Global Asymptotical Stability in a Stochastic Predator-Prey System With Variable Delays

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Abstract—**A stochastic predator-prey system involving time-varying delays is considered. Some new sufficient conditions for existence, extinction and global asymptotical stability are obtained. It is interesting that the results are based on the variable delays, which is different from the previous work (the results are delay-independent). Some numerical simulations are introduced to support the analytical findings.**

Keywords: existence, extinction, asymptotical stability, stochastic

1 Introduction

This paper is devoted to investigating the following stochastic differential system with variable delays:

$$\begin{cases} dx_1 &= x_1\Big(r_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2(t-\delta_1(t))\Big)dt \\ &\quad +\sigma_1(t)x_1 dB_1(t), \\ dx_2 &= x_2\Big(-r_2(t) + a_{21}(t)x_1(t-\delta_2(t)) - a_{22}(t)x_2\Big)dt \\ &\quad +\sigma_2(t)x_2 dB_2(t), \end{cases} \tag{SM}$$

on $t \geq 0$ with initial data $\{(x_1(\theta), x_2(\theta))^\top : -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; (0, +\infty)\times(0, +\infty))$. Here $\delta_i : R_+ \to [0, \tau],\ i = 1, 2$ is a continuous differentiable function, τ is given positive constant, $R_+ = [0, +\infty)$. In addition, throughout the present paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i. e., it is right continuous and $\mathcal{F}_0$ contains all $P-$null sets). Let $|\cdot|$ denote the Euclidean norm in R^n. For given a constant $\tau > 0$, let $C([-\tau, 0], R^n_+)$ denote the family of all continuous R^n_+-valued functions ξ with its norm $||\xi|| = \sup\{|\xi(\theta)| : \theta \in [-\tau, 0]\}$. Also, denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; R^n_+)$ the family of bounded, $\mathcal{F}_0-$measurable, $C^b_{\mathcal{F}_0}([-\tau, 0]; R^n_+)-$valued random variables.

A classical predator-prey model with time delays can be expressed as follows:

$$\begin{cases} \frac{dx_1}{dt} = x_1\Big(r_1 - a_{11}x_1 - a_{12}x_2(t-\delta_1)\Big), \\ \frac{dx_2}{dt} = x_2\Big(-r_2 + a_{21}x_1(t-\delta_2) - a_{22}x_2\Big), \end{cases} \tag{M}$$

with initial conditions

$$x_i(t) = \phi_i(t) > 0,\ t \in [-\tau, 0],\ i = 1, 2,$$

where x_1 and x_2 represent for the population sizes of the prey and the predator, respectively; $r_i,\ a_{ij},\ \tau$ and $\delta_i\ (i, j = 1, 2)$ are given positive constants. Owing to its theoretical and practical significance, there is an extensive literature investigated model (M); see e.g., [4–6]. Recently, due to environmental noises, the authors [7] studied a kind of stochastic predator-prey model with time delays as follows:

$$\begin{cases} dx_1 &= x_1\Big(r_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2(t-\tau_1)\Big)dt \\ &\quad +\sigma_1(t)x_1 dB_1(t), \\ dx_2 &= x_2\Big(-r_2(t) + a_{21}(t)x_1(t-\tau_2) - a_{22}(t)x_2\Big)dt \\ &\quad +\sigma_2(t)x_2 dB_2(t). \end{cases}$$

and obtained some existence and stability results. However, the delays in the above system are constants and the results are independent on delays. For more research methods for population dynamic system, see papers [8–13].

The main objective of this paper is to obtain sufficient conditions for the existence-and-uniqueness, extinction and global asymptotical stability of positive solutions for system (SM). It is interesting that the results obtained in this paper are based on the delays (or delay-dependent) which is different from the previous works that are delay-independent. Also, some numerical simulations are introduced to support the analytical findings.

2 Global positive solution of system (SM)

In this section, we will investigate the system (SM) with initial value $\xi \in C([-\tau, 0], R^2_+)$. In order for a stochastic differential equation to have a unique global solution (i.e.,

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no explosion in a finite time) for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition, see e.g., [14]. However, the coefficients of (SM) do not satisfy the above conditions, so the solutions of (SM) may explode at finite time. It is therefore useful to give some conditions under which the solution of (SM) is not only positive but also not explode to infinite at any finite time. For convenience of proof, if $f(t)$ is a continuous bounded function on R_+, define

$$f^u = \sup_{t\in R_+} f(t), \quad f^l = \inf_{t\in R_+} f(t).$$

The following theorem gives some sufficient conditions for the existence and uniqueness of the global positive solution of system (SM).

