Multiplicity of Solutions for Quasilinear Singular Euler-Lagrange Equations with Natural Growth

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Abstract—**This paper shows the existence of multiplicity solutions for quasilinear singular Euler-Lagrange equation**

$$-div((a(x)+\mid u\mid^{\gamma})\mid\nabla u\mid^{N-2}\nabla u)$$

$$+\frac{\gamma}{N}\mid u\mid^{\gamma-2}u\mid\nabla u\mid^{N}=\lambda\mid u\mid^{\theta-2}u+\mid u\mid^{q-2}u\quad in\quad\Omega,$$

with zero Dirichlet boundary condition. Under hypotheses $1<\theta<N<q<\gamma+N$; $\gamma>0$ and $\lambda>0$.
By using critical point methods we obtain the multiplicity of solutions for the above equation in the following cases:
If $1<\theta<N<q<\gamma+N$, $\gamma>1$ and there is a nonnegative constant λ^* such that $0<\lambda<\lambda^*$, such equation possesses an infinitely many bounded weak solutions.
If $1<\theta<N<q<\gamma+N$, $0<\gamma\le 1$ and $0<\lambda<\lambda^*$, the singular equation has an infinitely many bounded weak solutions.

Keywords: Euler-Lagrange equation, weak solution, truncated functional, nonsmooth critical point theory, singular lower order term.

1 Introduction

In this paper we study the following equation

$$-div((a(x)+\mid u\mid^{\gamma})\mid\nabla u\mid^{N-2}\nabla u)+\frac{\gamma}{N}\mid u\mid^{\gamma-2}u\mid\nabla u\mid^{N}$$

$$=\lambda\mid u\mid^{\theta-2}u+\mid u\mid^{q-2}u\quad in\quad\Omega \tag{1.1}$$

and

$$u=0\qquad on\quad\partial\Omega. \tag{1.2}$$

In this case, the functional corresponding to the quasilinear Euler-Lagrange J is

$$J(u)=\frac{1}{N}\int_{\Omega}(a(x)+\mid u\mid^{\gamma})\mid\nabla u\mid^{N}$$

$$-\frac{\lambda}{\theta}\int_{\Omega}\mid u\mid^{\theta}-\frac{1}{q}\int_{\Omega}\mid u\mid^{q}. \tag{1.3}$$

Where Ω is a bounded, open subset of R^N, $N>2$ and a(x) is a measurable function such that for some constants α and β we have

$$0<\alpha\le a(x)\le\beta\qquad a.e\quad x\in\Omega. \tag{1.4}$$

The main difficulty in solving this equation is due to the term $\mid u\mid^{\gamma}$ which, although we assume that $1<\theta<N<q<N+\gamma$, J is not well defined in all the space $W_0^{1,N}(\Omega)$. Similarly, this kind of non-differentiable functional J that combines a critical point theory has been investigated in [1] for $N=2$. The functional J is not Gâteau-differentiable in $W_0^{1,N}(\Omega)$ but is only differentiable through the direction of $W_0^{1,N}(\Omega)\cap Ł^{\infty}(\Omega)$. In that case, the functional J is well defined in $W_0^{1,N}(\Omega)\cap Ł^{\infty}(\Omega)$, if we impose an additional condition on γ, namely, $\gamma<N$.
In section 2, our technique for solving a quasilinear Euler-Lagrange equation (1.1)-(1.2) is based on approximating functional J with the sequence of functionals $J_{m,n}$ whose quadratic part in ∇v is bounded with respect to v. Similarly, our approach has been studied in [1], and L^{∞} priori estimate allows to prove that, when $\gamma>1$ the critical point $u_{\overline{m},\overline{n}}$ of $J_{\overline{m},\overline{n}}$ for $\overline{m},\overline{n}$ large enough, therefore, a solution to (1.1)-(1.2) is found without passing the limit of m and n. Then we use the theorem 2.8 in [2] to establish the existence of infinitely many solutions to equation (1.1)-(1.2) for $0<\lambda<\lambda^*$ and $1<\theta<N<q<\gamma+N$.
In section 3, we show the existence of multiplicity solutions to equation (1.1)-(1.2) for $0<\gamma\le 1$; $0<\lambda<\lambda^*$ and $1<\theta<N<q<\gamma+N$ using the theorem 2.3 in [2] and the method established in the theorem 3.1 in [1]. However, the difficulty of this case is that the zero Dirichlet boundary condition implies the singularity with respect to u in the lower order term $\frac{\gamma}{N}\frac{u}{|u|^{2-\gamma}}\mid\nabla u\mid^{N}$ of the Euler-Lagrange equation.
The multiplicity results for N-Laplacian with critical growth of concave-convex functions has been intensively studied (see [5,6]) in earlier studies. Recently, the existence of the nonnegative bounded weak solution to the quasilinear Euler-Lagrange equation involving concave-convex functions with $N=2$ has investigated by David Arcoya and Lucio Boccardo (see [1,15]).
Finally, the novelty of this work is that we study the existence of multiplicity of bounded weak solutions for quasilinear singular Euler-Lagrange equation with

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$p = N$.

Notation: in the rest of this work we use of the following notation. $L^N(\Omega)$, denote lebesgue spaces; the usual norm in $L^N(\Omega)$ is denoted by $|\ \ |_N$.
$W_0^{k,N}(\Omega)$ denote sobolev spaces ; the norm in $W_0^{1,N}(\Omega)$ is denoted by $\|\ \ \|_N$.
$C_0, C_1, C_2, C_3, ...$ denote (possibly different) positive constants.

