(0,0,2)} = b^{(0,2,0)} = b^{(0,0,2)} = c^{(0,2,0)} = c^{(0,0,2)} = 0,$$

$$a^{(2,0,0)} = \frac{m^2l(bl^3+ak^3)}{2k(bl+ak)},\ a^{(1,1,0)} = \mp\frac{m^2l(bl^3+ak^3)}{2k(bl+ak)}i,$$

$$a^{(0,1,1)} = \frac{ml(bl^3+ak^3)}{2k(bl+ak)},\ a^{(1,0,1)} = \pm\frac{ml(bl^3+ak^3)}{2k(bl+ak)}i,$$

$$b^{(0)} = b^{(0)},\ b^{(1,0,0)} = 0,\ b^{(0,1,0)} = 0,\ b^{(0,0,1)} = 0,$$

$$b^{(2,0,0)} = \frac{m^2(bl^3+ak^3)}{2(bl+ak)},\ b^{(1,1,0)} = \mp\frac{m^2(bl^3+ak^3)}{2(bl+ak)}i,$$

$$b^{(0,1,1)} = \frac{m(bl^3+ak^3)}{2(bl+ak)},\ b^{(1,0,1)} = \pm\frac{m(bl^3+ak^3)}{2(bl+ak)}i,$$

$$c^{(0)} = c^{(0)},\ c^{(1,0,0)} = 0,\ c^{(0,1,0)} = 0,\ c^{(0,0,1)} = 0,$$

$$c^{(2,0,0)} = \frac{l^2m^2(bl^3+ak^3)}{2k^2(bl+ak)},\ c^{(1,1,0)} = \mp\frac{l^2m^2(bl^3+ak^3)}{2k^2(bl+ak)}i,$$

$$c^{(0,1,1)} = \frac{l^2m(bl^3+ak^3)}{2k^2(bl+ak)},\ c^{(1,0,1)} = \pm\frac{l^2m(bl^3+ak^3)}{2k^2(bl+ak)}i.$$

Case 6:

$$a^{(0)} = -\frac{(ak^3+bl^3m^2-ck+3blb^{(0)}+ak^3m^2)l}{3k(bl+ak)} - \frac{(-m+3ab^{(0)}k-dl+bl^3)l}{3k(bl+ak)},$$

$$a^{(1,0,0)} = 0,\ a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,$$

$$a^{(0,2,0)} = a^{(0,0,2)} = b^{(0,2,0)} = b^{(0,0,2)} = c^{(0,2,0)} = c^{(0,0,2)} = 0,$$

$$a^{(2,0,0)} = \frac{m^2l(bl^3+ak^3)}{2k(bl+ak)},\ a^{(1,1,0)} = \pm\frac{m^2l(bl^3+ak^3)}{2k(bl+ak)}i,$$

$$a^{(0,1,1)} = -\frac{ml(bl^3+ak^3)}{2k(bl+ak)},\ a^{(1,0,1)} = \pm\frac{ml(bl^3+ak^3)}{2k(bl+ak)}i,$$

$$b^{(0)} = b^{(0)},\ b^{(1,0,0)} = 0,\ b^{(0,1,0)} = 0,\ b^{(0,0,1)} = 0,$$

$$b^{(2,0,0)} = \frac{m^2(bl^3+ak^3)}{2(bl+ak)},\ b^{(1,1,0)} = \pm\frac{m^2(bl^3+ak^3)}{2(bl+ak)}i,$$

$$b^{(0,1,1)} = -\frac{m(bl^3+ak^3)}{2(bl+ak)},\ b^{(1,0,1)} = \pm\frac{m(bl^3+ak^3)}{2(bl+ak)}i,$$

$$c^{(0)} = c^{(0)},\ c^{(1,0,0)} = 0,\ c^{(0,1,0)} = 0,\ c^{(0,0,1)} = 0,$$

$$c^{(2,0,0)} = \frac{l^2m^2(bl^3+ak^3)}{2k^2(bl+ak)},\ c^{(1,1,0)} = \pm\frac{l^2m^2(bl^3+ak^3)}{2k^2(bl+ak)}i,$$

$$c^{(0,1,1)} = -\frac{l^2m(bl^3+ak^3)}{2k^2(bl+ak)},\ c^{(1,0,1)} = \pm\frac{l^2m(bl^3+ak^3)}{2k^2(bl+ak)}i.$$

Substituting the results above into Eq. (23) we can obtain the following Jacobi elliptic functions solutions for Eqs. (20), where $\xi = \frac{m}{\Gamma(1+\alpha)}t^\alpha + \frac{k}{\Gamma(1+\beta)}x^\beta + \frac{l}{\Gamma(1+\gamma)}y^\gamma + \xi_0$.