Theorem 2.1. *For any given initial data $\{(x_1(\theta), x_2(\theta))^\top : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; (0, +\infty) \times (0, +\infty))$, if the population sizes of x_2 are no more than 1, there is an unique positive local solution $(x_1(t), x_2(t))$ on $t \ge 0$ with satisfying initial condition ξ and the solution will remain in R^2_+ with probability 1.*

Proof: By the biological meaning, we only focus on the positive solutions to system (SM). Thus it is reasonable to make the following change of variables, $x_1(t) = e^{u(t)}, x_2(t) = e^{v(t)}$. By using Itô formula, system (SM) can be reformulated in the following form:

$$\begin{cases} du &= [r_1(t) - a_{11}(t)e^u - a_{12}(t)e^{v(t-\delta_1(t))} - \frac{1}{2}\sigma_1^2(t)]dt \\ &\quad +\sigma_1(t)dB_1(t), \\ dv &= [-r_2(t) + a_{21}(t)e^{u(t-\delta_2(t))} - a_{22}(t)e^v - \frac{1}{2}\sigma_2^2(t)]dt \\ &\quad +\sigma_2(t)dB_2(t), \\ u(\theta) &= \ln x_1(\theta),\ v(\theta) = \ln x_2(\theta),\ \theta \in [-\tau, 0], \end{cases} \tag{SMA}$$

on $t \ge 0$. It is easy to see that the coefficients of (SMA) satisfy the local Lipschitz condition, then for any given initial values $u(\theta), v(\theta),\ \theta \in [-\tau, 0]$, there is a unique maximal local solution $u(t), v(t)$ on $[0, \tau_e)$, where τ_e is explosion time. By Itô formula, $x_1(t) = e^{u(t)}, x_2(t) = e^{v(t)}$ is the positive local solution to (SMA) with the initial value $u(\theta), v(\theta), \theta \in [-\tau, 0]$. In order to show this solution is global, we need to show that $\tau_e = \infty$ a. s.. For convenience of statement, let

$$F(x_1, x_2) = r_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2(t - \delta_1(t)),$$

$$G(x_1, x_2) = -r_2(t) + a_{21}(t)x_1(t - \delta_2(t)) - a_{22}(t)x_2.$$

The following proof is motivated by the work of Li and Mao [15]. Let $n_0 > 0$ be sufficiently large for $x_1(\theta)$ and $x_2(\theta)$ lying within the interval $[\frac{1}{n_0}, n_0]$. For each integer $n > n_0$, define the stopping times:

$$\tau_n = \inf\{t \in [0, \tau_e] : x_1(t) \notin (\frac{1}{n}, n) \text{ or } x_2(t) \notin (\frac{1}{n}, n)\}.$$

Throughout this paper, we set $\inf \emptyset = \infty$. Obviously, τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to} \tau_n$, whence $\tau_\infty \le \tau_e$ a. s.. Now, we only need to show $\tau_\infty = \infty$. If this statement is false, there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathcal{P}\{\tau_\infty \le T\} > \varepsilon$. Consequently, there exists an integer $n_1 \ge n_0$ such that

$$\mathcal{P}\{\tau_\infty \le T\} > \varepsilon, \quad n \ge n_1. \tag{2.1}$$

Define C^2-function $V : R^2_+ \to R_+$ by

$$V(x_1, x_2) = (\sqrt{x_1} - 1 - 0.5 \ln x_1) + (\sqrt{x_2} - 1 - 0.5 \ln x_2).$$

The nonnegativity of this function can be obtained from $u - 1 - \ln u \ge 0$ on $u > 0$. If $(x_1, x_2) \in R^2_+$, the Itô formula shows that

$$\begin{aligned} &dV(x_1, x_2) = 0.5(x_1^{-0.5} - x_1^{-1})F(x_1, x_2)dt \\ &+\tfrac{1}{8}(-x_1^{0.5} + 2)\sigma_1^2(t)dt + 0.5(x_1^{0.5} - 1)\sigma_1(t)dB_1(t) \\ &+0.5(x_2^{-0.5} - x_2^{-1})G(x_1, x_2)dt \\ &+\tfrac{1}{8}(-x_2^{0.5} + 2)\sigma_2^2(t)dt + 0.5(x_2^{0.5} - 1)\sigma_2(t)dB_2(t). \end{aligned} \tag{2.2}$$

