The Properties of Stochastic Mutualism Model with Time-lagged Delays

Huizhen Qu, Xiaorong Gan and Tianwei Zhang

***Abstract*—Sufficient conditions are gained for almost sure permanence, global asymptotic stability and mean square period of the stochastic mutualism model with time-lagged delays**

$$\begin{cases} \mathrm{d}N_1(t) = r_1(t)N_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2(t-\tau_2)}{1+N_2(t-\tau_2)} - N_1(t)\right]\mathrm{d}t \\ \qquad +\sigma_1 N_1(t)\mathrm{d}B_{1t}, \\ \mathrm{d}N_2(t) = r_2(t)N_2(t)\left[\frac{K_2(t)+\alpha_2(t)N_1(t-\tau_1)}{1+N_1(t-\tau_1)} - N_2(t)\right]\mathrm{d}t \\ \qquad +\sigma_2 N_2(t)\mathrm{d}B_{2t}, \end{cases}$$

where $r_i(t), K_i(t), \alpha_i(t) \in C(\mathbb{R}, \mathbb{R}^+)$ and $\alpha_i(t) > K_i(t), i = 1, 2$. This paper implies that under the condition $\frac{1}{2}\sigma_i^2 < r_i^- k_i^-, i = 1, 2$, the intensity of white noise has a negative impact on almost sure permanence, but in any case, it makes no difference on global asymptotic stability. And the system is mean square periodic if ω is the period of $r_i(t), K_i(t), \alpha_i(t), i = 1, 2$.

***Index Terms*—Stochastic mutualism model; Almost sure permanence; Itô formula; Global asymptotic stability; Mean square period.**

I. Introduction

Consider the mutualism model

$$\begin{cases} \frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = r_1 N_1(t)\left[\frac{K_1+\alpha_1 N_2(t)}{1+N_2(t)} - N_1(t)\right], \\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = r_2 N_2(t)\left[\frac{K_2+\alpha_2 N_1(t)}{1+N_1(t)} - N_2(t)\right], \end{cases} \tag{1.1}$$

where $\alpha_i, K_i, r_i \in \mathbb{R}^+$ are constants and $\alpha_i > K_i, i = 1, 2$. Counting on the nature of $K_i (i = 1, 2)$, we classify system (1.1) as facultative, obligate or a combination of both. We refer to Dean[1], Boucher[2], Vandermeer and Boucher[3], Wolin and Lawlor[4], and Boucher *et al.*[5] for more details of mutualistic interactions. A modification of system (1.1) leads to the time-lagged model

$$\begin{cases} \frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = r_1 N_1(t)\left[\frac{K_1+\alpha_1 N_2(t-\tau_2)}{1+N_2(t-\tau_2)} - N_1(t)\right], \\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = r_2 N_2(t)\left[\frac{K_2+\alpha_2 N_1(t-\tau_1)}{1+N_1(t-\tau_1)} - N_2(t)\right], \end{cases} \tag{1.2}$$

where $\tau_1, \tau_2 \in [0, \infty)$ are constants. The cooperative or mutualistic effects in system (1.2) are not immediately realized, but happened with time goes on. However, due to environmental noise, may point out that the random fluctuation[6] should be showed by the rate of growth in the mutualism model. Assume that environmental noise disturbs the growth rate r with

$$r \to r + \sigma \dot{B}_t,$$

where σ^2 is the intensity of white noise and B_t is a standard Brownian motion. Then we obtain the stochastic model:

$$\begin{cases} \mathrm{d}N_1(t) = r_1 N_1(t)\left[\frac{K_1+\alpha_1 N_2(t-\tau_2)}{1+N_2(t-\tau_2)} - N_1(t)\right]\mathrm{d}t \\ \qquad +\sigma_1 N_1(t)\mathrm{d}B_{1t}, \\ \mathrm{d}N_2(t) = r_2 N_2(t)\left[\frac{K_2+\alpha_2 N_1(t-\tau_1)}{1+N_1(t-\tau_1)} - N_2(t)\right]\mathrm{d}t \\ \qquad +\sigma_2 N_2(t)\mathrm{d}B_{2t}. \end{cases}$$

Actually, the natural growth rate of many populations varies with t, such as, due to the temperature. Therefore it is significant and reasonable to consider the stochastic non-autonomous logistic model

$$\begin{cases} \mathrm{d}N_1(t) = r_1(t)N_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2(t-\tau_2)}{1+N_2(t-\tau_2)} \right. \\ \qquad \left. -N_1(t)\right]\mathrm{d}t + \sigma_1 N_1(t)\mathrm{d}B_{1t}, \\ \mathrm{d}N_2(t) = r_2(t)N_2(t)\left[\frac{K_2(t)+\alpha_2(t)N_1(t-\tau_1)}{1+N_1(t-\tau_1)} \right. \\ \qquad \left. -N_2(t)\right]\mathrm{d}t + \sigma_2 N_2(t)\mathrm{d}B_{2t}, \end{cases} \tag{1.3}$$

where $\alpha_i(t), K_i(t), r_i(t) \in C(\mathbb{R}, \mathbb{R}^+)$ and $\alpha_i(t) > K_i(t), i = 1, 2$. In recent years, Eq.$(1.3)$ has been researched intensively, see e.g.$[7, 8, 9, 10, 11]$.

It is well-know that, in mathematical ecology, permanence is a very important and interesting subject, which means that a population system will survive forever. Generally speaking, a definitive population system is permanent, if a system has the following property

$$0 < N \le \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) \le M < \infty,$$

while $i = 1, 2, \ldots, n$.

On the other hand, studies on mutualism model not only involve permanence but also involve other dynamic behaviors such as stability and periodicity. In recent years, on the basis of permanence result, many scholars studied the global asymptotic stability and the positive periodic solutions of some kinds of nonlinear ecosystems by using periodic theory. For more details we refer to $[12, 13, 14, 15, 16]$ and the references therein. However, according to nowaday's literature, there are few people obtain the permanence of system (1.3). Therefore, the main purpose of this paper is to establish some new sufficient conditions for the global asymptotic stability and positive periodic solutions of system(1.3).

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Since all species suffer from the fluctuation of the environment such as food supplies, harvest and seasonal effects of weather etc. So it is usual to assume the periodic parameters in the system. However, in application, if the various constituent components of the temporally nonuniform environment are in commensurable periods, then one has to consider the temporally environment to be mean square periodic. Hence, if we consider the effects of the environment factors, mean square periodicity is sometimes more realistic and more general than periodicity. Recently, there are many papers dealing with periodic solutions [17, 18, 19, 20, 21, 22, 23] and the references therein. It deserves to be mentioned that there have no results on mutualism model with mean square periodic solutions.