2 The case $\gamma > 1$

Definition 2.1 A measurable function u is called a weak solution to the problem (1.1)-(1.2) if $u \in W_0^{1,N}(\Omega)$ such that $|\ u\ |^{\gamma-2}\ u\ |\ \nabla u\ |^N \in L^1(\Omega)$ and,

$$\int_\Omega (a(x)+ |\ u\ |^\gamma)\ |\ \nabla u\ |^{N-2}\ \nabla u \nabla v$$

$$+\frac{\gamma}{N}\int_\Omega |\ u\ |^{\gamma-2}\ u\ |\ \nabla u\ |^N\ v$$

$$= \lambda \int_\Omega |\ u\ |^{\theta-2}\ uv + \int_\Omega |\ u\ |^{q-2}\ uv, \qquad (2.1)$$

holds for every $u \in W_0^{1,N}(\Omega) \cap L^\infty(\Omega)$.

Theorem 2.2 Suppose $\gamma > 1$ and if q, θ verifies the hypothesis

$$1 < \theta < N < q < \gamma + N. \qquad (2.3)$$

Moreover, there exists $\lambda^* > 0$ such that

$$0 < \lambda < \lambda^*.$$

Then, the problem (1.1)-(1.2) possesses an infinitely many bounded weak solutions.

Proof. We use a similar argument as in [7] to prove the existence of multiplicity weak solutions to equation (1.1)-(1.2), then we argue this proof by splitting it in several steps

- **Step 1**: A truncated function $J_{m,n}$
- **Step 2**: $J_{m,n}(u) \geq a_{n,\lambda}$ for all $\|\ u\ \|_N = r_{n,\lambda}$ and $J_{m,n}$ is bounded from below on $B_{r_{n,\lambda}}$
- **Step 3**: Compactness of the truncated functional $J_{m,n}$
- **Step 4**: Existence of critical points of the truncated functional $J_{m,n}$
- **Step 5**: Uniformly L^∞ - estimates
- **Step 6**: Conclusion

• **Step 1: Truncated functional**
We define the truncated functional $J_{m,n}$ from the definition of the functional J, if m is a positive integer, we consider the C^2 regularization of the truncation at level m, $T_m(t)$ is given

$$T_m(t) = \begin{cases} -m - \frac{1}{2} \ \ if \ \ t \leq -m-1 \\ (m+1)t + \frac{t^2+m^2}{2} \ \ if \ \ -m-1 \leq t \leq -m \\ t \ \ if \ \ -m \leq t \leq m \\ (m+1)t - \frac{t^2+m^2}{2} \ \ if \ \ m \leq t \leq m+1 \\ m + \frac{1}{2} \ \ if \ \ t \geq m+1 \end{cases} \qquad (2.4)$$

see [1],[16]
Assuming that q_1 and q_0 are numbers such that $1 < q_0 < \theta < N < q_1 < q < N + \gamma$ and the C^2 regularization of the truncated function $f_{n,\lambda}(t)$ is defined by

$$f_{n,\lambda}(t) = \lambda h_n(t) + g_n(t),$$

where

$$h_n(t) = \begin{cases} \frac{|t|^\theta}{\theta} \ \ if \ \ |\ t\ |< n \\ n^\theta\left(\frac{1}{\theta} - \frac{1}{q_0}\right) + n^{\theta-q_0}\frac{|t|^{q_0}}{q_0} \ \ if \ \ |\ t\ |\geq n. \end{cases} \qquad (2.5)$$

$$g_n(t) = \begin{cases} \frac{|t|^q}{q} \ \ if \ \ |\ t\ |< n \\ n^q\left(\frac{1}{q} - \frac{1}{q_1}\right) + n^{q-q_1}\frac{|t|^{q_1}}{q_1} \ \ if \ \ |\ t\ |\geq n. \end{cases} \qquad (2.6)$$

By observing the definition of $h_n(t)$ and $g_n(t)$, we deduce the following inequalities

$$0 \leq h_n(t) \leq \frac{n^{\theta-q_0}}{q_0} |\ t\ |^{q_0} \ \ and \ \ 0 \leq h_n(t) \leq \frac{|\ t\ |^\theta}{\theta}. \quad (2.7)$$

$$0 \leq g_n(t) \leq \frac{n^{q-q_1}}{q_1} |\ t\ |^{q_1} \ \ and \ \ 0 \leq g_n(t) \leq \frac{|\ t\ |^q}{q}. \quad (2.8)$$

Consequently, we are able to deduce the estimate of $f_{n,\lambda}(t)$ by

$$0 \leq f_{n,\lambda}(t) \leq \frac{\lambda n^{\theta-q_0}}{q_0}\ |\ t\ |^{q_0} + \frac{n^{q-q_1}}{q_1}\ |\ t\ |^{q_1}. \qquad (2.9)$$

Let us consider the truncated functional

$$J_{m,n}(u) = \frac{1}{N}\int_\Omega (a(x)+ |\ T_m(u)\ |^\gamma)\ |\ \nabla u\ |^N - \int_\Omega f_{n,\lambda}(u) \ \ for \ \ u \in W_0^{1,N}(\Omega), \qquad (2.10)$$

which is clearly well defined since
$1 < q_0 < \theta < N < q_1 < q < \min(\frac{N^2}{N-1}, \gamma + N)$
and $\theta + q + \gamma \leq \frac{4N^2}{N-1}$.