Family 1:

$$\begin{cases} u_1(x,y,t) = a^{(0)} + \frac{m^2l(bl^3+ak^3)}{k(bl+ak)}sn^2(\xi) \\ \qquad \pm\frac{ml(bl^3+ak^3)i}{k(bl+ak)}sn(\xi)dn(\xi), \\ v_1(x,y,t) = -\frac{4ak^3m^2l+ak^3l+3a^{(0)}k^2a+3a^{(0)}kbl}{3l(bl+ak)} \\ \qquad -\frac{-ckl-dl^2+bl^4+4bl^4m^2-ml}{3l(bl+ak)} \\ \qquad +\frac{m^2(bl^3+ak^3)}{bl+ak}sn^2(\xi) \\ \qquad \pm\frac{m(bl^3+ak^3)i}{bl+ak}sn(\xi)dn(\xi), \\ w_1(x,y,t) = c^{(0)} + \frac{l^2m^2(bl^3+ak^3)}{k^2(bl+ak)}sn^2(\xi) \\ \qquad \pm\frac{l^2(bl^3+ak^3)mi}{k^2(bl+ak)}sn(\xi)dn(\xi). \end{cases} \tag{24}$$

Family 2:

$$\begin{cases} u_2(x,y,t) = a^{(0)} + \dfrac{m^2 l(bl^3+ak^3)}{k(bl+ak)} sn^2(\xi) \\ \qquad \pm \dfrac{m^2 l(bl^3+ak^3)i}{k(bl+ak)} sn(\xi)cn(\xi), \\ v_2(x,y,t) = -\dfrac{ak^3m^2l + 4ak^3l + 3a^{(0)}k^2a + 3a^{(0)}kbl}{3l(bl+ak)} \\ \qquad - \dfrac{-ckl + bl^4m^2 - ml - dl^2 + 4bl^4}{3l(bl+ak)} \\ \qquad + \dfrac{m^2(bl^3+ak^3)}{bl+ak} sn^2(\xi) \\ \qquad \pm \dfrac{m^2(bl^3+ak^3)i}{bl+ak} sn(\xi)cn(\xi), \\ w_2(x,y,t) = c^{(0)} + \dfrac{l^2m^2(bl^3+ak^3)}{k^2(bl+ak)} sn^2(\xi) \\ \qquad \pm \dfrac{l^2(bl^3+ak^3)m^2 i}{k^2(bl+ak)} sn(\xi)cn(\xi). \end{cases} \tag{25}$$

Family 3:

$$\begin{cases} u_3(x,y,t) = a^{(0)} + \dfrac{2m^2 l(bl^3+ak^3)}{k(bl+ak)} sn^2(\xi), \\ v_3(x,y,t) = -\dfrac{4ak^3l + 4ak^3m^2l + 3a^{(0)}k^2a + 3a^{(0)}kbl}{3l(bl+ak)} \\ \qquad - \dfrac{-ckl - dl^2 + 4bl^4m^2 + 4bl^4 - ml}{3l(bl+ak)} \\ \qquad + \dfrac{2m^2(bl^3+ak^3)}{bl+ak} sn^2(\xi), \\ w_3(x,y,t) = c^{(0)} + \dfrac{2l^2m^2(bl^3+ak^3)}{k^2(bl+ak)} sn^2(\xi). \end{cases} \tag{26}$$

Family 4:

$$\begin{cases} u_4(x,y,t) = -\dfrac{(ak^3 + bl^3m^2 - ck + 3blb^{(0)} + ak^3m^2)l}{3k(bl+ak)} \\ \qquad - \dfrac{(-m + 3ab^{(0)}k - dl + bl^3)l}{3k(bl+ak)} \\ \qquad + \dfrac{m^2 l(bl^3+ak^3)}{k(bl+ak)} sn^2(\xi) \\ \qquad \pm \dfrac{ml(bl^3+ak^3)}{k(bl+ak)} cn(\xi)dn(\xi), \\ v_4(x,y,t) = b^{(0)} + \dfrac{m^2(bl^3+ak^3)}{bl+ak} sn^2(\xi) \\ \qquad \pm \dfrac{m(bl^3+ak^3)}{bl+ak} cn(\xi)dn(\xi), \\ w_4(x,y,t) = c^{(0)} + \dfrac{l^2m^2(bl^3+ak^3)}{k^2(bl+ak)} sn^2(\xi) \\ \qquad \pm \dfrac{l^2m(bl^3+ak^3)}{k^2(bl+ak)} cn(\xi)dn(\xi). \end{cases} \tag{27}$$