We set

$$\begin{aligned} r_i^+ &= \sup_{t\in(0,\infty)} r_i(t), \quad r_i^- = \inf_{t\in(0,\infty)} r_i(t),\\ k_i^+ &= \sup_{t\in(0,\infty)} k_i(t), \quad k_i^- = \inf_{t\in(0,\infty)} k_i(t),\\ \alpha_i^+ &= \sup_{t\in(0,\infty)} \alpha_i(t), \quad \alpha_i^- = \inf_{t\in(0,\infty)} \alpha_i(t), \end{aligned}$$

where $i = 1, 2$.

Definition 1. A stochastic population system is said to be almost surely stochastically permanent if for any initial value $x_0 \in \mathbb{R}_+^n$, the solution $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ has the property

$$0 < N_i \le \liminf_{t\to\infty} |x_i(t)| \le \limsup_{t\to\infty} |x_i(t)| \le M_i < \infty,$$

while $i = 1, 2, \dots, n$.

Lemma 1. [24] *Brownian motion satisfies the law of iterated logarithm, that is* $\lim\limits_{t\to\infty} \frac{B(t)}{t^\alpha} = 0, \alpha > \frac{1}{2}$.

Remark 1. Setting $\alpha = 1$, we get $\lim\limits_{t\to\infty} \frac{B(t)}{t} = 0$. Therefore, for $\forall\epsilon > 0$, it exists positive constant T_0 such that $|B(t)| < \epsilon t$ for all $t > T_0$.

Remark 2. By Remark 1 and the continuity of Brownian motion, for $\forall\epsilon > 0$, there exists $l \in \mathbb{R}^+$, such that $|B_t| \le \epsilon t + l$, $\forall t \in \mathbb{R}^+$.

In Section 2, through the prove of Lemma 2 and Lemma 3, we yield the Theorem 1, i.e., assuming $\frac{1}{2}\sigma_i^2 < r_i^- k_i^-, i = 1, 2$, almost sure permanence of the stochastic mutualism model is considered, in Section 3, we study the global asymptotic stability of system (1.3), in Section 4, we discuss the system's mean square period, in Section 5, we give an example to illustrate the main results in the section 2 and 3. Finally, we close the paper with conclusions.

II. Permanence

Lemma 2. *If* $\frac{1}{2}\sigma_i^2 < r_i^+\alpha_i^+$, $i = 1, 2$, *then the solution to Eq.*(1.3) *satisfies the following inequalities*

$$\limsup_{t\to\infty} N_1(t) \le \frac{r_1^-\alpha_1^- - \frac{1}{2}\sigma_1^2}{r_1^-} := M_1,$$
$$\limsup_{t\to\infty} N_2(t) \le \frac{r_2^-\alpha_2^- - \frac{1}{2}\sigma_2^2}{r_2^-} := M_2.$$

Proof: Denote $N_1(t) = \frac{1}{x_1(t)}$, by Itô formula to the first equation of $Eq.(1.3)$, we obtain

$$\begin{aligned} \mathrm{d}x_1(t) &= -\frac{1}{N_1^2(t)}\mathrm{d}N_1(t) + \frac{1}{N_1^3(t)}(\mathrm{d}N_1(t))^2\\ &= -\frac{1}{N_1(t)}r_1(t)\Big[\frac{K_1(t) + \alpha_1(t)N_2(t-\tau_2)}{1 + N_2(t-\tau_2)}\\ &\quad -N_1(t)\Big]\mathrm{d}t - \sigma_1\frac{1}{N_1(t)}\mathrm{d}B_{1t} + \sigma_1^2\frac{1}{N_1(t)}\mathrm{d}t\\ &= -r_1(t)x_1(t)\frac{K_1(t) + \alpha_1(t)N_2(t-\tau_2)}{1 + N_2(t-\tau_2)}\mathrm{d}t\\ &\quad +r_1(t)\mathrm{d}t - \sigma_1 x_1(t)\mathrm{d}B_{1t} + \sigma_1^2 x_1(t)\mathrm{d}t\\ &\ge -r_1(t)\alpha_1(t)x_1(t)\mathrm{d}t + r_1(t)\mathrm{d}t - \sigma_1 x_1(t)\mathrm{d}B_{1t}\\ &\quad +\sigma_1^2 x_1(t)\mathrm{d}t. \end{aligned}$$

Thus

$$\mathrm{d}x_1(t) + (r_1(t)\alpha_1(t) - \sigma_1^2)x_1(t)\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}B_{1t} \ge r_1(t)\mathrm{d}t. \quad (2.1)$$

Setting

$$\mathrm{d}x_1(t) + [r_1(t)\alpha_1(t) - \sigma_1^2]x_1(t)dt + \sigma_1 x_1(t)\mathrm{d}B_{1t} = 0.$$

We rewrite the above equation as

$$\frac{1}{x_1(t)}\mathrm{d}x_1(t) = (-r_1(t)\alpha_1(t) + \sigma_1^2)\mathrm{d}t - \sigma_1\mathrm{d}B_{1t}. \quad (2.2)$$

By Itô formula, it follows

$$\mathrm{d}\ln x_1(t) = \frac{1}{x_1(t)}\mathrm{d}x_1(t) - \frac{1}{2}\frac{1}{x_1^2(t)}(\mathrm{d}x_1(t))^2. \quad (2.3)$$

From (2.2) and (2.3), it leads

$$\begin{aligned} \frac{1}{x_1(t)}\mathrm{d}x_1(t) &= \mathrm{d}\ln x_1(t) + \frac{1}{2}\sigma_1^2\mathrm{d}t\\ &= (-r_1(t)\alpha_1(t) + \sigma_1^2)\mathrm{d}t - \sigma_1\mathrm{d}B_{1t}. \end{aligned}$$

Then

$$\mathrm{d}\ln x_1(t) = (-r_1(t)\alpha_1(t) + \frac{1}{2}\sigma_1^2)\mathrm{d}t - \sigma_1\mathrm{d}B_{1t}.$$

Integrating both sides from 0 to t gets

$$\begin{aligned} x_1(t) &= x_1(0)\exp\{-\int_0^t r_1(u)\alpha_1(u)\mathrm{d}u + \frac{1}{2}\sigma_1^2 t\\ &\quad -\sigma_1 B_{1t}\}. \end{aligned} \quad (2.4)$$

From (2.1) and (2.4), it yields

$$\begin{aligned} &\mathrm{d}x_1(t)\exp\left\{\int_0^t r_1(u)\alpha_1(u)\mathrm{d}u - \frac{1}{2}\sigma_1^2 t + \sigma_1 B_{1t}\right\} \ge\\ &r_1(t)\exp\left\{\int_0^t r_1(u)\alpha_1(u)\mathrm{d}u - \frac{1}{2}\sigma_1^2 t + \sigma_1 B_{1t}\right\}\mathrm{d}t \end{aligned} \quad (2.5)$$