• **Step 2: Geometry of truncated functional**
Consider a positive real constant $0 < r$ such that

$$B_r = \left\{u \in W_0^{1,N}(\Omega) / \|\ u\ \|_N \leq r\right\}$$

Integrating over Ω both sides of the inequality (2.9), we have

$$\int_\Omega f_{n,\lambda}(u) \leq \lambda \frac{n^{q-q_0}}{q_0} \int_\Omega |\, u\,|^{q_0} + \frac{n^{q-q_1}}{q_1} \int_\Omega |\, u\,|^{q_1}.$$

Combining Hölder and Poincare inequalities, we obtain the following result

$$\int_\Omega f_{n,\lambda}(u) \leq C_0 n^{q-q_1} \parallel u \parallel_N^{q_0} + C_1 n^{q-q_1} \parallel u \parallel_N^{q_1}, \quad (2.11)$$

where C_0 and C_1 are nonnegative constants.
Performing calculations and taking into account the inequality (2.11), we infer that

$$J_{m,n}(u) \geq \frac{\alpha}{N} \parallel u \parallel_N^N - C_0 n^{\theta - q_0} \parallel u \parallel_N^{q_0} - C_1 n^{q-q_1} \parallel u \parallel_N^{q_1},$$

with $a(x) + |\, T_m(u)\,|^\gamma \geq \alpha$.
Thereby, there exist nonnegative constants $r_{n,\lambda}$, $\overline{r}_{n,\lambda}$ and λ^* such that

$$J_{m,n}(u) > 0 \; in \quad B_{r_{n,\lambda}} \quad and \quad J_{m,n}(u) \geq \overline{r}_{n,\lambda} \quad in\; \partial B_{r_{n,\lambda}}$$

for all $0 < \lambda < \lambda^*$

• Step 3: Compactness of the truncated functional $J_{m,n}$

Lemma 2.12 Let $\{w_k\}$ be a sequence in $W_0^{1,N}(\Omega) \cap \L^\infty(\Omega)$ satisfying, for every $n \in N$ the following conditions:

$$J_{m,n}(w_k) \leq C_1$$

$$|\, w_k \,|_{\,\infty} \leq 2b_k$$

$$\langle J'_{m,n}(w_k), w \rangle \leq \varepsilon_k \left(\frac{|\, w\,|_\infty}{b_k} + \parallel w \parallel_N \right)$$

$\forall w \in W_0^{1,N}(\Omega) \cap \L^\infty(\Omega)$.
Where C_1 is a nonnegative constant, $\{b_k\} \subset R^+ - \{0\}$ is any nonnegative sequence and $\{\varepsilon_k\} \subset R^+ - \{0\}$ is a sequence converging to zero, then $\{w_k\}$ has a strongly convergent subsequence in $W_0^{1,N}(\Omega)$ (see [8]).

Considering X_m as a test function defined such that

$$X_m(t) = \begin{cases} \frac{T_m(t)}{T'_m(t)} \quad if \quad -m-1 \leq t \leq -m \\ 0 \quad otherwise \end{cases}$$

Remark 2.13 Since the following inequalities

$$\frac{T''_m(t) T_m(t)}{\left[T'_m(t)\right]^2} \leq 1 \quad and \quad \frac{T_m(t)}{T'_m(t)} \leq -t\theta$$

hold, then it is easy to verify that the function $X_m \in W_0^{1,N}(\Omega)$.
After computing terms below

$$J_{m,n}(w_k) + \frac{1}{\theta} \langle J'_{m,n}(w_k), X_m(w_k) \rangle$$

we have

$$\int_\Omega \left(\frac{1}{N} + \frac{1}{\theta} - \frac{T''_m(w_k) T_m(w_k)}{\theta \left[T'_m(w_k)\right]^2} \right) a(x) \mid \nabla w_k \mid^N$$

$$+ \int_\Omega \left(\frac{1}{N} + \frac{1}{\theta} - \frac{T''_m(w_k) T_m(w_k)}{\theta \left[T'_m(w_k)\right]^2} + \frac{\gamma}{\theta N} \right)$$

$$\times |\, T_m(w_k)\,|^\gamma \mid \nabla w_k \mid^N$$

$$+ \int_\Omega \left(-\frac{T_m(w_k)}{\theta T'_m(w_k)} f'_{n,\lambda}(w_k) - f_{n,\lambda}(w_k) \right)$$

$$\leq C_1 + \varepsilon_k \left(\frac{|\, w_k\,|_\infty}{b_k} + \parallel w_k \parallel_N \right).$$

We notice that the left hand side terms are positives. The first and second terms are positive due to the hypothesis $\frac{T''_m(t) T_m(t)}{\left[T'_m(t)\right]^2} \leq 1$.
The positiveness of the third term is given by the definition of $f_{m,n}(t)$ function and the assumption $\frac{T_m(t)}{T'_m(t)} \leq -t\theta$.
The sequence $\{w_k\}$ is bounded in $W_0^{1,N}(\Omega)$. Then it weakly converges into $W_0^{1,N}(\Omega)$ up to the subsequence that we still denote $\{w_k\}$ converging to a function w.