Compute

$$\begin{aligned} &(x_1^{-0.5} - x_1^{-1})F(x_1, x_2) = (x_1^{-0.5} - x_1^{-1}) \\ &\times[r_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2(t - \delta_1(t))] \\ &= r_1(t)x_1^{-0.5} - a_{11}(t)x_1^{0.5} - a_{12}(t)x_2(t - \delta_1(t))x_1^{-0.5} \\ &-r_1(t)x_1^{-1} + a_{11}(t) + a_{12}(t)x_2(t - \delta_1(t))x_1^{-1} \\ &\le r_1^u + a_{11}^u =: K_1 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &(x_2^{-0.5} - x_2^{-1})G(x_1, x_2) \\ &= (x_2^{-0.5} - x_2^{-1})[-r_2(t) + a_{21}(t)x_1(t - \delta_2(t)) - a_{22}(t)x_2] \\ &= -r_2(t)x_2^{-0.5} + a_{21}(t)x_1(t - \delta_2(t))x_2^{-0.5} - a_{22}(t)x_2^{0.5} \\ &+r_2(t)x_2^{-1} - a_{21}(t)x_1(t - \delta_2(t))x_2^{-1} + a_{22}(t) \\ &\le r_2^u + a_{22}^u =: K_2. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), integrating both sides of (2.2) from 0 to $\tau_n \wedge T$ and then taking the expectation leads to

$$\begin{aligned} &V(x_1(\tau_n \wedge T), x_2(\tau_n \wedge T)) \\ &\le V(x_1(0), x_2(0)) \\ &+[0.5(K_1 + K_2) + 0.25(\sigma_1^u)^2 + 0.25(\sigma_2^u)^2]T. \end{aligned} \tag{2.5}$$

Set $\Omega_n = \{\tau_n \le T\}$, by (2.1) we have $P(\Omega_n) \ge \varepsilon$. Note that for each $\omega \in \Omega_n$, there is some i such that $x_i(\tau_n, \omega)$ equals n or $\frac{1}{n}$ for $i = 1, 2$. Hence $V(x_1(\tau_n \wedge T), x_2(\tau_n \wedge T))$ is no less than

$$\min\{\sqrt{n} - 1 - 0.5 \ln n, \sqrt{1/n} - 1 - 0.5 \ln 1/n\}.$$

By (2.5) we have

$$\begin{aligned} &V(x_1(0), x_2(0) + [0.5K_1 + 0.5K_2 + 0.25(\sigma_1^u)^2 + 0.25(\sigma_2^u)^2]T \\ &\ge E[1_{\Omega_n}(\omega)V(x_1(\tau_n), x_2(\tau_n))] \\ &\ge \varepsilon \min\{\sqrt{n} - 1 - 0.5 \ln n, \sqrt{1/n} - 1 - 0.5 \ln 1/n\}, \end{aligned}$$

where 1_{Ω_n} is the indicator function of Ω_n. Letting $n \to \infty$, leads to the contradiction

$$\begin{aligned} &\infty > V(x_1(0), x_2(0)) + [0.5K_1 \\ &+0.5K_2 + 0.25(\sigma_1^u)^2 + 0.25(\sigma_2^u)^2]T = \infty. \end{aligned}$$

The proof is completed.

3 Extinction

One of the important issues in the study of population systems is the extinction. In this section we shall investigate the extinction of system (SM). The following theorem gives a sufficient criterion for the population extinct.

Theorem 3.1. *Let conditions $r_1(t) - 0.5\sigma_1^2(t) < 0$ and $a_{21}(t) - a_{11}(t) < 0$ hold. Then the populations x_1 and x_2 with initial value $\{(x_1(\theta), x_2(\theta))^\top : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; (0, +\infty) \times (0, +\infty))$ by (SM) will become extinct exponentially with probability one.*

Proof: Applying Itô formula to system (SM), we have

$$\begin{aligned} d[\ln(x_1(t))] &= (r_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2(t - \delta_1(t)) \\ &-\tfrac{1}{2}\sigma_1^2(t))dt + \sigma_1(t)dB_1(t) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} d[\ln(x_2(t))] &= (-r_2(t) + a_{21}(t)x_1(t - \delta_2(t)) - a_{22}(t)x_2 \\ &-\tfrac{1}{2}\sigma_2^2(t))dt + \sigma_2(t)dB_2(t). \end{aligned} \tag{3.2}$$