Since B_{1t} is Brown motion, $\lim\limits_{t\to\infty} B_{1t}t^{-1} = 0$. So there exists $\epsilon > 0$ small enough and $T_0 = T_0(\epsilon) > 0$ such that $|B_{1t}| \le \epsilon t$, $\forall t \ge T_0$. Letting

$$\begin{aligned} X_1(T_0) &= x_1(T_0)\exp\left\{\int_0^{T_0} r_1(u)\alpha_1(u)\mathrm{d}u - \frac{1}{2}\sigma_1^2 T_0\right.\\ &\quad \left. +\sigma_1 B_{1T_0}\right\}. \end{aligned}$$

Hence, integrating (2.5) from T_0 to t leads to

$$
\begin{aligned}
&x_1(t)\exp\left\{\int_0^t r_1(u)\alpha_1(u)\mathrm{d}u-\frac{1}{2}\sigma_1^2t+\epsilon\sigma_1 t\right\}\\
&\geq x_1(t)\exp\left\{\int_0^t r_1(u)\alpha_1(u)\mathrm{d}u-\frac{1}{2}\sigma_1^2t+\sigma_1|B_{1t}|\right\}\\
&\geq x_1(t)\exp\left\{\int_0^t r_1(u)\alpha_1(u)\mathrm{d}u-\frac{1}{2}\sigma_1^2t+\sigma_1 B_{1t}\right\}\\
&\geq X_1(T_0)+\int_{T_0}^t r_1(s)\exp\left\{\int_0^s r_1(u)\alpha_1(u)\mathrm{d}u\right.\\
&\quad\left.-\frac{1}{2}\sigma_1^2 s+\sigma_1 B_{1s}\right\}ds\\
&\geq X_1(T_0)+\int_{T_0}^t r_1(s)\exp\left\{\int_0^s r_1(u)\alpha_1(u)\mathrm{d}u\right.\\
&\quad\left.-\frac{1}{2}\sigma_1^2 s-\sigma_1 |B_{1s}|\right\}\mathrm{d}s\\
&\geq X_1(T_0)+\int_{T_0}^t r_1(s)\exp\left\{\int_0^s r_1(u)\alpha_1(u)\mathrm{d}u\right.\\
&\quad\left.-\frac{1}{2}\sigma_1^2 s-\epsilon\sigma_1 s\right\}\mathrm{d}s,
\end{aligned}
$$

which yields

$$
\begin{aligned}
x_1(t)\ \geq\ & X_1(T_0)\exp\left\{-\int_0^t r_1(u)\alpha_1(u)\mathrm{d}u+\frac{1}{2}\sigma_1^2 t\right.\\
&\left.-\epsilon\sigma_1 t\right\}+\int_{T_0}^t r_1(s)\exp\left\{\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u\right.\\
&\left.-\frac{1}{2}\sigma_1^2(s-t)-\epsilon\sigma_1(t+s)\right\}\mathrm{d}s\\
\geq\ & X_1(T_0)\exp\left\{(\frac{1}{2}\sigma_1^2-r_1^+\alpha_1^+)t-\epsilon\sigma_1 t\right\}\\
&+r_1^-\int_{T_0}^t\exp\left\{r_1^-\alpha_1^-(s-t)-\frac{1}{2}\sigma_1^2(s-t)\right.\\
&\left.-\epsilon\sigma_1(s+t)\right\}\mathrm{d}s
\end{aligned}
$$

Letting $t\to\infty$ follows

$$
\begin{aligned}
\liminf_{t\to\infty} x_1(t)\ \geq\ & \lim_{t\to\infty}\left\{X_1(T_0)\exp\left\{(\frac{1}{2}\sigma_1^2-r_1^+\alpha_1^+\right.\right.\\
&\left.-\epsilon\sigma_1)t\right\}+r_1^-\int_{T_0}^t\exp\{r_1^-\alpha_1^-(s-t)\\
&\left.-\frac{1}{2}\sigma_1^2(s-t)-\epsilon\sigma_1(s+t)\}\mathrm{d}s\right\}\\
=\ & \lim_{t\to\infty} r_1^-\int_{T_0}^t\exp\left\{r_1^-\alpha_1^-(s-t)\right.\\
&\left.-\frac{1}{2}\sigma_1^2(s-t)\right\}\mathrm{d}s\\
=\ & \frac{r_1^-}{r_1^-\alpha_1^- -\frac{1}{2}\sigma_1^2}.
\end{aligned}
$$

Consequently

$$
\limsup_{t\to\infty} N_1(t)=\frac{1}{\liminf\limits_{t\to\infty} x_1(t)}\leq\frac{r_1^-\alpha_1^- -\frac{1}{2}\sigma_1^2}{r_1^-}:=M_1.
$$

By the same way, we get

$$
\limsup_{t\to\infty} N_2(t)\leq\frac{r_2^-\alpha_2^- -\frac{1}{2}\sigma_2^2}{r_2^-}:=M_2.
$$

This completes the proof. ■

In the following, we give a crucial assumption for the permanence of system (1.3):

(H_1) $r_i^- K_i^- > \frac{1}{2}\sigma_i^2$, $i=1,2$.

From (H_1), there exists $\epsilon_0>0$ small enough, such that

$$
r_i^- K_i^- -\frac{1}{2}\sigma_i^2-\epsilon_0\sigma_i>0,\quad i=1,2.
$$

By Remarks 1-2, there must exist $T_0>0$ and $l_i>0$ such that

$$
|B_{it}|\leq\epsilon_0 t\ \text{ for all } t\geq T_0,\quad |B_{it}|\leq\epsilon_0 t+l_i\ \text{ for all } t\geq 0,
$$

where $l_i:=\sup\limits_{s\in[0,T_0]}|B_{is}|$, $i=1,2$.

Lemma 3. *If (H_1) holds, then the solution to Eq.(1.3) satisfies the following inequalities*

$$
\begin{aligned}
\liminf_{t\to\infty} N_1(t)\geq\frac{r_1^+K_1^+ -\frac{1}{2}\sigma_1^2-\epsilon_0\sigma_1}{r_1^+e^{\sigma_1 l_1}}:=N_1,\\
\liminf_{t\to\infty} N_2(t)\geq\frac{r_2^+K_2^+ -\frac{1}{2}\sigma_2^2-\epsilon_0\sigma_2}{r_2^+e^{\sigma_2 l_2}}:=N_2.
\end{aligned}
$$