• Step 4: Existence of critical points of the truncated functional $J_{m,n}$

we suppose that $\overline{H}_k$ be a k-dimensional subspace of $W_0^{1,N}(\Omega)$.
Let

$$\Sigma = \left\{ C \subset W_0^{1,N}(\Omega) / \quad 0 \overline{\in} C,\, C = -C \right\}.$$

According to the geometry of truncated functional $J_{m,n}$ and previous remarks, the assumptions (I_1) and (I_3) of theorem 2.8 in [2]are satisfied. Moreover, considering the following set $A_{m,n}$ defined as

$$A_{m,n} = B_{r_{n,\lambda}} \cup \{J_{m,n} > 0\}.$$

Therefore we assert that $\overline{H}_k \cap A_{m,n}$ is bounded for all $n \in N$, consequently the hypothesis (I_5) is achieved
We now build the following set

$$\Gamma^* = \{h \in C(W_0^{1,N}(\Omega), W_0^{1,N}(\Omega)) :$$

$$h \textit{ is an odd homeomorphism } h(0) = 0 \textit{ and } h(B_1) \subset A_{m,n}\}$$

and

$$\Gamma_k = \{K \in \Sigma : \gamma(K \cap h(\partial B_1)) \geq k \quad \forall h \in \Gamma^*\}$$

and then

$$S_k = \inf_{K \in \Gamma_k} \max_{u \in K} J_{m,n}(u).$$

Since the conditions of lemma 2.1 in [2] still holds. we choose $h(u) = r_{n,\lambda} u$, where h lies in Γ^* from this we can deduce that $K \cap B_{r_{n,\lambda}} \neq \emptyset$ for all $K \in \Gamma_k$. $J_{m,n}$ is bounded from below on $\partial B_{r_{n,\lambda}}$, then

$$S_k = \inf_{K \in \Gamma_k} \max_{u \in K} J_{m,n}(u) \geq \overline{r}_{n,\lambda} > 0.$$

Finally, whole assumptions of theorem 2.8 in [2] holds true. Consequently, there are infinitely many nontrivial critical points of $J_{m,n}$ belonging to $W_0^{1,N}(\Omega)\cap L^\infty(\Omega)$ such that

$$J_{m,n}(u_{m,n})>0 \qquad \forall n,m\in N.$$

Hence, the Dirichlet problem (2.14)-(1.2) has an infinitely many nontrivial weak solutions.

• **Step 5: Uniformly L^∞-estimate**
Consider the following equation

$$-div\{(a(x)+\mid T_m(u_{m,n})\mid^\gamma)\mid\nabla u_{m,n}\mid^{N-2}\nabla u_{m,n}\}$$

$$+\frac{\gamma}{N}\frac{T_m'(u_{m,n})}{T_m(u_{m,n})}\mid T_m(u_{m,n})\mid^\gamma\mid\nabla u_{m,n}\mid^N=f'_{n,\lambda}(u_{m,n}). \tag{2.14}$$

Assuming that $u_{m,n}$ consists of all those solutions to equation (2.14)-(1.4) and setting $T_m(w_{m,n})=w_{m,n}$, thus the equation (2.14) can be written as

$$-div\left\{(a(x)+\mid u_{m,n}\mid^\gamma)\mid\nabla u_{m,n}\mid^{N-2}\nabla u_{m,n}\right\}$$

$$+\frac{\gamma}{N}\mid u_{m,n}\mid^{\gamma-2}u_{m,n}\mid\nabla u_{m,n}\mid^N=f'_{n,\lambda}(u_{m,n}). \tag{2.15}$$

In order to show that the solutions to equation (2.15)-(1.2) are uniformly bounded, we construct an embedding of the Orlicz-Sobolev space by using the theorem 3.1 in [9]. Then we deduce the boundedness of solutions to such equation in $L^\infty(\Omega)$.
Suppose that $W(X)=W_0^{1,N}(\Omega)$, the sobolev space which has the property such that every bounded sequence in $W_0^{1,N}(\Omega)$ has a subsequence that is convergent almost everywhere.
We consider two Young functions in relation with our purpose $F:t\mapsto t^{\gamma+N}$ and $\psi:t\mapsto t^{\frac{\gamma+N}{N}}$ such that $F\gg\psi$. Since $u\in W_0^{1,N}(\Omega)$ such that $\mid u\mid^\gamma\mid\nabla u\mid^N\in L^1(\Omega)$, then the embedding $W_0^{1,N}(\Omega)\subset L^F(\Omega)$ holds.

Remark 2.16 Recalling that the embedding space $L^\varphi(\Omega)$ equipped with the *Luxemburg* norm

$$\mid u\mid_\varphi=\inf\left\{k>0/\int_\Omega\varphi\left(\frac{\mid u\mid}{k}\right)d\mu\le 1\right\}$$

is a Banach space (see [9]).
Let $\{t_m\}\subset R^+-\{0\}$ be an increasing sequence which diverges to infinity. By applying the theorem 3.1 in [9], we get the following embedding

$$W_0^{1,N}(\Omega)\subset\subset L^\psi(\Omega)\quad is\quad compact.$$

Thus

$$\int_\Omega\psi\left(\frac{\mid u_{m,n}\mid}{k}\right)\le\int_\Omega\mid u_{m,n}\mid^{\frac{\gamma+N}{N}}$$

$$\le C\int_\Omega\mid u_{m,n}\mid^\gamma\mid\nabla u_{m,n}\mid^N.$$

Therefore, there exist a nonnegative constant $C_{m,n}$, because of

$$\int_\Omega\mid u_{m,n}\mid^\gamma\mid\nabla u_{m,n}\mid^N\,is\,bounded\,with\,respect\,to\,m\,and\,n,$$

thus

$$\mid u_{m,n}\mid_{\psi(t_m)}=inf\left\{k>0;\int_\Omega\psi\left(\frac{\mid u_{m,n}\mid}{k}\right)dx\le 1\right\},$$

it follows that

$$\mid u_{m,n}\mid_{\psi(t_m)}\le C_{m,n}.$$

An adaptation to the quasilinear case of the proof of a result of Stampacchia (see [10, theorem 4.1 and 4.2]) implies that there exists $\widetilde{M}_n>0$ such that