Integrating both sides of (3.1) and (3.2) from 0 to t leads to

$$\begin{aligned} &\tfrac{\ln(x_1(t)/x_1(0))}{t} = \tfrac{\int_0^t [r_1(s) - 0.5\sigma_1^2(s)]ds}{t} \\ &-\tfrac{\int_0^t a_{11}(s)x_1(s)ds}{t} - \tfrac{\int_0^t a_{12}(s)x_2(s-\delta_1(s))ds}{t} + \tfrac{\int_0^t \sigma_1(s)dB_1(s)}{t}. \end{aligned} \tag{3.3}$$

In the same way we can show that

$$\begin{aligned} &\tfrac{\ln(x_2(t)/x_2(0))}{t} = \tfrac{\int_0^t [-r_2(s) - 0.5\sigma_2^2(s)]ds}{t} - \tfrac{\int_0^t a_{22}(s)x_2(s)ds}{t} \\ &+\tfrac{\int_0^t a_{21}(s)x_1(s-\delta_2(s))ds}{t} + \tfrac{\int_0^t \sigma_2(s)dB_2(s)}{t}. \end{aligned} \tag{3.4}$$

Set $M_i(t) = \int_\theta^t \sigma_i(t)dB_i(t),\ i = 1, 2$. Then $M_i(t)$ is a local martingale whose quadratic variation is

$$\langle M_i, M_i\rangle_t = \int_0^t \sigma_i^2(s)ds \le (\sigma_i^2)^u t.$$

By the strong law of large numbers for Martingales (see e.g., [15]) we have

$$\lim_{t\to+\infty} \frac{M_i(t)}{t} = 0 \quad a.s.. \tag{3.5}$$

From (3.3), (3.5) and $r_1(t) - 0.5\sigma_1^2(t) < 0$, we have

$$\lim_{t\to+\infty} \sup \frac{\ln(x_1(t))}{t} \le \frac{\int_0^t [r_1(s) - 0.5\sigma_1^2(s)]ds}{t} < 0 \quad a.s..$$

From (3.4), we have

$$\begin{aligned} &\tfrac{\ln(x_2(t))}{t} \le \tfrac{\int_0^t a_{21}(s)x_1(s-\delta_2(s))ds}{t} - \tfrac{\int_0^t a_{11}(s)x_1(s)ds}{t} \\ &+\tfrac{\ln(x_2(0))}{t} + \tfrac{\ln(x_1(0))}{t} + \tfrac{\int_0^t \sigma_1(s)dB_1(s)}{t} + \tfrac{\int_0^t \sigma_2(s)dB_2(s)}{t} \\ &\le \tfrac{a_{21}^u \tau ||\xi||}{t} + \tfrac{\int_0^t (a_{21}(s) - a_{11}(s))x_1(s)ds}{t} \\ &+\tfrac{\ln(x_2(0))}{t} + \tfrac{\ln(x_1(0))}{t} + \tfrac{\int_0^t \sigma_1(s)dB_1(s)}{t} + \tfrac{\int_0^t \sigma_2(s)dB_2(s)}{t}. \end{aligned} \tag{3.6}$$

From (3.5), (3.6) and $a_{21}(t) - a_{11}(t) < 0$, we have

$$\lim_{t\to+\infty} \sup \frac{\ln(x_2(t))}{t} < 0 \quad a.s..$$

Hence the populations x_1 and x_2 become extinct exponentially with probability one.

4 Global asymptotical stability

In this section, we will establish sufficient criteria for the global asymptotical stability of system (SM).

Definition 4.1. *System (SM) is said to be global asymptotical stability if*

$$\lim_{t\to+\infty} |x_1(t) - x_2(t)| = \lim_{t\to+\infty} |y_1(t) - y_2(t)| = 0 \quad a.s.$$

for any two positive solutions $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ of system (SM). To begin with, we first give some lemmas.

Lemma 4.1. *Let $x_1(t), x_2(t)$ be a solution to (SM) with initial value $\{(x_1(\theta), x_2(\theta))^\top : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; (0, +\infty) \times (0, +\infty))$. If $a_{22}^l - a_{21}^u > 0$, then for all $t \ge 0,\ p > 1$, there exist constants $L(p)$ and $G(p)$ such that*

$$E[x_1^p(t)] \le L(p),\ E[x_2^p(t)] \le G(p).$$

Proof: For $t \ge 0$, define $V(u(t)) = u^p(t)$ for all $u \in R_+, p > 1$. By the Itô formula, we have

$$\begin{aligned} dV(x_1) &= px_1^{p-1}dx_1 + 0.5p(p-1)x_1^{p-2}(dx_1)^2 \\ &= px_1^p[r_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2(t - \delta_1(t)) \\ &+0.5(p-1)\sigma_1^2(t)]dt + px_1^p\sigma_1(t)dB_1(t). \end{aligned}$$