Proof: Denote $N_1(t)=\frac{1}{x_1(t)}$, by Itô formula to the first equation of $Eq.(1.3)$, it leads

$$
\begin{aligned}
\mathrm{d}x_1(t)\ =\ & -\frac{1}{N_1^2(t)}\mathrm{d}N_1(t)+\frac{1}{N_1^3(t)}(dN_1(t))^2\\
=\ & -\frac{1}{N_1(t)}r_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2(t-\tau_2)}{1+N_2(t-\tau_2)}\right.\\
&\left.-N_1(t)\right]\mathrm{d}t-\sigma_1\frac{1}{N_1(t)}\mathrm{d}B_{1t}+\frac{1}{N_1(t)}\sigma_1^2dt\\
\leq\ & -\frac{1}{N_1(t)}r_1(t)[K_1(t)-N_1(t)]\mathrm{d}t\\
&-\sigma_1\frac{1}{N_1(t)}\mathrm{d}B_{1t}+\frac{1}{N_1(t)}\sigma_1^2\mathrm{d}t\\
=\ & -r_1(t)K_1(t)x_1(t)\mathrm{d}t+r_1(t)\mathrm{d}t+\sigma_1^2x_1(t)\mathrm{d}t\\
&-\sigma_1 x_1(t)\mathrm{d}B_{1t}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathrm{d}x_1(t)+[r_1(t)K_1(t)-\sigma_1^2]x_1(t)\mathrm{d}t+\sigma_1 x_1(t)\mathrm{d}B_{1t}\\
\leq r_1(t)\mathrm{d}t.
\end{aligned}
$$

That is

$$
\begin{aligned}
&\mathrm{d}x_1(t)\exp\left\{\int_0^t r_1(u)K_1(u)\mathrm{d}u-\frac{1}{2}\sigma_1^2 t+\sigma_1 B_{1t}\right\}\\
&\leq r_1(t)\exp\left\{\int_0^t r_1(u)K_1(u)\mathrm{d}u-\frac{1}{2}\sigma_1^2 t+\sigma_1 B_{1t}\right\}\mathrm{d}t.
\end{aligned}
$$

Integrating both sides from T_0 to t obtains

$$
\begin{aligned}
&x_1(t)\exp\left\{\int_0^t r_1(u)K_1(u)\mathrm{d}u-\frac{1}{2}\sigma_1^2 t+\sigma_1 B_{1t}\right\}-X_1(T_0)\\
&\leq\int_{T_0}^t r_1(s)\exp\left\{\int_0^s r_1(u)K_1(u)\mathrm{d}u-\frac{1}{2}\sigma_1^2+\sigma_1 B_{1s}\right\}\mathrm{d}s,
\end{aligned}
$$

where

$$X_1(T_0) = x_1(T_0)\exp\left\{\int_0^{T_0} r_1(u)K_1(u)\mathrm{d}u - \frac{1}{2}\sigma_1^2 T_0 + \sigma_1 B_{1T_0}\right\}.$$

By Remark 2, it follows that

$$\begin{aligned} x_1(t) &\le X_1(T_0)\exp\left\{-\int_0^t r_1(u)K_1(u)\mathrm{d}u + \frac{1}{2}\sigma_1^2 t - \sigma_1 B_{1t}\right\} + \int_{T_0}^t r_1(s)\exp\left\{\int_t^s r_1(u)K_1(u)\mathrm{d}u - \frac{1}{2}\sigma_1^2(s-t) + \sigma_1(B_{1s}-B_{1t})\right\}\mathrm{d}s \\ &\le X_1(T_0)\exp\left\{(\frac{1}{2}\sigma_1^2 - r_1^- K_1^-)t + \epsilon_0\sigma_1 t\right\} + r_1^+ e^{\sigma_1 l_1}\int_{T_0}^t \exp(r_1^+K_1^+ - \frac{1}{2}\sigma_1^2 - \epsilon_0\sigma_1)(s-t)ds \\ &= X_1(T_0)\exp\left\{\frac{1}{2}\sigma_1^2 - r_1^-K_1^- + \epsilon_0\sigma_1\right\}t + \frac{r_1^+e^{\sigma_1 l_1}}{r_1^+K_1^+ - \frac{1}{2}\sigma_1^2 - \epsilon_0\sigma_1}\left\{1 - \exp\left[-(r_1^+K_1^+ - \frac{1}{2}\sigma_1^2 - \epsilon_0\sigma_1)(t-T_0)\right]\right\}. \end{aligned}$$

Letting $t \to \infty$, we get

$$\limsup_{t\to\infty} x_1(t) \le \frac{r_1^+e^{\sigma_1 l_1}}{r_1^+K_1^+ - \frac{1}{2}\sigma_1^2 - \epsilon_0\sigma_1}.$$

Consequently

$$\liminf_{t\to\infty} N_1(t) = \frac{1}{\limsup\limits_{t\to\infty} x_1(t)} \ge \frac{r_1^+K_1^+ - \frac{1}{2}\sigma_1^2 - \epsilon_0\sigma_1}{r_1^+e^{\sigma_1 l_1}} := N_1.$$

By the same way, we get

$$\liminf_{t\to\infty} N_2(t) \ge \frac{r_2^+K_2^+ - \frac{1}{2}\sigma_2^2 - \epsilon_0\sigma_2}{r_2^+e^{\sigma_2 l_2}} := N_2.$$

This completes the proof. ∎

According to Lemma 2 and Lemma 3, we obtain the following theorem.

Theorem 1. *If (H_1) holds, then the solution to Eq. (1.3) is almost surely stochastically permanent, that is,*

$$N_1 =: \frac{r_1^+K_1^+ - \frac{1}{2}\sigma_1^2 - \epsilon_0\sigma_1}{r_1^+e^{\sigma_1 l_1}} \le \liminf_{t\to\infty} N_1(t) \le \limsup_{t\to\infty} N_1(t) \le \frac{r_1^-\alpha_1^- - \frac{1}{2}\sigma_1^2}{r_1^-} := M_1, \qquad (2.6)$$

$$N_2 =: \frac{r_2^+K_2^+ - \frac{1}{2}\sigma_2^2 - \epsilon_0\sigma_2}{r_2^+e^{\sigma_2 l_2}} \le \liminf_{t\to\infty} N_2(t) \le \limsup_{t\to\infty} N_2(t) \le \frac{r_2^-\alpha_2^- - \frac{1}{2}\sigma_2^2}{r_2^-} := M_2. \qquad (2.7)$$

III. Global Asymptotic Stability

Theorem 2. *Assume that*

(H_2) *there exists two positive constant λ_1 and λ_2 such that*

$$\Gamma_1 = \lambda_1 r_1^- - \lambda_2 r_2^+(K_2^+ + \alpha_2^+) > 0,$$

$$\Gamma_2 = \lambda_2 r_2^- - \lambda_1 r_1^+(K_1^+ + \alpha_1^+) > 0.$$

Then system (1.3) is globally asymptotically stable.