$$\mid u_{m,n}\mid_\infty\le\widetilde{M}_n.$$

Let now m_n be an integer such that
$m_n\ge\max(\widetilde{M}_n+p,\bar{t})$ and if we define $u_n=u_{m_n,n}$, then $T_{m_n}(u_n)=u_n\quad and\quad T'_{m_n}(u_n)=1$
Accordingly, the function u_n verifies the equation

$$-div\left\{(a(x)+\mid u_n\mid^\gamma)\mid\nabla u_n\mid^{N-2}\nabla u_n\right\}$$

$$+\frac{\gamma}{N}\mid u_n\mid^{\gamma-2}u_n\mid\nabla u_n\mid^N=f'_{n,\lambda}(u_n), \tag{2.17}$$

with zero Dirichlet boundary condition.
Assuming the sequence $\{t_n\}\subset R^+-\{0\}$ is an increasing sequence which converges to infinity. By induction on the Orlicz space $L^{\psi(t_n)}(\Omega)$, we can similarly prove that

$$\mid u_n\mid_{\psi(t_n)}=inf\left\{k>0;\int_\Omega\psi\left(\frac{\mid u_n\mid}{k}\right)dx\le 1\right\}\le\widehat{C}.$$

Using again an adaptation of the proof of theorem 4.1 and 4.2 in [10] yields that there exists $\widehat{C}_*>0$ such that

$$\mid u_n\mid_\infty\le\widehat{C}_*\quad\forall n\ge\max(\bar{t},\overline{n}).$$

• **Step 6**: Conclusion
Finally, if $\forall n\ge\max(\widehat{C}_*,\bar{t},\overline{n})$
then $f'_n(u_n)=\lambda\mid u_n\mid^{\theta-2}u_n+\mid u_n\mid^{q-2}u_n$
and then $u\overset{def}{=}u_{\overline{n}}$.
Hence we conclude that the Dirichlet problem (1.1)-(1.2) has an infinitely many bounded weak solutions.

3 The case $0<\gamma\le 1$

In this section we suppose that $0<\gamma\le 1$, and the assumption

$$1<\theta<N<q<\gamma+N$$

still holds.
For a solution u of (1.1)-(1.2), we remark that $u\in W_0^{1,N}(\Omega)$ such that $\frac{|\nabla u|^N}{|u|^{2-\gamma}}u$ lies in $L^1(\Omega)$, therefore

$$\int_\Omega (a(x)+\mid u\mid^\gamma)\,|\nabla u\mid^{N-2}\nabla u\nabla v+\frac{\gamma}{N}\int_\Omega \frac{\mid\nabla u\mid^N}{\mid u\mid^{2-\gamma}}uv$$
$$=\int_\Omega \mid u\mid^{\theta-2}uv+\int_\Omega\mid u\mid^{q-2}uv \qquad (3.1)$$

holds for every $u\in W_0^{1,N}(\Omega)\cap L^\infty(\Omega)$.
The following theorem enables us to establish the existence of infinitely many bounded weak solutions of Dirichlet problem (1.1)-(1.2).

Theorem 3.2 Assume that $0<\gamma\le 1$ and if q,θ satisfies the condition

$$1<\theta<N<q<\gamma+N.$$

Moreover, there exists $\lambda^*>0$ such that

$$0<\lambda<\lambda^*.$$

Then there exist infinitely many bounded weak solutions to Dirichlet problem (1.1)-(1.2).

Proof. The main idea of this proof is given in [1,13], The fact that the function $t\mapsto\mid t\mid^\gamma$ is not differentiable, will force us to choose a C^2 approximation of a truncature function $t\mapsto\left(\frac{1}{m}+\mid t\mid^N\right)^{\frac{\gamma}{N}}$, and a passage to the limit with respect to m will be necessary to deduce solutions to equation (1.1)-(1.2). We consider the truncated functional $\widetilde{J}_{m,n}$ for v in $W_0^{1,N}(\Omega)$ as follows

$$\widetilde{J}_{m,n}(v)=\frac{1}{N}\int_\Omega\left[a(x)+\left(\frac{1}{m}+\mid T_m(v)\mid^N\right)^{\frac{\gamma}{N}}\right]\mid\nabla v\mid^N$$
$$-\int_\Omega f_{n,\lambda}(v), \qquad (3.3)$$

where T_m and $f_{n,\lambda}$ would be found in the section 2 and $1<q_0<\theta<N<q_1<q<N+\gamma$.
We observe that $\widetilde{J}_{m,n}$ is well defined if $q_1<N+\gamma$ and $\theta+p+q\le\frac{4N^2}{N-1}$.
The Euler-Lagrange equation in relation with the above functional is defined for $v\in W_0^{1,N}(\Omega)$

$$-div\left\{\left(a(x)+\left(\frac{1}{m}+\mid T_m(v)\mid^N\right)^{\frac{\gamma}{N}}\right)\mid\nabla v\mid^{N-2}\nabla v\right\}$$
$$+\frac{\gamma}{N}\frac{T_m'(v)\mid T_m(v)\mid^{N-2}T_m(v)}{\left(\frac{1}{m}+\mid T_m(v)\mid^N\right)^{\frac{N-\gamma}{N}}}\mid\nabla v\mid^N=f_{n,\lambda}'(v), \quad (3.4)$$

with zero Dirichlet boundary condition. We establish the compactness condition of the truncated functional $\widetilde{J}_{m,n}$ by repeating the argument used in the Theorem 2.2 Section 2, we obtain