Making use of Itô formula again to $e^tV(x_1)$ results in

$$\begin{aligned} d[e^tV(x_1)] &= e^tV(x_1)dt + e^tdV(x_1) \\ &= \{e^tx_1^p + pe^tx_1^p[r_1(t) - a_{11}(t)x_1 \\ &-a_{12}(t)x_2(t - \delta_1(t)) \\ &+0.5(p-1)\sigma_1^2(t)]\}dt + pe^tx^p\sigma_1(t)dB_1(t). \end{aligned}$$

Integrating both sides of the above equality from 0 to t and taking expectations, we have

$$\begin{aligned} &E[e^tx_1^p(t)] \le x_1^p(0) \\ &+pE\int_0^t e^s\{x_1^p(s)[\tfrac{1}{p} + r_1^u + 0.5(p-1)(\sigma_1^2)^u - a_{11}^l x_1]\}ds \\ &\le x_1^p(0) + \int_0^t e^s L_1(p)ds = x_1^p(0) + L_1(p)(e^t - 1), \end{aligned}$$

where $L_1(p) = \frac{[1 + r_1^u p + 0.5p(p-1)(\sigma_1^2)^u]^{p+1}}{(p+1)^{p+1}(a_{11}^l)^p}$. Thus there exists a $T > 0$ such that $E[x_1^p(t)] \le 1.5L_1(p)$ for all $t \ge T$. At the same time, an application of the continuity of $E[x_1^p(t)]$ results in that there exists $\tilde{L}_1(p) > 0$ such that $E[x_1^p(t)] \le \tilde{L}_1(p)$ for all $t \le T$. Let $L(p) = \max\{1.5L_1(p), \tilde{L}_1(p)\}$, then for all $t \ge 0$, we have $E[x^p(t)] \le L(p)$. On the other hand, we can show that

$$\begin{aligned} &d[e^tV(x_2)] = e^tV(x_2)dt + e^tdV(x_2) \\ &= \{e^tx_2^p + pe^tx_2^p[-r_2(t) + a_{21}(t)x_1(t - \delta_2(t)) \\ &-a_{22}(t)x_2 + 0.5(p-1)\sigma_2^2(t)]\}dt + pe^tx_2^p\sigma_2(t)dB_2(t) \\ &\le \{e^tx_2^p + pe^tx_2^p[-r_2(t) + a_{21}^u||\xi|| + a_{21}^u(t)x_1(t) \\ &-a_{22}(t)x_2 + 0.5(p-1)\sigma_2^2(t)]\}dt + pe^tx_2^p\sigma_2(t)dB_2(t). \end{aligned}$$

Integrating both sides of the above inequality from 0 to t and taking expectations, we have

$$\begin{aligned} E[e^t x_2^p(t)] &\le x_2^p(0) + pE\int_0^t e^s\{x_2^p(s)[\tfrac{1}{p} + 0.5(p-1)(\sigma_2^2)^u \\ &+a_{21}^u||\xi|| - a_{22}^l x_2 + a_{21}^u x_1(s)]\}ds \\ &\le x_2^p(0) + p\int_0^t e^s E\big[x_2^p(s)[\tfrac{1}{p} + 0.5(p-1)(\sigma_2^2)^u \\ &+a_{21}^u||\xi|| - a_{22}^l x_2]\big]ds \\ &+pa_{21}^u\int_0^t e^s E[x_2^{p+1}(s)] + \tfrac{pa_{21}^u}{p+1}\int_0^t e^s E[x_1^{p+1}(s)]ds \\ &= x_2^p(0) + pE\int_0^t e^s x_2^p(s)[\tfrac{1}{p} + 0.5(p-1)(\sigma_2^2)^u \\ &+a_{21}^u||\xi|| - (a_{22}^l - a_{21}^u)x_2(s)]ds \\ &+\tfrac{pa_{21}^u}{p+1}\int_0^t e^s E[x_1^{p+1}(s)]ds \\ &\le x_2^p(0) + \int_0^t e^s L_2(p)ds + \tfrac{pa_{21}^u}{p+1}L(p+1)\int_0^t e^s ds \\ &= x_2^p(0) + [L_2(p) + \tfrac{pa_{21}^u}{p+1}L(p+1)](e^t-1), \end{aligned}$$

where $L_2(p) = \frac{[1+0.5p(p-1)(\sigma_2^2)^u + pa_{21}^u||\xi||]^{p+1}}{(p+1)^{p+1}(a_{22}^l - a_{21}^u)^p}$. Thus we get

$$\lim_{t\to+\infty}\sup E[x_2^p(t)] \le L_2(p) + \frac{pa_{21}^u}{p+1}L(p+1) =: L_3(p).$$