Proof: Assuming $(N_1(t), N_2(t))^T$ and $(\bar N_1(t), \bar N_2(t))^T$ are any two solutions of $Eq.(1.3)$. Let $(y_1, y_2)^T = (\ln N_1(t), \ln N_2(t))^T$ and $(\bar y_1, \bar y_2)^T = (\ln \bar N_1(t), \ln \bar N_2(t))^T$. Denote $y_1(t) = \ln N_1(t)$, by Itô formula to the first equation of system (1.3) , we obtain

$$\begin{aligned} \mathrm{d}y_1(t) &= \frac{1}{N_1(t)}\mathrm{d}N_1(t) - \frac{1}{2}\frac{1}{N_1^2(t)}(\mathrm{d}N_1(t))^2 \\ &= \frac{1}{N_1(t)}\mathrm{d}N_1(t) - \frac{1}{2}\sigma_1^2\mathrm{d}t \\ &= r_1(t)\left[\frac{K_1(t) + \alpha_1(t)N_2(t-\tau_2)}{1+N_2(t-\tau_2)} - N_1(t)\right]\mathrm{d}t + \sigma_1\mathrm{d}B_{1t} - \frac{1}{2}\sigma_1^2\mathrm{d}t, \end{aligned}$$

by the same way, it transforms system (1.3) into the following system,

$$\begin{cases} \mathrm{d}y_1(t) = r_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2(t-\tau_2)}{1+N_2(t-\tau_2)} - N_1(t)\right]\mathrm{d}t + \sigma_1\mathrm{d}B_{1t} - \frac{1}{2}\sigma_1^2\mathrm{d}t, \\ \mathrm{d}y_2(t) = r_2(t)\left[\frac{K_2(t)+\alpha_2(t)N_1(t-\tau_1)}{1+N_1(t-\tau_1)} - N_2(t)\right]\mathrm{d}t + \sigma_2\mathrm{d}B_{2t} - \frac{1}{2}\sigma_2^2\mathrm{d}t, \\ \mathrm{d}\bar y_1(t) = r_1(t)\left[\frac{K_1(t)+\alpha_1(t)\bar N_2(t-\tau_2)}{1+\bar N_2(t-\tau_2)} - \bar N_1(t)\right]\mathrm{d}t + \sigma_1\mathrm{d}B_{1t} - \frac{1}{2}\sigma_1^2\mathrm{d}t, \\ \mathrm{d}\bar y_2(t) = r_2(t)\left[\frac{K_2(t)+\alpha_2(t)\bar N_1(t-\tau_1)}{1+\bar N_1(t-\tau_1)} - \bar N_2(t)\right]\mathrm{d}t + \sigma_2\mathrm{d}B_{2t} - \frac{1}{2}\sigma_2^2\mathrm{d}t. \end{cases} \qquad (3.1)$$

Define

$$V(t) = V_0(t) + V_1(t) + V_2(t), \qquad (3.2)$$

where

$$\begin{aligned} V_0(t) &= \lambda_1|y_1(t) - \bar y_1(t)| + \lambda_2|y_2(t) - \bar y_2(t)|, \\ V_1(t) &= \lambda_2\int_{t-\tau_1}^t r_2^+(K_2^+ + \alpha_2^+)|N_1(s) - \bar N_1(s)|\mathrm{d}s, \\ V_2(t) &= \lambda_1\int_{t-\tau_2}^t r_1^+(K_1^+ + \alpha_1^+)|N_2(s) - \bar N_2(s)|\mathrm{d}s. \end{aligned}$$

Calculating the upper right derivative of $V_0(t)$ along system (3.1),

$$\begin{aligned} D^+V_0(t) &= \lambda_1\mathrm{sgn}[y_1(t) - \bar y_1(t)][y_1'(t) - \bar y_1'(t)] + \lambda_2\mathrm{sgn}[y_2(t) - \bar y_2(t)][y_2'(t) - \bar y_2'(t)] \\ &\le -\lambda_1 r_1(t)|N_1(t) - \bar N_1(t)| + \lambda_1 r_1(t)[K_1(t) + \alpha_1(t)]|N_2(t-\tau_2) - \bar N_2(t-\tau_2)| - \lambda_2 r_2(t)|N_2(t) - \bar N_2(t)| \\ &\quad + \lambda_2 r_2(t)[K_2(t) + \alpha_2(t)]|N_1(t-\tau_1) - \bar N_1(t-\tau_1)| \\ &\le -\lambda_1 r_1^-|N_1(t) - \bar N_1(t)| + \lambda_1 r_1^+(K_1^+ + \alpha_1^+)|N_2(t-\tau_2) - \bar N_2(t-\tau_2)| - \lambda_2 r_2^-|N_2(t) - \bar N_2(t)| \\ &\quad + \lambda_2 r_2^+(K_2^+ + \alpha_2^+)|N_1(t-\tau_1) - \bar N_1(t-\tau_1)|. \end{aligned} \qquad (3.3)$$

Further, calculating the upper right derivative of $V_1(t), V_2(t)$ along system (3.1), it follows that

$$\begin{aligned} D^+V_1(t) &= \lambda_2 r_2^+(K_2^+ + \alpha_2^+)|N_1(t) - \bar{N}_1(t)| \\ &\quad -\lambda_2 r_2^+(K_2^+ + \alpha_2^+)|N_1(t-\tau_1) \\ &\quad -\bar{N}_1(t-\tau_1)|, \end{aligned} \tag{3.4}$$

$$\begin{aligned} D^+V_2(t) &= \lambda_1 r_1^+(K_1^+ + \alpha_1^+)|N_2(t) - \bar{N}_2(t)| \\ &\quad -\lambda_1 r_1^+(K_1^+ + \alpha_1^+)|N_2(t-\tau_2) \\ &\quad -\bar{N}_2(t-\tau_2)|. \end{aligned} \tag{3.5}$$

Together with (3.2) – (3.5), for $\forall t \in \mathbb{R}$, we get

$$\begin{aligned} D^+V(t) &\le [-\lambda_1 r_1^- + \lambda_2 r_2^+(K_2^+ + \alpha_2^+)]|N_1(t) - \bar{N}_1(t)| \\ &\quad +[-\lambda_2 r_2^- + \lambda_1 r_1^+(K_1^+ + \alpha_1^+)]|N_2(t) - \bar{N}_2(t)| \\ &\le -\Gamma_1|N_1(t) - \bar{N}_1(t)| - \Gamma_2|N_2(t) - \bar{N}_2(t)|, \end{aligned}$$

Hence, for $\forall t \in \mathbb{R}$, $V(t)$ is nonincreasing, integrating the above formula from 0 to t yields

$$\begin{aligned} &V(t) + \Gamma_1 \int_0^t |N_1(s) - \bar{N}_1(s)|\mathrm{d}s \\ &+\Gamma_2 \int_0^t |N_2(s) - \bar{N}_2(s)|\mathrm{d}s \le V(0) < +\infty, \quad \forall t \ge 0, \end{aligned}$$

implies that,

$$\begin{aligned} &\int_0^t |N_1(s) - \bar{N}_1(s)|\mathrm{d}s < +\infty, \\ &\int_0^t |N_2(s) - \bar{N}_2(s)|\mathrm{d}s < +\infty, \end{aligned}$$

that is,

$$\lim_{s\to+\infty} |N_1(s) - \bar{N}_1(s)| = \lim_{s\to+\infty} |N_2(s) - \bar{N}_2(s)| = 0.$$

This completes the proof. ■

Remark 3. The theorem illustrates that the intensity of white noise has a negative impact on almost sure permanence, but it makes no difference on global asymptotic stability.