$$\int_\Omega\left(\frac{1}{N}+\frac{1}{\theta}-\frac{T_m''(w_k)T_m(w_k)}{\theta\,[T_m'(w_k)]^2}\right)a(x)\mid\nabla w_k\mid^N$$
$$+\int_\Omega\left(\frac{1}{N}+\frac{1}{\theta}-\frac{T_m''(w_k)T_m(w_k)}{\theta\,[T_m'(w_k)]^2}+\frac{\gamma}{\theta N}\frac{\mid T_m(w_k)\mid^N}{\frac{1}{m}+\mid T_m(w_k)\mid^N}\right)$$
$$\times\left(\frac{1}{m}+\mid T_m(w_k)\mid^N\right)^{\frac{\gamma}{N}}\mid\nabla w_k\mid^N$$
$$+\int_\Omega\left(-\frac{T_m(w_k)}{\theta T_m'(w_k)}f_{n,\lambda}'(w_k)-f_{n,\lambda}(w_k)\right)$$
$$\le C_1+\varepsilon_k\left(\frac{\mid w_k\mid_\infty}{b_k}+\parallel w_k\parallel_N\right).$$

Reasoning as before, the three left hand side terms are positives (the second one term is positive because $\frac{|T_m(w_k)|^N}{\frac{1}{m}+|T_m(w_k)|^N}\ge 0$) thereby, we can conclude that the sequence $\{w_k\}$ is bounded in $W_0^{1,N}(\Omega)$. Moreover, it weakly converges into $W_0^{1,N}(\Omega)$ up to the subsequence which we still denote $\{w_k\}$ converging to a function w. The fact that $a(x)+\left(\frac{1}{m}+\mid T_m(v)\mid^N\right)^{\frac{\gamma}{N}}\ge\alpha$, we infer from the proof of theorem 2.15 section 2, the functional $\widetilde{J}_{m,n}$ is bounded from below. Accordingly,we are able to deduce that $\widetilde{J}_{m,n}$ satisfies all assumptions of theorem 2.3 in [3],and then there exist infinitely many nontrivial solutions $u_{m,n}^k$ with $k=1,2,\ldots\ldots$ in $W_0^{1,N}(\Omega)\cap L^\infty(\Omega)$ to equation (3.4)-(1.2).
In the following steps every solution of Dirichlet problem (3.4)-(1.2) could be represented by $w_{m,n}$ for all $k=1,2,\ldots\ldots$
According to the proof of theorem 2.15, we infer that there is $M_n>0$ such that for every m, we have

$$\mid w_{m,n}\mid_\infty\le M_n. \qquad (3.5)$$

The fact that $f_{n,\lambda}'(t)=\lambda h_n'(t)+g_n'(t)$, there exists a nonnegative constant $\widetilde{N}_n$ such that

$$\mid f_{n,\lambda}'(w_{m,n})\mid_\infty\le\widetilde{N}_n. \qquad (3.6)$$

Choosing m sufficiently large, we therefore see that $T_m(w_{m,n})=w_{m,n}$ and then, $w_{m,n}$ solves the following equation

$$-div\left\{\left(a(x)+\left(\frac{1}{m}+\mid w_{m,n}\mid^N\right)^{\frac{\gamma}{N}}\right)\mid\nabla w_{m,n}\mid^{N-2}\nabla w_{m,n}\right\}$$
$$+\frac{\gamma}{N}\frac{\mid w_{m,n}\mid^{N-2}w_{m,n}}{\left(\frac{1}{m}+\mid w_{m,n}^N\mid\right)^{\frac{N-\gamma}{N}}}\mid\nabla w_{m,n}\mid^N=f_{n,\lambda}'(w_{m,n}). \quad (3.7)$$

Notice that, in contract with the case $\gamma>1$, here we still have an explicit dependance on m in the equation.
Since $a(x)\mid\nabla w_{m,n}\mid^N\in L^1(\Omega)$ and $w_{m,n}\in L^\infty(\Omega)$ then

the sequence $\{w_{m,n}\}$ is bounded in $W_0^{1,N}(\Omega)\cap L^\infty(\Omega)$ up to subsequence, it weakly converges in $W_0^{1,N}(\Omega)$ to a function w_n which lies in $W_0^{1,N}(\Omega)\cap L^\infty(\Omega)$.
Consider a test function $\widetilde{T}_k$ such that

$$\widetilde{T}_k(z)=1+kz.$$

Taking into account $\widetilde{T}_k(w_{m,n})$ as test function in (3.7) and dropping nonnegative terms, and so letting k tends to zero, we get

$$\frac{\gamma}{N}\int_\Omega \frac{|w_{m,n}|^{N-2}w_{m,n}}{\left(\frac{1}{m}+|w_{m,n}|^N\right)^{\frac{N-\gamma}{N}}}|\nabla w_{m,n}|^N\le\int_\Omega f'_{n,\lambda}(w_{m,n}),$$

which implies that

$$\frac{\gamma}{N}\int_\Omega \frac{|w_{m,n}|^{N-2}w_{m,n}}{\left(\frac{1}{m}+|w_{m,n}|^N\right)^{\frac{N-\gamma}{N}}}|\nabla w_{m,n}|^N\le\widetilde{P}_n, \qquad (3.8)$$

because (3.5) holds true.
Consequently, the term $\frac{|w_{m,n}|^{N-2}w_{m,n}}{\left(\frac{1}{m}+|w_{m,n}|^N\right)^{\frac{N-\gamma}{N}}}|\nabla w_{m,n}|^N$ is bounbed in $L^1(\Omega)$ and the fact that (3.6) holds, then this term $a(x)+\left(\frac{1}{m}+|w_{m,n}|^N\right)^{\frac{\gamma}{N}}$ is bounded in $L^\infty(\Omega)$ with respect to m. We amalgamate the theorem 2.1 in [11] with the theorem A.O.6 in [12] to deduce that the sequence $\{\Delta_N w_{m,n}\}$ *weak** converges $\Delta_N w_n$ for all ϕ lies in $W_0^{1,N}(\Omega)$ as m tends to infinity. Therefore, by using Fatou lemma we can pass the limit in (3.8) to obtain that

$$\frac{\gamma}{N}\int_\Omega\frac{|\nabla w_n|^N}{|w_n|^{2-\gamma}}w_n\le\widetilde{P}_n$$