Then there exists a $T>0$ such that $E[x_2^p(t)] \le 1.5L_3(p)$ for all $t \ge T$. There also exists $E[x_2^p(t)] \le \tilde{L}_3(p)$ for $t<T$. Let $G(p) = \max\{1.5L_3(p), \tilde{L}_3(p)\}$, then for all $t \ge 0$, $E[x_2^p(t)] \le G(p)$.

Lemma 4.2. *Let $(x_1(t), x_2(t))$ be a solution of (SM) with initial value $\{(x_1(\theta), x_2(\theta))^\top : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau,0];(0,+\infty)\times(0,+\infty))$. If $a_{22}^l - a_{21}^u > 0$, then almost every sample path of $(x_1(t), x_2(t))$ is uniformly continuous on $t \ge 0$.*

Proof: The first equation of system (SM) is equivalent to the following stochastic equation

$$\begin{aligned} x_1(t) &= x_1(0) + \int_0^t x_1(s)(r_1(s) - a_{11}(s)x_1 \\ &-a_{12}(s)x_2(s-\delta_1(s)))ds + \int_0^t \sigma_1(s)x_1(s)dB_1(s). \end{aligned} \tag{4.1}$$

Note that

$$\begin{aligned} &E|x_1[r_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2(t-\delta_1(t))]|^p \\ &= E[|x_1|^p|r_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2(t-\delta_1(t))|^p] \\ &\le 0.5E|x_1|^{2p} + 0.5E|r_1(t) - a_{11}(t)x_1 \\ &-a_{12}(t)x_2(t-\delta_1(t))|^{2p} \\ &\le 0.5\{L(2p) + 3^{2p-1}[(r_1^u)^{2p} \\ &+(a_{11}^u)^{2p}E|x_1^{2p}(t)| + (a_{12}^u + ||\xi||)^{2p}E|x_2^{2p}(t)|]\} \\ &\le 0.5\{L(2p) + 3^{2p-1}[(r_1^u)^{2p} + (a_{11}^u)^{2p}L(2p) \\ &+(a_{12}^u + ||\xi||)^{2p}G(2p)]\} := K_1(p). \end{aligned} \tag{4.2}$$

Moreover, from the moment inequality for stochastic integrals and Lemma 4.1 one can obtain that for $0 \le t_1 \le t_2$ and $p>2$,

$$\begin{aligned} &E|\int_{t_1}^{t_2}\sigma_1(s)x_1(s)dB_1(s)|^p \le [(\sigma_1^2)^u]^p[\tfrac{p(p-1)}{2}]^{p/2} \\ &\times(t_2-t_1)^{(p-2)}\int_{t_1}^{t_2}E|x_1(s)|^p ds \\ &\le [(\sigma_1^2)^u]^p[\tfrac{p(p-1)}{2}]^{p/2}(t_2-t_1)^{p/2}L(p). \end{aligned} \tag{4.3}$$

Then for $0<t_1<t_2<\infty, t_2-t_1 \le 1, 1/p+1/q=1$, from (4.1)-(4.3) we have

$$\begin{aligned} &E(|x_1(t_2) - x_1(t_1)|^p) = E|\int_{t_1}^{t_2}x_1(s)[r_1(s) - a_{11}(s)x_1 \\ &-a_{12}(s)x_2(s-\delta_1(s))]ds \\ &+\int_{t_1}^{t_2}\sigma_1(s)x_1(s)dB_1(s)|^p \\ &\le 2^{p-1}E|\int_{t_1}^{t_2}x_1(s)[r_1(s) - a_{11}(s)x_1 \\ &-a_{12}(s)x_2(s-\delta_1(s))]ds|^p \\ &+2^{p-1}E|\int_{t_1}^{t_2}\sigma_1(s)x_1(s)dB_1(s)|^p \\ &\le 2^{p-1}(t_2-t_1)^{p/q}|\int_{t_1}^{t_2}E|x_1(s)[r_1(s) - a_{11}(s)x_1 \\ &-a_{12}(s)x_2(s-\delta_1(s))]|^p ds \\ &+2^{p-1}[\tfrac{p(p-1)}{2}]^{p/2}(t_2-t_1)^{p/2}[(\sigma_1^2)^u]^p L(p) \\ &\le 2^{p-1}(t_2-t_1)^{p/q+1}K_1(p) \\ &+2^{p-1}[\tfrac{p(p-1)}{2}]^{p/2}(t_2-t_1)^{p/2}[(\sigma_1^2)^u]^p L(p) \\ &\le 2^{p-1}(t_2-t_1)^{p/2}[(t_2-t_1)^{p/2} + (\tfrac{p(p-1)}{2})^{p/2}]K_2(p) \\ &\le 2^{p-1}(t_2-t_1)^{p/2}[1 + (\tfrac{p(p-1)}{2})^{p/2}]K_2(p), \end{aligned}$$