IV. Periodic Solution

In this section, we assume that

(H_3) there exists a positive constant ω such that

$$\begin{aligned} &r_i(t+\omega) = r_i(t), \quad K_i(t+\omega) = K_i(t), \\ &\alpha_i(t+\omega) = \alpha_i(t), \quad i = 1, 2. \end{aligned}$$

(H_4) $\max\{\frac{A_1+B_1}{r_1^-\alpha_1^-}, \frac{A_2+B_2}{r_2^-\alpha_2^-}\} < 1$, where

$$\begin{aligned} A_1 &= \frac{2}{r_1^-\alpha_1^-}\Big[4[r_1^+(K_1^+ - \alpha_1^-)]^2 + 4[r_1^+(M_1+\epsilon)]^2 \\ &\quad +\sigma_1^2\Big], \\ B_1 &= \frac{8}{r_1^-\alpha_1^-}[r_1^+(M_1+\epsilon)(K_1^+ - \alpha_1^-)]^2, \\ A_2 &= \frac{2}{r_2^-\alpha_2^-}\Big[4[r_2^+(K_2^+ - \alpha_2^-)]^2 + 4[r_2^+(M_2+\epsilon)]^2 \\ &\quad +\sigma_2^2\Big], \\ B_2 &= \frac{8}{r_2^-\alpha_2^-}[r_2^+(M_2+\epsilon)(K_2^+ - \alpha_2^-)]^2. \end{aligned}$$

Definition 2. A function f is called mean square periodic if there exists a positive constant ω such that

$$\begin{aligned} &E|N_1(t+\omega) - N_1(t)|^2 = 0, \\ &E|N_2(t+\omega) - N_2(t)|^2 = 0, \forall t \in \mathbb{R}. \end{aligned}$$

Theorem 3. *Assume that (H_3) and (H_4) hold, and for any $\epsilon > 0$, there exists $t > T_0$, such that $N_i(t) < M_i + \epsilon, i = 1, 2$, then system (1.3) is mean square periodic, that is*

$$\begin{aligned} &E|N_1(t+\omega) - N_1(t)|^2 = 0, \\ &E|N_2(t+\omega) - N_2(t)|^2 = 0. \end{aligned}$$

Proof: From the first equation of system (1.3), we get

$$\begin{aligned} \mathrm{d}N_1(t) &= r_1(t)N_1(t)\Big[\frac{K_1(t) + \alpha_1(t)N_2(t-\tau_2)}{1 + N_2(t-\tau_2)} \\ &\quad -N_1(t)\Big]\mathrm{d}t + \sigma_1 N_1(t)\mathrm{d}B_{1t} \\ &= r_1(t)\alpha_1(t)N_1(t)\mathrm{d}t + \sigma_1 N_1(t)\mathrm{d}B_{1t} \\ &\quad +r_1(t)\frac{K_1(t) - \alpha_1(t)}{1 + N_2(t-\tau_2)}N_1(t)\mathrm{d}t - r_1(t)N_1^2(t)\mathrm{d}t \\ &= r_1(t)\alpha_1(t)N_1(t)\mathrm{d}t + f(t)\mathrm{d}t + \sigma_1 N_1(t)\mathrm{d}B_{1t}, \end{aligned}$$

where

$$f(t) = r_1(t)\frac{K_1(t) - \alpha_1(t)}{1 + N_2(t-\tau_2)}N_1(t) - r_1(t)N_1^2(t).$$

Therefore

$$\begin{aligned} \mathrm{d}N_1(t+\omega) &= r_1(t+\omega)\alpha_1(t+\omega)N_1(t+\omega)\mathrm{d}t \\ &\quad +f(t+\omega)\mathrm{d}t + \sigma_1 N_1(t+\omega)\mathrm{d}B_{1(t+\omega)} \\ &= r_1(t)\alpha_1(t)N_1(t+\omega)\mathrm{d}t + f(t+\omega)\mathrm{d}t \\ &\quad +\sigma_1 N_1(t+\omega)\mathrm{d}B_{1(t+\omega)}. \end{aligned}$$

So

$$\begin{aligned} &\mathrm{d}N_1(t+\omega) - \mathrm{d}N_1(t) \\ &= r_1(t)\alpha_1(t)[N_1(t+\omega) - N_1(t)]\mathrm{d}t + [f(t+\omega) - f(t)]\mathrm{d}t \\ &\quad +\sigma_1[N_1(t+\omega)\mathrm{d}B_{1(t+\omega)} - N_1(t)\mathrm{d}B_{1t}]. \end{aligned} \tag{4.1}$$

Setting $Y_i(t) = N_i(t+\omega) - N_i(t), i = 1, 2$, from (4.1) we get

$$\begin{aligned} &\mathrm{d}Y_1(t) = r_1(t)\alpha_1(t)Y_1(t)\mathrm{d}t + [f(t+\omega) - f(t)]\mathrm{d}t \\ &+\sigma_1[N_1(t+\omega)\mathrm{d}\tilde{B}_{1t} - N_1(t)\mathrm{d}B_{1t}], \end{aligned}$$

where $\tilde{B}_{1t} = B_{1t+\omega} - B_{1t} = B_{1t} - B_{10} \overset{d}{=} B_{1t}$.
Then, it follows

$$\begin{aligned} &\mathrm{d}Y_1(t)e^{-\int_0^t r_1(s)\alpha_1(s)\mathrm{d}s} \\ &= e^{-\int_0^t r_1(s)\alpha_1(s)\mathrm{d}s}[f(t+\omega) - f(t)]\mathrm{d}t \\ &\quad +\sigma_1 e^{-\int_0^t r_1(s)\alpha_1(s)\mathrm{d}s}[N_1(t+\omega)\mathrm{d}\tilde{B}_{1t} - N_1(t)\mathrm{d}B_{1t}]. \end{aligned}$$

Integrating both sides from t to T_0 gets

$$\begin{aligned} &Y_1(T_0)e^{-\int_0^{T_0} r_1(s)\alpha_1(s)\mathrm{d}s} - Y_1(t)e^{-\int_0^t r_1(s)\alpha_1(s)\mathrm{d}s} \\ &= \int_t^{T_0} e^{-\int_0^s r_1(u)\alpha_1(u)\mathrm{d}u}[f(s+\omega) - f(s)]\mathrm{d}s \\ &\quad +\sigma_1 \int_t^{T_0} e^{-\int_0^s r_1(u)\alpha_1(u)\mathrm{d}u}[N_1(s+\omega)\mathrm{d}\tilde{B}_{1s} \\ &\quad -N_1(s)\mathrm{d}B_{1s}], \end{aligned}$$

that is

$$\begin{aligned}Y_1(t) &= Y_1(T_0)e^{-\int_t^{T_0} r_1(s)\alpha_1(s)\mathrm{d}s}\\&\quad -\int_t^{T_0} e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}[f(s+\omega)-f(s)]\mathrm{d}s\\&\quad -\int_t^{T_0} \sigma_1 e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}[N_1(s+\omega)\mathrm{d}\tilde{B}_{1s}\\&\quad -N_1(s)\mathrm{d}B_{1s}].\end{aligned}$$