According to the previous proof we can deduce that

$\int_\Omega |w_n|^\gamma|\nabla w_n|^N$ *is bounded with respect to* n

for w_k instead of w_n.
Remark 3.9 Repeating the same procedure as in section 2, the sequence $\{w_k\}$ is bounded in $L^\infty(\Omega)$.
In order to show that the equation (3.7)-(1.2) has a supersolution, we choose $\phi\ge0$ in $W_0^{1,N}(\Omega)\cap L^\infty(\Omega)$ and define $H_m(t)=\frac{1}{\alpha N}\left(\frac{1}{m}+|t|^N\right)^{\frac{\gamma}{N}}$.
Taking $\phi e^{-w_{m,n}H_m(w_{m,n})}$ as test a function in (3.7), we get

$$\int_\Omega\left[a(x)+\left(\frac{1}{m}+|w_{m,n}|^N\right)^{\frac{\gamma}{N}}\right]|\nabla w_{m,n}|^{N-2}$$
$$\times\nabla w_{m,n}\nabla\phi e^{-w_{m,n}H_m(w_{m,n})}$$
$$=\int_\Omega w_{m,n}H'_m(w_{m,n})\left[a(x)+\left(\frac{1}{m}+|w_{m,n}|^N\right)^{\frac{\gamma}{N}}\right]|\nabla w_{m,n}|^N$$
$$\times\phi e^{-w_{m,n}H_m(w_{m,n})}$$
$$+\int_\Omega H_m(w_{m,n})\left[a(x)+\left(\frac{1}{m}+|w_{m,n}|^N\right)^{\frac{\gamma}{N}}\right]|\nabla w_{m,n}|^N$$
$$\times\phi e^{-w_{m,n}H_m(w_{m,n})}$$
$$-\frac{\gamma}{N}\int_\Omega\frac{|w_{m,n}|^{N-2}w_{m,n}}{\left(\frac{1}{m}+|w_{m,n}|^N\right)^{\frac{N-\gamma}{N}}}|\nabla w_{m,n}|^N\phi e^{-w_{m,n}H_m(w_{m,n})}$$
$$+\int_\Omega f'_{n,\lambda}(w_{m,n})\phi e^{-w_{m,n}H_m(w_{m,n})}.$$

By combining the assumption $0<\alpha\le a(x)\le\beta$ with the definition of H_m, we have

$$tH'_m(t)\left[a(x)+\left(\frac{1}{m}+|t|^N\right)^{\frac{\gamma}{N}}\right]-\frac{\gamma}{N}\frac{|t|^{N-2}t}{\left(\frac{1}{m}+|t|^N\right)^{\frac{N-\gamma}{N}}}$$
$$\ge tH'(t)\left(\frac{1}{m}+|t|^N\right)^{\frac{\gamma}{N}}>0.$$

Consequently,

$$\int_\Omega\left[a(x)+\left(\frac{1}{m}+|w_{m,n}|^N\right)^{\frac{\gamma}{N}}\right]|\nabla w_{m,n}|^{N-2}\nabla w_{m,n}$$
$$\times\nabla\phi e^{-w_{m,n}H_m(w_{m,n})}$$
$$\ge\int_\Omega f'_{n,\lambda}(w_{m,n})\phi e^{-w_{m,n}H_m(w_{m,n})}.$$

Now we pass the limit, as m tends to infinity, thanks to the weak convergence of the sequence $\{w_{m,n}\}$ to w_n and its boundedness in $L^\infty(\Omega)$.
Defining $H(t)=\frac{1}{\alpha N}|t|^\gamma$ as the limit of $H_m(t)$ when m tends to infinity, we have

$$\int_\Omega[a(x)+|w_n|^\gamma]|\nabla w_n|^{N-2}\nabla w_n\nabla\phi e^{-w_nH_m(w_n)}$$
$$\ge\int_\Omega f'_{n,\lambda}(w_n)\phi e^{-w_nH_m(w_n)}.$$

The fact that the sequence $\{w_n\}$ is bounded in $L^\infty(\Omega)$, we can choose $\overline{n}$ large enough so that $w\overset{def}{=}w_n\le\overline{n}$.
We notice that $w=w_{\overline{n}}$ in Ω yields a singular set of function $\frac{1}{|w|^{2-\gamma}}$ is the boundary of Ω.
Since $0\le\phi\in W_0^{1,N}(\Omega)\cap L^\infty(\Omega)$ is taken as in the equation (3.7) in which $w_{m,\overline{n}}$ a solution, and so using Fatou lemma the sequence $\{\Delta_N w_n\}$ *weak** converges to $\Delta_N w$ for all ϕ belongs to $W_0^{1,N}(\Omega)$ yields,

$$\int_\Omega\left[a(x)+\left(\frac{1}{m}+|w|^N\right)^{\frac{\gamma}{N}}\right]|\nabla w|^{N-2}\nabla w\nabla\phi$$
$$\ge\int_\Omega f'_{\overline{n},\lambda}(w)\phi.$$