where $K_2(p) = \max\{K_1(p), [(\sigma_1^2)^u]^p L(p)\}$. Then from [16], almost every path of $x_1(t)$ is locally but uniformly Hölder-continuous with exponent ϑ for every $\vartheta \in (0,(p-2)/2p)$ and therefore almost every sample path of $x_1(t)$ is uniformly continuous on $t \ge 0$. In the same way we can verify that almost every path of $x_2(t)$ is uniformly continuous.

Lemma 4.3.(see e.g., [17])*Let f be a non-negative function defined on R_+ such that f is integrable and is uniformly continuous. Then* $\lim_{t\to+\infty} f(t) = 0$.

Theorem 4.1. If $a_{22}^l - a_{21}^u > 0$, $a_{11}^l - a_{21}^u > 0$, $a_{22}^l - a_{12}^u > 0$ and $\int_0^t [a_{12}(s) + a_{21}(s)]ds < +\infty$, then system (SM) is globally asymptotically stable.

Proof: Define

$$V(t) = |\ln x_1(t) - \ln x_2(t)| + |\ln y_1(t) - \ln y_2(t)|,$$

then $V(t)$ continuous and positive function on $t \ge 0$. A direct calculation of the right differential $d^+V(t)$ of $V(t)$, then applying Itös formula yields

$$\begin{aligned} &d^+V(t) = sgn(x_1 - x_2)\{[\tfrac{dx_1}{x_1} - \tfrac{(dx_1)^2}{2x_1^2}] - [\tfrac{dx_2}{x_2} - \tfrac{(dx_2)^2}{2x_2^2}]\} \\ &+sgn(y_1 - y_2)\{[\tfrac{dy_1}{y_1} - \tfrac{(dy_1)^2}{2y_1^2}] - [\tfrac{dy_2}{y_2} - \tfrac{(dy_2)^2}{2y_2^2}]\} \\ &= sgn(x_1 - x_2)\{-a_{11}(t)(x_1 - x_2) - a_{12}(t) \\ &\times[y_1(t-\delta_1(t)) - y_2(t-\delta_1(t))]\}dt \\ &+sgn(y_1 - y_2)\{a_{21}(t)[x_1(t-\delta_2(t)) \\ &-x_2(t-\delta_2(t))] - a_{22}(t)(y_1 - y_2)\}dt \\ &\le \{-a_{11}(t)|x_1 - x_2| + 2||\xi||(a_{12}(t) + a_{21}(t)) \\ &+a_{12}(t)|y_1 - y_2| - a_{22}(t)|y_1 - y_2| + a_{21}(t)|x_1 - x_2|\}dt. \end{aligned}$$

Integrating both sides of the above inequality leads to

$$\begin{aligned} V(t) &\le V(0) + \int_0^t\{[a_{21}(t) - a_{11}(t)]|x_1 - x_2| \\ &+[a_{12}(t) - a_{22}(t)]|y_1 - y_2| + 2||\xi||[a_{12}(t) + a_{21}(t)]\}ds. \end{aligned}$$

Consequently, we have

$$\begin{aligned} &V(t) + \int_0^t\{[a_{11}(t) - a_{21}(t)]|x_1 - x_2| \\ &+[a_{22}(t) - a_{12}(t)]|y_1 - y_2|\}ds \\ &\le \int_0^t 2||\xi||[a_{12}(t) + a_{21}(t)]ds + V(0) < \infty. \end{aligned}$$

From $V(t) \geq 0$, $a_{11}^l - a_{21}^u > 0$, $a_{22}^l - a_{12}^u > 0$ and $\int_0^t [a_{12}(s) + a_{21}(s)]ds < +\infty$, we have

$$|x_1(t) - x_2(t)| \in L^1[0, \infty), \ |y_1(t) - y_2(t)| \in L^1[0, \infty).$$

Then the desired assertion follows from Lemmas 4.2 and 4.3 immediately.

