New Stability Criteria for a Class of Stochastic Systems with Time Delays

Jian Wang

Abstract—**In this paper, the problem of exponential stability is investigated for a class of grey stochastic time-delay systems. However, to date, few authors have considered the stability analysis problem of grey systems. The main aim of the paper is to fill gaps. By constructing a suitable Lyapunov-Krasovskii functional, and using decomposition technique, novel sufficient stability conditions are obtained, which ensure our considered system in the mean-square exponential stability. In addition, an example is given to show the correctness and effectiveness of the obtained criteria. The results will appear in the near future.**

Index Terms—**Exponential Stability, Time-Delay, Grey Systems, Lyapunov-Krasovskii Functional, Decomposition Technique**

I. INTRODUCTION

It is well known that time delays exist in many practical systems, such as communication, electronics, and chemical systems. Time delays frequently occur in various engineering systems, which lead to oscillation, instability, or performance degradation of the systems [1]. For instance, on account of limited bandwidth, the current networks often cause possible time delays via network communication. Therefore, the study of systems with time delays has become a subject of intensive research activity, and a great number of results have been obtained, see [2-10]. In [4], based on the Lyapunov stability theory and the linear matrix inequality approach, the author has proposed the stability criteria of dynamic systems with mulitple time-varying delays and nonlinear uncertainties. In addition, stochastic modeling also has important effects in many fields of science or industry, for example, economics, ecology, and information systems, etc [11-12]. Thus, the study of stochastic systems has become very important, and much increasing attention has been focused on the stability problem of stochastic systems in recent years. Up to now, many important results have been reported [13-18]. In [15], the authors have provided several sufficient criteria to ensure stochastic systems in the mean-square exponential stability.

Practically, due to various reasons such as the lack of information source or unknown uncertainties, we will not establish an exact mathematical model of an object or process easily, and we can not obtain some parameters of systems accurately. So, we have to evaluate the parameters of systems. It should be pointed that, when the parameters are established by grey numbers, the systems will become grey (uncertain) systems, see [19]. Hence, it is necessary to study the stability problem of grey systems. However, there are few results on this problem [19-22, 24]. In [19], the authors have studied the exponential stability for the grey stochastic systems with distributed delays and interval parameters, and have proposed several stability criteria, which guarantee the grey system in p-moment exponential stability.

Inspired by the above works, we are concerned with the exponential stability problem for a class of grey time-delay stochastic systems in this paper. First, we construct a suitable Lyapunov-Krasovskii functional. Then, using decomposition technique of the continuous matrix-covered sets of grey matrix, we will obtain two novel stability criteria to ensure the grey system in the mean-square exponential stability. In the end of this paper, an example is presented to illustrate the effectiveness of the obtained stability criteria.

Notations: The superscript "T" represents the transpose, R^n and $R^{n\times n}$ denote the n-dimensional Euclidean space and the set of all n ×n real matrices. $\|\cdot\|$ denotes the Euclidean norm for vector or the spectral norm of matrices. The notation $X\geq Y$ (respectively $X>Y$) where X and Y are real symmetric matrices, means that $X-Y$ is positive semi-definite (respectively positive definite). Moreover, let $\left(\Omega, F, \{F_t\}_{t\geq 0}, P\right)$ denote a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions. Let $C([-\tau,0];R^n)$ denote the family of all continuous R^n -valued functions φ on $[-\tau,0]$. Let $L^2_{F_0}([-\tau,0];R^n)$ denote the family of all F_0 -measurable bounded $C([-\tau,0];R^n)$-valued random variables $\xi=\{\xi(\theta):-\tau\leq\theta\leq 0\}$.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider the following grey stochastic time-delay system:

$$\begin{cases} dx(t)=\left[A(\otimes)x(t)+B(\otimes)x(t-\tau)\right]dt \\ \qquad\qquad +\sigma\left(x(t),x(t-\tau),t\right)dw(t), \quad t\geq 0 \\ \\ x_0=\xi, \;\; \xi\in L^2_{F_0}([-\tau,0];R^n), \quad -\tau\leq t\leq 0 \end{cases} \tag{2.1}$$

where $A(\otimes)$ and $B(\otimes)$ are grey (**uncertain**) n×n matrices, $A(\otimes)=(\otimes^a_{ij})$, $B(\otimes)=(\otimes^b_{ij})$, and $\otimes^a_{ij}$, $\otimes^b_{ij}$ are called grey elements of $A(\otimes)$ and $B(\otimes)$.

The research is supported by the National Natural Science Foundation of China (No. 11301009) and the Key Project of Natural Science Foundation of Educational Committee of Henan Province (No. 16B110001).

Jian Wang is with School of Mathematics and Statistics, Anyang Normal University, Anyang 455002, PR China e-mail: ss2wangxin@sina.com

(Advance online publication: 14 May 2016)

$$[L_a, U_a] = \{A(\hat{\otimes}) = (a_{ij}) : \underline{a_{ij}} \le a_{ij} \le \overline{a_{ij}}, i, j = 1,2,..n\}$$

$$[L_b, U_b] = \{B(\hat{\otimes}) = (b_{ij}) : \underline{b_{ij}} \le b_{ij} \le \overline{b_{ij}}, i, j = 1,2,..n\}$$

are the continuous matrix-covered sets of $A(\otimes)$ and $B(\otimes)$, $A(\hat{\otimes})$ and $B(\hat{\otimes})$ are whitened (**deterministic**) matrices of $A(\otimes)$ and $B(\otimes)$, $[\underline{a_{ij}}, \overline{a_{ij}}]$ and $[\underline{b_{ij}}, \overline{b_{ij}}]$ are the number-covered sets of $\otimes_{ij}^a, \otimes_{ij}^b$.