We follow the idea as in [13], therefore we choose $v=\phi e^{H(w)-H_m(w_{m,\overline{n}})}$ as test function in (3.7) to obtain

$$\int_\Omega\left[a(x)+\left(\frac{1}{m}+|w_{m,\overline{n}}|^N\right)^{\frac{\gamma}{N}}\right]|\nabla w_{m,\overline{n}}|^{N-2}\nabla w_{m,\overline{n}}$$

$$\times\nabla\phi e^{H(w)-H_m(w_{m,\overline{n}})}$$
$$+\int_\Omega\left[a(x)+\left(\frac{1}{m}+\mid w_{m,\overline{n}}\mid^N\right)^{\frac{\gamma}{N}}\right]\mid\nabla w_{m,\overline{n}}\mid^{N-2}\nabla w_{m,\overline{n}}$$
$$\times\nabla\phi e^{H(w)-H_m(w_{m,\overline{n}})}H'(w)-\int_\Omega f'_{\overline{n},\lambda}(w_n)\phi e^{H(w)-H_m(w_{m,\overline{n}})}$$
$$=\int_\Omega\left[a(x)+\left(\frac{1}{m}+\mid w_{m,\overline{n}}\mid^N\right)^{\frac{\gamma}{N}}\right]\mid\nabla w_{m,\overline{n}}\mid^{N-2}\nabla w_{m,\overline{n}}$$
$$\times\nabla\phi e^{H(w)-H_m(w_{m,\overline{n}})}H'(w_{m,\overline{n}})$$
$$-\frac{\gamma}{N}\int_\Omega\frac{\mid w_{m,\overline{n}}\mid^{N-2}w_{m,\overline{n}}}{\left(\frac{1}{m}+\mid w_{m,\overline{n}}\mid^N\right)^{\frac{N-\gamma}{N}}}\mid\nabla w_{m,\overline{n}}\mid^N\phi e^{H(w)-H_m(w_{m,\overline{n}})}.$$

Reasoning as before, the right hand side is nonnegative, consequently, letting m tends to infinity and then applying Fatou lemma, thereby we have,

$$\int_\Omega[a(x)+\mid w\mid^\gamma]\mid\nabla w\mid^{N-2}\nabla w\nabla\phi$$
$$+\int_\Omega[a(x)+\mid w\mid^\gamma]\mid\nabla w\mid^N H'(w)\phi$$
$$-\int_\Omega\left(\lambda\mid w\mid^{\theta-2}w+\mid w\mid^{q-2}w\right)\phi$$
$$\geq\int_\Omega[a(x)+\mid w\mid^\gamma]\mid\nabla w\mid^N H'(w)\phi-\frac{\gamma}{N}\int_\Omega\frac{\mid\nabla w\mid^N}{\mid w\mid^{2-\gamma}}w\phi,$$

therefore

$$\int_\Omega[a(x)+\mid w\mid^\gamma]\mid\nabla w\mid^{N-2}\nabla w\nabla\phi+\frac{\gamma}{N}\int_\Omega\frac{\mid\nabla w\mid^N}{\mid w\mid^{2-\gamma}}w\phi$$
$$\geq\int_\Omega\left(\lambda\mid w\mid^{\theta-2}w+\mid w\mid^{q-2}w\right)\phi.\qquad(3.10)$$

$e^{H(w)-H_m(w_{m,\overline{n}})}$ equals to 1 as m tends to infinity.
We prove the converse inequality of (3.10)
Indeed, taking $0\leq\phi\in W_0^{1,N}(\Omega)\cap L^\infty(\Omega)$ as the test function in the equation for $w_{m,\overline{n}}$ solution to equation (3.7)and applying Fatou lemma to equation (3.7). Moreover, $f'_{\overline{n}}(w_{\overline{n}})=f'_{\overline{n}}(w)=\lambda\mid w\mid^{\theta-2}w+\mid w\mid^{q-2}w$ then, we deduce by letting m tends to infinity that

$$\int_\Omega[a(x)+\mid w\mid^\gamma]\mid\nabla w\mid^{N-2}\nabla w\nabla\phi+\frac{\gamma}{N}\int_\Omega\frac{\mid\nabla w\mid^N}{\mid w\mid^{2-\gamma}}w\phi$$
$$\leq\int_\Omega\left(\lambda\mid w\mid^{\theta-2}w+\mid w\mid^{q-2}w\right)\phi.\qquad(3.11)$$

By combining (3.10) with (3.11), we get

$$\int_\Omega[a(x)+\mid w\mid^\gamma]\mid\nabla w\mid^{N-2}\nabla w\nabla\phi+\frac{\gamma}{N}\int_\Omega\frac{\mid\nabla w\mid^N}{\mid w\mid^{2-\gamma}}w\phi$$
$$=\int_\Omega\left(\lambda\mid w\mid^{\theta-2}w+\mid w\mid^{q-2}w\right)\phi.$$

for every $\phi\in W_0^{1,N}(\Omega)\cap L^\infty(\Omega)$ The solution w of (1.3)-(1.4) is either $w\overset{def}{=}u^0$ or $w\overset{def}{=}u^1$ or …… or $w\overset{def}{=}u^k$ or ……
Hence, we can assert that the Dirichlet problem (1.1)-(1.2) has infinitely many positive bounded weak solution.□

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