5 Numerical simulations

Now let us use Milstein's numerical method [18] to support our results. In Fig. 1(a), we choose $r_1 = 0.9$, $r_2 = 0.1$, $a_{11} = a_{12} = 0.3$, $a_{21} = a_{22} = 0.2$, $\delta_1(t) = \delta_2(t) = 1$, $\sigma_1(t) = \sigma_2(t) = 0$. Then

$$\Delta = a_{11}a_{22} + a_{12}a_{21} = 0.12, \ \Delta_1 = r_1 a_{22} + r_2 a_{12} = 0.21,$$

$$\Delta_2 = r_1 a_{21} - r_2 a_{11} = 0.15 > 0.$$

Then by the work of Kuang [3], the positive equilibrium

$$x^* = (\frac{\Delta_1}{\Delta}, \frac{\Delta_2}{\Delta}) = (1.75, 1.25)$$

is globally asymptotically stable. Fig. 1(a) confirms these. In Fig. 1(b), we choose $r_1 = 0.8$, $r_2 = 0.1$, $a_{11} = 0.5$, $a_{12} = 0.02 + 0.1 \sin t$, $a_{21} = 0.2 + 0.1 \sin t$, $a_{22} = 0.4$, $\delta_1(t) = \delta_2(t) = 2 - \sin t$, $\sigma_1(t) = \sigma_2(t) = 0.5$. By Theorem 4.1, system (SM) is global asymptotically stable. See Fig. 1(b).

6 Conclusions and future directions

In this paper, a stochastic predator-prey system involving time-varying delays is considered. Some new sufficient conditions for existence, extinction and global asymptotical stability are obtained.

There are still many interesting and challenging questions that need to study. In this paper, we only consider that the white noise affects the growth rate $r_i(t)$, $i = 1, 2$, for other parameters, for example, $a_{ij}(t)$, $i, j = 1, 2$ are affected by the white noise which not be studied. We wish that such questions will be investigated by some authors.

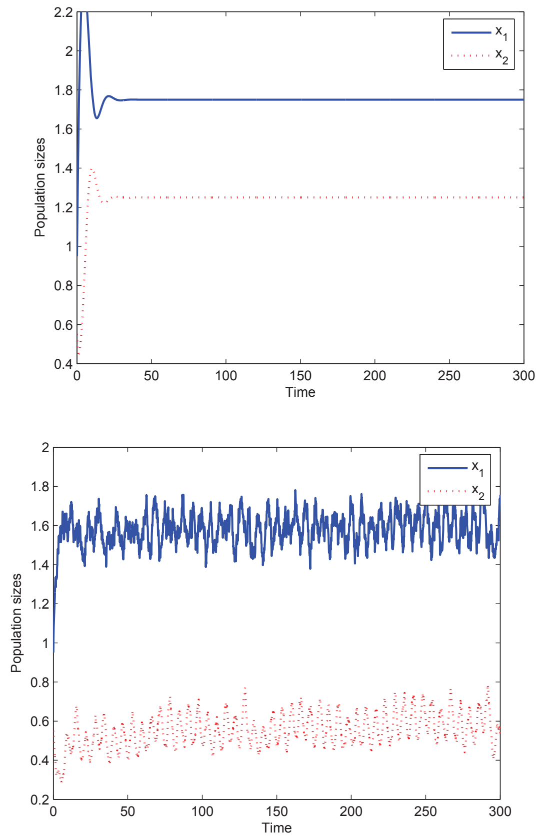

Figure 1: Solutions of system (SM) for $r_1 = 0.9$, $r_2 = 0.1$, $a_{11} = a_{12} = 0.3$, $a_{21} = a_{22} = 0.2$, $\delta_1(t) = \delta_2(t) = 1$, $\sigma_1(t) = \sigma_2(t) = 0$ in Fig. 1(a), and $r_1 = 0.8$, $r_2 = 0.1$, $a_{11} = 0.5$, $a_{12} = 0.02 + 0.1 \sin t$, $a_{21} = 0.2 + 0.1 \sin t$, $a_{22} = 0.4$, $\delta_1(t) = \delta_2(t) = 2 - \sin t$, $\sigma_1(t) = \sigma_2(t) = 0.5$ in Fig. 1(b). The horizontal axis represents the time t, the vertical axis represents the population sizes.

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