Family 5:

$$\begin{cases} u_5(x,y,t) = -\dfrac{(ak^3 + bl^3m^2 - ck + 3blb^{(0)} + ak^3m^2)l}{3k(bl+ak)} \\ \qquad - \dfrac{(-m + 3ab^{(0)}k - dl + bl^3)l}{3k(bl+ak)} \\ \qquad + \dfrac{m^2 l(bl^3+ak^3)}{k(bl+ak)} sn^2(\xi) \\ \qquad \mp \dfrac{m^2 l(bl^3+ak^3)i}{2k(bl+ak)} sn(\xi)cn(\xi) \\ \qquad + \dfrac{ml(bl^3+ak^3)}{2k(bl+ak)} cn(\xi)dn(\xi) \\ \qquad \pm \dfrac{ml(bl^3+ak^3)i}{2k(bl+ak)} sn(\xi)dn(\xi), \\ v_5(x,y,t) = b^{(0)} + \dfrac{m^2(bl^3+ak^3)}{2(bl+ak)} sn^2(\xi) \\ \qquad \mp \dfrac{m^2(bl^3+ak^3)i}{2(bl+ak)} sn(\xi)cn(\xi) \\ \qquad + \dfrac{m(bl^3+ak^3)}{2(bl+ak)} cn(\xi)dn(\xi) \\ \qquad \pm \dfrac{m(bl^3+ak^3)i}{2(bl+ak)} sn(\xi)dn(\xi), \\ w_5(x,y,t) = c^{(0)} + \dfrac{l^2m^2(bl^3+ak^3)}{2k^2(bl+ak)} sn^2(\xi) \\ \qquad \mp \dfrac{l^2m^2(bl^3+ak^3)i}{2k^2(bl+ak)} sn(\xi)cn(\xi) \\ \qquad + \dfrac{l^2m(bl^3+ak^3)}{2k^2(bl+ak)} cn(\xi)dn(\xi) \\ \qquad \pm \dfrac{l^2m(bl^3+ak^3)}{2k^2(bl+ak)} isn(\xi)dn(\xi). \end{cases} \tag{28}$$

Family 6:

$$\begin{cases} u_6(x,y,t) = -\dfrac{(ak^3 + bl^3m^2 - ck + 3blb^{(0)} + ak^3m^2)l}{3k(bl+ak)} \\ \qquad - \dfrac{(-m + 3ab^{(0)}k - dl + bl^3)l}{3k(bl+ak)} \\ \qquad + \dfrac{m^2 l(bl^3+ak^3)}{k(bl+ak)} sn^2(\xi) \\ \qquad \mp \dfrac{m^2 l(bl^3+ak^3)i}{2k(bl+ak)} sn(\xi)cn(\xi) \\ \qquad - \dfrac{ml(bl^3+ak^3)}{2k(bl+ak)} cn(\xi)dn(\xi) \\ \qquad \pm \dfrac{ml(bl^3+ak^3)i}{2k(bl+ak)} sn(\xi)dn(\xi), \\ v_6(x,y,t) = b^{(0)} + \dfrac{m^2(bl^3+ak^3)}{2(bl+ak)} sn^2(\xi) \\ \qquad \mp \dfrac{m^2(bl^3+ak^3)i}{2(bl+ak)} sn(\xi)cn(\xi) \\ \qquad - \dfrac{m(bl^3+ak^3)}{2(bl+ak)} cn(\xi)dn(\xi) \\ \qquad \pm \dfrac{m(bl^3+ak^3)i}{2(bl+ak)} sn(\xi)dn(\xi), \\ w_6(x,y,t) = c^{(0)} + \dfrac{l^2m^2(bl^3+ak^3)}{2k^2(bl+ak)} sn^2(\xi) \\ \qquad \mp \dfrac{l^2m^2(bl^3+ak^3)i}{2k^2(bl+ak)} sn(\xi)cn(\xi) \\ \qquad - \dfrac{l^2m(bl^3+ak^3)}{2k^2(bl+ak)} cn(\xi)dn(\xi) \\ \qquad \pm \dfrac{l^2m(bl^3+ak^3)}{2k^2(bl+ak)} isn(\xi)dn(\xi). \end{cases} \tag{29}$$

**Remark 2**. To our best knowledge, the Jacobi elliptic function solutions established in (24)-(29) are new exact solutions to the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System (20).