Letting $T_0 \to +\infty$, we obtain

$$\begin{aligned}Y_1(t) &= -\int_t^{+\infty} e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}[f(s+\omega)-f(s)]\mathrm{d}s\\&\quad -\int_t^{+\infty} \sigma_1 e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}[N_1(s+\omega)\mathrm{d}\tilde{B}_{1s}\\&\quad -N_1(s)\mathrm{d}B_{1s}].\end{aligned}$$

Using Hölder inequality and isometric transformation, it follows

$$\begin{aligned}&E|Y_1(t)|^2\\&= E\bigg|\int_t^{+\infty} e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}[f(s+\omega)-f(s)]\mathrm{d}s\\&\quad +\sigma_1\int_t^{+\infty} e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}[N_1(s+\omega)\mathrm{d}\tilde{B}_{1s}\\&\quad -N_1(s)\mathrm{d}B_{1s}]\bigg|^2\\&\le 2E\bigg|\int_t^{+\infty} e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}[f(s+\omega)-f(s)]\mathrm{d}s\bigg|^2\\&\quad +2E\bigg|\sigma_1\int_t^{+\infty} e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}[N_1(s+\omega)\mathrm{d}\tilde{B}_{1s}\\&\quad -N_1(s)\mathrm{d}B_{1s}]\bigg|^2\\&\le 2E\int_t^{+\infty} e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}\mathrm{d}s\int_t^{+\infty} e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}\\&\quad [f(s+\omega)-f(s)]^2\mathrm{d}s+2\sigma_1^2E\int_t^{+\infty} e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}\mathrm{d}s\\&\quad \times\int_t^{+\infty} e^{-\int_t^s r_1(u)\alpha_1(u)\mathrm{d}u}[N_1(s+\omega)-N_1(s)]^2\mathrm{d}s\\&\le \frac{2}{r_1^-\alpha_1^-}\int_t^{+\infty} e^{-r_1^-\alpha_1^-(s-t)}E[f(s+\omega)-f(s)]^2\mathrm{d}s\\&\quad +\frac{2\sigma_1^2}{r_1^-\alpha_1^-}\int_t^{+\infty} e^{-r_1^-\alpha_1^-(s-t)}EY_1^2(s)\mathrm{d}s, \qquad (4.2)\end{aligned}$$

where

$$\begin{aligned}&[f(t+\omega)-f(t)]^2\\&= \bigg\{[r_1(t)(K_1(t)-\alpha_1(t))]\bigg[\frac{N_1(t+\omega)}{1+N_2(t+\omega-\tau_2)}\\&\quad -\frac{N_1(t)}{1+N_2(t-\tau_2)}\bigg]-r_1(t)[N_1^2(t+\omega)-N_1^2(t)]\bigg\}^2\\&\le 2[r_1(t)(K_1(t)-\alpha_1(t))]^2\bigg[\frac{N_1(t+\omega)}{1+N_2(t+\omega-\tau_2)}\\&\quad -\frac{N_1(t)}{1+N_2(t-\tau_2)}\bigg]^2+2r_1^2(t)[N_1^2(t+\omega)-N_1^2(t)]^2\\&\le 2[r_1^+(K_1^+-\alpha_1^-)]^2\bigg[\frac{N_1(t+\omega)}{1+N_2(t+\omega-\tau_2)}\\&\quad -\frac{N_1(t)}{1+N_2(t-\tau_2)}\bigg]^2+2(r_1^+)^2[N_1^2(t+\omega)\\&\quad -N_1^2(t)]^2. \qquad (4.3)\end{aligned}$$

Since

$$\begin{aligned}&\bigg[\frac{N_1(t+\omega)}{1+N_2(t+\omega-\tau_2)}-\frac{N_1(t)}{1+N_2(t-\tau_2)}\bigg]^2\\&= \bigg[\frac{1}{1+N_2(t+\omega-\tau_2)}(N_1(t+\omega)-N_1(t))\\&\quad -N_1(t)\frac{1}{(1+\xi)^2}[N_2(t+\omega-\tau_2)-N_2(t-\tau_2)]\bigg]^2\\&\le \bigg[[N_1(t+\omega)-N_1(t)]-N_1(t)[N_2(t+\omega-\tau_2)\\&\quad -N_2(t-\tau_2)]\bigg]^2\\&\le 2[N_1(t+\omega)-N_1(t)]^2\\&\quad +2N_1^2(t)[N_2(t+\omega-\tau_2)-N_2(t-\tau_2)]^2\\&= 2Y_1^2(t)+2(M_1+\epsilon)^2Y_2^2(t-\tau_2), \qquad (4.4)\end{aligned}$$

where ξ is between $N_2(t+\omega-\tau_2)$ and $N_2(t-\tau_2)$, and

$$\begin{aligned}&N_1^2(t+\omega)-N_1^2(t)\\&= [N_1(t+\omega)+N_1(t)][N_1(t+\omega)-N_1(t)]\\&\le 2(M_1+\epsilon)[N_1(t+\omega)-N_1(t)]\\&= 2(M_1+\epsilon)Y_1(t). \qquad (4.5)\end{aligned}$$

From (4.3)-(4.5) we get

$$\begin{aligned}&[f(t+\omega)-f(t)]^2\\&\le 2[r_1^+(K_1^+-\alpha_1^-)]^2[2Y_1^2(t)+2(M_1+\epsilon)^2Y_2^2(t-\tau_2)]\\&\quad +2(r_1^+)^2[2(M_1+\epsilon)Y_1(t)]^2\\&= \bigg[4[r_1^+(K_1^+-\alpha_1^-)]^2+4[r_1^+(M_1+\epsilon)]^2\bigg]Y_1^2(t)\\&\quad +4[r_1^+(M_1+\epsilon)(K_1^+-\alpha_1^-)]^2Y_2^2(t-\tau_2). \qquad (4.6)\end{aligned}$$

From (4.2) and (4.6) it leads

$$\begin{aligned}E|Y_1(t)|^2 &\le \frac{2}{r_1^-\alpha_1^-}\int_t^{+\infty} e^{-r_1^-\alpha_1^-(s-t)}E\bigg\{\bigg[4[r_1^+(K_1^+\\&\quad -\alpha_1^-)]^2+4[r_1^+(M_1+\epsilon)]^2\bigg]Y_1^2(t)\\&\quad +4[r_1^+(M_1+\epsilon)(K_1^+-\alpha_1^-)]^2Y_2^2(t-\tau_2)\bigg\}\mathrm{d}s\\&\quad +\frac{2\sigma_1^2}{r_1^-\alpha_1^-}\int_t^{+\infty} e^{-r_1^-\alpha_1^-(s-t)}EY_1^2(s)\mathrm{d}s\\&= A_1\int_t^{+\infty} e^{-r_1^-\alpha_1^-(s-t)}EY_1^2(s)\mathrm{d}s\\&\quad +B_1\int_t^{+\infty} e^{-r_1^-\alpha_1^-(s-t)}EY_2^2(s-\tau_2)\mathrm{d}s. \qquad (4.7)\end{aligned}$$