## V. Further results for the space-time fractional KP-BBM equation

In this section, as an extension of the method used in Sections II-IV, we propose a different approach to deduce new exact solutions for the following space-time fractional KP-BBM equation

$$D_x^{\alpha}[D_t^{\alpha}u + D_x^{\alpha}u - aD_x^{\alpha}u^2 - bD_t^{\alpha}(D_x^{2\alpha}u)] + eD_y^{2\alpha}u = 0. \quad (30)$$

In order to obtain new exact solutions for Eq. (30), suppose $u(x,y,t) = U(\eta)$, where $\eta = ct + kx + ky + \eta_0$, $c$, $k$, $\eta_0$ are all constants with $k,\ c \neq 0$. Then by use of Eq. (4) one can deduce that

$$\begin{cases} D_t^{\alpha}u = D_t^{\alpha}U(\eta) = D_\eta^{\alpha}U(\eta)(\eta'(t))^{\alpha} = c^{\alpha}D_\eta^{\alpha}U(\eta), \\ D_x^{\alpha}u = D_x^{\alpha}U(\eta) = D_\eta^{\alpha}U(\eta)(\eta'(x))^{\alpha} = k^{\alpha}D_\eta^{\alpha}U(\eta), \\ D_y^{\alpha}u = D_y^{\alpha}U(\eta) = D_\eta^{\alpha}U(\eta)(\eta'(y))^{\alpha} = k^{\alpha}D_\eta^{\alpha}U(\eta). \end{cases} \quad (31)$$

Then Eq. (30) can be turned into the following form with respect to the new variable $\eta$:

$$c^{\alpha}k^{\alpha}D_\eta^{2\alpha}U(\eta) + k^{2\alpha}D_\eta^{2\alpha}U(\eta) - 2ak^{2\alpha}[(D_\eta^{\alpha}U(\eta))^2 + U(\eta)D_\eta^{2\alpha}U(\eta)] - bc^{\alpha}k^{3\alpha}D_\eta^{4\alpha}U(\eta) + ek^{2\alpha}D_\eta^{2\alpha}U(\eta) = 0. \quad (32)$$

Set $\xi = \frac{\eta^{\alpha}}{\Gamma(1+\alpha)}$, $sn(\xi) = \overline{sn}(\eta)$, $cn(\xi) = \overline{cn}(\eta)$, $dn(\xi) = \overline{dn}(\eta)$, $cs(\xi) = \overline{cs}(\eta)$, $sd(\xi) = \overline{sd}(\eta)$, $dc(\xi) = \overline{dc}(\eta)$, $sc(\xi) = \overline{sc}(\eta)$, $ds(\xi) = \overline{ds}(\eta)$, $cd(\xi) = \overline{cd}(\eta)$, $nd(\xi) = \overline{nd}(\eta)$, $ns(\xi) = \overline{ns}(\eta)$, $nc(\xi) = \overline{nc}(\eta)$. Then by Eqs. (1) and (3) one can obtain that $D_\eta^{\alpha}\overline{sn}(\eta) = D_\eta^{\alpha}\overline{sn}(\eta) = D_\eta^{\alpha}sn(\xi) = sn'(\xi)D_\eta^{\alpha}\xi = sn'(\xi)$. So according to Eq. (9) we deduce that

$$D_\eta^{\alpha}\overline{sn}(\eta) = \overline{cn}(\eta)\overline{dn}(\eta), \quad (33)$$

and similarly

$$D_\eta^{\alpha}\overline{cn}(\eta) = -\overline{sn}(\eta)\overline{dn}(\eta), \quad (34)$$

$$D_\eta^{\alpha}\overline{dn}(\eta) = -m^2\overline{sn}(\eta)\overline{cn}(\eta). \quad (35)$$

Suppose that the solution of Eq. (32) can be expressed by a polynomial in the Jacobi elliptic functions as follows:

$$U(\eta) = a^{(0)} + \sum_{n+p+q=1}^{m} a^{(n,p,q)}\overline{sn}^n(\eta)\overline{cn}^p(\eta)\overline{dn}^q(\eta). \quad (36)$$