By the same way, we obtain

$$\begin{aligned}&E|Y_2(t)|^2 \le A_2\int_t^{+\infty} e^{-r_2^-\alpha_2^-(s-t)}EY_2^2(s)\mathrm{d}s\\&+B_2\int_t^{+\infty} e^{-r_2^-\alpha_2^-(s-t)}EY_1^2(s-\tau_1)\mathrm{d}s. \qquad (4.8)\end{aligned}$$

Setting

$$X_0 = \max_{T_0\le s\le +\infty}\{EY_1^2(s), EY_2^2(s)\},$$

from (4.7) and (4.8), we yield

$$\begin{aligned} E|Y_1(t)|^2 &\le A_1 \int_t^{+\infty} e^{-r_1^- \alpha_1^- (s-t)} \mathrm{d}s X_0 \\ &\quad + B_1 \int_t^{+\infty} e^{-r_1^- \alpha_1^- (s-t)} \mathrm{d}s X_0 \\ &= \frac{A_1 + B_1}{r_1^- \alpha_1^-} X_0, \end{aligned} \tag{4.9}$$

$$\begin{aligned} E|Y_2(t)|^2 &\le A_2 \int_t^{+\infty} e^{-r_2^- \alpha_2^- (s-t)} \mathrm{d}s X_0 \\ &\quad + B_2 \int_t^{+\infty} e^{-r_2^- \alpha_2^- (s-t)} \mathrm{d}s X_0 \\ &= \frac{A_2 + B_2}{r_2^- \alpha_2^-} X_0, \end{aligned} \tag{4.10}$$

From (4.9) and (4.10), we have

$$\begin{aligned} &\max_{T_0 \le t \le +\infty} \{E|Y_1(t)|^2, E|Y_2(t)|^2\} \\ &\le \max\{\frac{A_1 + B_1}{r_1^- \alpha_1^-}, \frac{A_2 + B_2}{r_2^- \alpha_2^-}\} X_0, \end{aligned}$$

that is

$$X_0 \le \max\{\frac{A_1 + B_1}{r_1^- \alpha_1^-}, \frac{A_2 + B_2}{r_2^- \alpha_2^-}\} X_0.$$

By (H_4) we get

$$X_0 = 0,$$

thus

$$E|Y_1(t)|^2 = 0, \quad E|Y_2(t)|^2 = 0,$$

that is

$$\begin{aligned} E|N_1(t+\omega) - N_1(t)|^2 = 0, \\ E|N_2(t+\omega) - N_2(t)|^2 = 0. \end{aligned}$$

This completes the proof. ■

V. An example

In this section we use an example to illustrate the main results. Consider the system

$$\begin{cases} \mathrm{d}N_1(t) = (\frac{1}{2} + \frac{1}{2}\cos^2 t) N_1(t) \\ \qquad \left[\frac{(\frac{1}{8} + \frac{1}{16}\cos^2 t) + (\frac{1}{8} + \frac{1}{16}\cos^2 t) N_2(t - e^{-100})}{1 + N_2(t - e^{-100})} \right. \\ \qquad \left. - N_1(t) \right] \mathrm{d}t + \frac{1}{8} N_1(t) \mathrm{d}B_{1t}, \\ \mathrm{d}N_2(t) = (\frac{1}{2} + \frac{1}{2}\sin^2 t) N_2(t) \\ \qquad \left[\frac{(\frac{1}{8} + \frac{1}{16}\sin^2 t) + (\frac{1}{8} + \frac{1}{16}\sin^2 t) N_1(t - e^{-100})}{1 + N_1(t - e^{-100})} \right. \\ \qquad \left. - N_2(t) \right] \mathrm{d}t + \frac{1}{8} N_2(t) \mathrm{d}B_{2t}. \end{cases} \tag{5.1}$$

Since, $\frac{1}{2}\sigma_i^2 < r_i^- k_i^-, i = 1, 2$, satisfies the condition of Theorem 1, we choose $\sigma_i = \frac{1}{8}, l_i \le 1, i = 1, 2$. From system (5.1) and (2.6), (2.7), we yield

$$N_1 =: \frac{r_1^+ K_1^+ - \frac{1}{2}\sigma_1^2 - \epsilon_0 \sigma_1}{r_1^+ e^{\sigma_1 l_1}} \ge \frac{11}{64} e^{-\frac{1}{8}},$$

$$N_2 =: \frac{r_2^+ K_2^+ - \frac{1}{2}\sigma_2^2 - \epsilon_0 \sigma_2}{r_2^+ e^{\sigma_2 l_2}} \ge \frac{11}{64} e^{-\frac{1}{8}},$$

$$M_1 =: \frac{r_1^- \alpha_1^- - \frac{1}{2}\sigma_1^2}{r_1^-} = \frac{7}{64},$$

$$M_2 =: \frac{r_2^- \alpha_2^- - \frac{1}{2}\sigma_2^2}{r_2^-} = \frac{7}{64}.$$

So,

$$\frac{11}{64} e^{-\frac{1}{8}} \le \liminf_{t \to \infty} N_1(t) \le \limsup_{t \to \infty} N_1(t) \le \frac{7}{64},$$

$$\frac{11}{64} e^{-\frac{1}{8}} \le \liminf_{t \to \infty} N_2(t) \le \limsup_{t \to \infty} N_2(t) \le \frac{7}{64}.$$

Therefore, system(5.1) is almost surely stochastically permanent.

On the other hand, setting $\lambda_1 = \lambda_2 = 1$, we yield

$$\Gamma_1 = \lambda_1 r_1^- - \lambda_2 r_2^+ (K_2^+ + \alpha_2^+) > 0,$$

$$\Gamma_2 = \lambda_2 r_2^- - \lambda_1 r_1^+ (K_1^+ + \alpha_1^+) > 0,$$

therefore, system (5.1) is global asymptotic stability.

VI. Conclusion

This paper concerns the stochastic and time-lagged mutualism model. We know that permanence is a very important and interesting subject in mathematical ecology, which means that a population system will survive forever. A definition of almost sure permanence is presented here, which is similar to the definition in definitive models. Under the condition $\frac{1}{2}\sigma_i^2 < r_i^- k_i^-, i = 1, 2$, the stochastic model (1.3) is almost surely stochastically permanent and the intensity of white noise has a negative impact on it, but makes no difference on global asymptotic stability. And in some certain conditions, we deduce the system (1.3) is mean square periodic.

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