By balancing the order of $D_\eta^{4\alpha}U(\eta)$ and $U(\eta)D_\eta^{2\alpha}U(\eta)$ in (32) one can obtain $m = 2$. So

$$U(\eta) = a^{(0)} + a^{(1,0,0)}\overline{sn}(\eta) + a^{(0,1,0)}\overline{cn}(\eta) + a^{(0,0,1)}\overline{dn}(\eta) + a^{(2,0,0)}\overline{sn}^2(\eta) + a^{(1,1,0)}\overline{sn}(\eta)\overline{cn}(\eta) + a^{(0,1,1)}\overline{cn}(\eta)\overline{dn}(\eta) + a^{(1,0,1)}\overline{sn}(\eta)\overline{dn}(\eta) + a^{(0,2,0)}cn^2(\xi) + a^{(0,0,2)}dn^2(\xi). \quad (37)$$

Substituting (37) into (32), using (33)-(35) and collecting all the terms with the same power of $\overline{sn}^n(\eta)\overline{cn}^p(\eta)\overline{dn}^q(\eta)$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations with the aid of mathematical software, yields the following values, where $i$ denotes the unit of the imaginary numbers.

Case 1:

$$a^{(0)} = \frac{k^{\alpha} + bc^{\alpha}k^{2\alpha}m^2 + 4bc^{\alpha}k^{2\alpha} + ek^{\alpha} + c^{\alpha}}{2ak^{\alpha}},\ a^{(1,0,0)} = 0,$$
$$a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,\ a^{(2,0,0)} = -\frac{3m^2k^{\alpha}c^{\alpha}b}{a},$$
$$a^{(1,1,0)} = \pm\frac{3c^{\alpha}bk^{\alpha}m^2}{a}i,\ a^{(0,1,1)} = 0,\ a^{(1,0,1)} = 0,$$
$$a^{(0,2,0)} = a^{(0,0,2)} = 0.$$

Case 2:

$$a^{(0)} = \frac{k^{\alpha} + 4bc^{\alpha}k^{2\alpha}m^2 + 4bc^{\alpha}k^{2\alpha} + ek^{\alpha} + c^{\alpha}}{2ak^{\alpha}},\ a^{(1,0,0)} = 0,$$
$$a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,\ a^{(2,0,0)} = -\frac{6m^2k^{\alpha}c^{\alpha}b}{a},$$
$$a^{(1,1,0)} = 0,\ a^{(0,1,1)} = 0,\ a^{(1,0,1)} = 0,$$
$$a^{(0,2,0)} = a^{(0,0,2)} = 0.$$

Case 3:

$$a^{(0)} = \frac{k^{\alpha} + 4bc^{\alpha}k^{2\alpha}m^2 + bc^{\alpha}k^{2\alpha} + ek^{\alpha} + c^{\alpha}}{2ak^{\alpha}},\ a^{(1,0,0)} = 0,$$
$$a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,\ a^{(2,0,0)} = -\frac{3m^2k^{\alpha}c^{\alpha}b}{a},$$
$$a^{(1,1,0)} = 0,\ a^{(0,1,1)} = 0,\ a^{(1,0,1)} = \pm\frac{3c^{\alpha}bk^{\alpha}m}{a}i,$$
$$a^{(0,2,0)} = a^{(0,0,2)} = 0.$$

Case 4:

$$a^{(0)} = \frac{k^{\alpha} + bc^{\alpha}k^{2\alpha}m^2 + bc^{\alpha}k^{2\alpha} + ek^{\alpha} + c^{\alpha}}{2ak^{\alpha}},\ a^{(1,0,0)} = 0,$$
$$a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,\ a^{(2,0,0)} = -\frac{3m^2k^{\alpha}c^{\alpha}b}{a},$$
$$a^{(1,1,0)} = 0,\ a^{(0,1,1)} = \pm\frac{3mk^{\alpha}c^{\alpha}b}{a},\ a^{(1,0,1)} = 0,$$
$$a^{(0,2,0)} = a^{(0,0,2)} = 0.$$

Case 5:

$$a^{(0)} = \frac{k^{\alpha} + bc^{\alpha}k^{2\alpha}m^2 + bc^{\alpha}k^{2\alpha} + ek^{\alpha} + c^{\alpha}}{2ak^{\alpha}},\ a^{(1,0,0)} = 0,$$
$$a^{(0,1,0)} = 0,\ a^{(0,0,1)} = 0,\ a^{(2,0,0)} = -\frac{3m^2k^{\alpha}c^{\alpha}b}{2a},$$
$$a^{(1,1,0)} = \pm\frac{3c^{\alpha}bk^{\alpha}m^2}{2a}i,\ a^{(0,1,1)} = \pm\frac{3mk^{\alpha}c^{\alpha}b}{2a},$$
$$a^{(1,0,1)} = \pm\frac{3c^{\alpha}bk^{\alpha}m}{2a}i,\ a^{(0,2,0)} = a^{(0,0,2)} = 0.$$

Substituting the results above into Eq. (37), considering such equality holds $\overline{sn}(\eta) = sn(\xi)$, where $\xi = \frac{\eta^{\alpha}}{\Gamma(1+\alpha)}$, and $\eta = ct + kx + ky + \eta_0$, we can obtain the following exact solutions in the forms of the Jacobi elliptic functions for Eq. (30).

Family 1:

$$u_1(x,y,t) = \frac{k^{\alpha} + bc^{\alpha}k^{2\alpha}m^2 + 4bc^{\alpha}k^{2\alpha} + ek^{\alpha} + c^{\alpha}}{2ak^{\alpha}} - \frac{3m^2k^{\alpha}c^{\alpha}b}{a}sn^2(\xi) \pm \frac{3c^{\alpha}bk^{\alpha}m^2i}{a}sn(\xi)cn(\xi). \quad (38)$$

Family 2:

$$u_2(x,y,t) = \frac{k^\alpha + 4bc^\alpha k^{2\alpha} m^2 + 4bc^\alpha k^{2\alpha} + ek^\alpha + c^\alpha}{2ak^\alpha} - \frac{6m^2 k^\alpha c^\alpha b}{a} sn^2(\xi). \quad (39)$$

Family 3:

$$u_3(x,y,t) = \frac{k^\alpha + 4bc^\alpha k^{2\alpha} m^2 + bc^\alpha k^{2\alpha} + ek^\alpha + c^\alpha}{2ak^\alpha} - \frac{3m^2 k^\alpha c^\alpha b}{a} sn^2(\xi) \pm \frac{3c^\alpha bk^\alpha m}{a} isn(\xi)dn(\xi). \quad (40)$$

Family 4:

$$u_4(x,y,t) = \frac{k^\alpha + bc^\alpha k^{2\alpha} m^2 + bc^\alpha k^{2\alpha} + ek^\alpha + c^\alpha}{2ak^\alpha} - \frac{3m^2 k^\alpha c^\alpha b}{a} sn^2(\xi) \pm \frac{3mk^\alpha c^\alpha b}{a} cn(\xi)dn(\xi). \quad (41)$$

Family 5:

$$u_5(x,y,t) = \frac{k^\alpha + bc^\alpha k^{2\alpha} m^2 + bc^\alpha k^{2\alpha} + ek^\alpha + c^\alpha}{2ak^\alpha} - \frac{3m^2 k^\alpha c^\alpha b}{2a} sn^2(\xi) \pm \frac{3c^\alpha bk^\alpha m^2 i}{2a} sn(\xi)cn(\xi) \pm \frac{3mk^\alpha c^\alpha b}{2a} cn(\xi)dn(\xi) \pm \frac{3c^\alpha bk^\alpha mi}{2a} sn(\xi)dn(\xi). \quad (42)$$

**Remark 3**. We note that the value of $\xi$ in (38)-(42) is essentially different from that in (15)-(19), and the Jacobi elliptic function solutions established in (38)-(42) are different exact solutions from those in Section III. So the approach used here is essentially different from that in Sections II-IV. Moreover, as one can see, the method mentioned in this section can also be applied to deduce new Jacobi elliptic function solutions for the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System (20) under the condition $\alpha = \beta = \gamma = 1$.

## VI. CONCLUSIONS

In this paper, we extend the Jacobi elliptic function method to seek exact solutions for fractional partial differential equations. This method belongs to the categories of the sub-equation methods, and the fractional complex transformation used here for $\xi$ plays the most important role in the solving process. In order to demonstrate the validity of this method, we apply it to seek exact solutions in the forms of the Jacobi elliptic functions for the space-time fractional KP-BBM equation and the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System. With the aid of mathematical software, a series of Jacobi elliptic functions solutions for the two equations have been successfully found. Finally, as an extension of this method, we propose a new approach for seeking new Jacobi elliptic function solutions for space-time fractional differential equations. Being concise and powerful, this method can also be applied to seek exact solutions for many other fractional differential equations.

## VII. ACKNOWLEDGEMENTS

This work is partially supported by Natural Science Foundation of Shandong Province (China) (ZR2013AQ009).

The author would like to thank the referees very much for their valuable suggestions on improving this paper.

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