Throughout the paper, we make the following assumptions and definition on system (2.1):

(A1) $H : R^n \times R^n \times R_+ \to R^{n\times n}$, and satisfies the local Lipschitz condition.

(A2) if there exist scalars $\alpha \ge 0$, $\beta \ge 0$, for arbitrary $(x, y, t) \in H : R^n \times R^n \times R_+$, the following inequality holds:

$$Trace[\sigma^T(x, y, t)\sigma(x, y, t) \le \alpha|x|^2 + \beta|y|^2$$

Definition 2.1. [19] System (2.1) is exponentially stable in mean square, if for all $\xi \in L^2_{F_0}([-\tau,0]; R^n)$ and whitened matrices $A(\hat{\otimes}) \in [L_a, U_a]$, $B(\hat{\otimes}) \in [L_b, U_b]$, there exist scalars $r > 0$ and $C > 0$, such that

$$E|x(t;\xi)|^2 \le Ce^{-rt} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2, t \ge 0$$

Now, let us introduce the following lemmas, in particular, lemma 2.1, which will be used in the proof of our main results.

Lemma 2.1. [19] If $A(\otimes) = (\otimes_{ij})_{m\times n}$ is a grey matrix, $[\underline{a_{ij}}, \overline{a_{ij}}]$ is a number-covered sets of grey element $\otimes_{ij}$, then for arbitrary whitened matrix $A(\hat{\otimes}) \in [L_a, U_a]$, we have

(1) $A(\hat{\otimes}) = \dfrac{U_a + L_a}{2} + \Delta A$

(2) $0 \le \Delta A \le \dfrac{U_a - L_a}{2}$

(3) $\left\|A(\hat{\otimes})\right\| \le \left\|\dfrac{U_a + L_a}{2}\right\| + \left\|\dfrac{U_a - L_a}{2}\right\|$

where $L_a = (\underline{a_{ij}})_{m\times n}$, $U_a = (\overline{a_{ij}})_{m\times n}$, $\Delta A = (\dfrac{\overline{a_{ij}} - \underline{a_{ij}}}{2}\ \hat{r}_{ij})_{m\times n}$, $\hat{r}_{ij}$ is a whitened number of γ_{ij}, $\hat{r}_{ij} \in [-1,1]$, $[-1,1]$ is a number-covered sets of γ_{ij}, and γ_{ij} is a unit grey number.

Lemma 2.2. [23] Let N be real matrix of appropriate dimensions, then for any vectors x, y and $\varepsilon > 0$, we have

$$2x^T Ny \le \varepsilon x^T x + \varepsilon^{-1} y^T N^T Ny$$

III. Main Results and Proofs

In the following theorems, two novel stability criteria will be derived, which guarantee system (2.1) in the mean-square exponential stability.

Theorem 3.1. If there exist scalars $\varepsilon_1 > 0, \varepsilon_2 > 0$, such that

$$\lambda_{\max}\left(\frac{U_a + L_a}{2} + \frac{U_a^T + L_a^T}{2}\right) + k_1 + k_2 < 0 \quad (3.1)$$

where

$$k_1 = 1 + 2\left\|\frac{U_a - L_a}{2}\right\| + \varepsilon_1 + \alpha$$

$$k_2 = \varepsilon_1^{-1}(1+\varepsilon_2)\lambda_{\max}\left(\left(\frac{U_b + L_b}{2}\right)^T\left(\frac{U_b + L_b}{2}\right)\right) + \varepsilon_1^{-1}(1+\varepsilon_2^{-1})\left\|\frac{U_b - L_b}{2}\right\|^2 + \beta$$

Then, for all $\xi \in L^2_{F_0}([-\tau,0]; R^n)$, we have

$$E|x(t,\xi)|^2 \le (1 + k_2\tau e^{r\tau})e^{-rt} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2,\ t \ge 0 \quad (3.2)$$

Here, r is the unique positive solution of the following equation

$$r + \lambda_{\max}\left(\frac{U_a + L_a}{2} + \frac{U_a^T + L_a^T}{2}\right) + k_1 + k_2 e^{r\tau} = 0 \quad (3.3)$$

That is, system (2.1) is exponentially stable in mean square.

Proof First, we introduce a Lyapunov-Krasovskii functional:

$$V(x(t),t) = e^{rt}x^T(t)x(t) + \int_{t-\tau}^{t} e^{rs}x^T(s)x(s)ds \quad (3.4)$$

Then, by using the weak infinitesimal operator along the trajectories of system (2.1), we have

$$\begin{aligned} &LV(x(t),t) \\ &= (r+1)e^{rt}x^T(t)x(t) - e^{r(t-\tau)}x^T(t-\tau)x(t-\tau) \\ &+ e^{rt}\{2x^T(t)A(\hat{\otimes})x(t) + 2x^T(t)B(\hat{\otimes})x(t-\tau) \\ &+ Trace[\sigma^T(x(t), x(t-\tau), t)\,\sigma(x(t), x(t-\tau), t)]\} \end{aligned} \quad (3.5)$$

By Lemma 2.1 and Lemma 2.2, it follows that

$$\begin{aligned} &2x^T(t)A(\hat{\otimes})x(t) \\ &\le \lambda_{\max}\left(\frac{U_a + L_a}{2} + \frac{U_a^T + L_a^T}{2}\right)x^T(t)x(t) \\ &+ 2\left\|\frac{U_a - L_a}{2}\right\|x^T(t)x(t) \end{aligned} \quad (3.6)$$

and

$$2x^T(t)B(\hat{\otimes})x(t-\tau)$$
$$\le \varepsilon_1 x^T(t)x(t)$$
$$+\varepsilon_1^{-1}(1+\varepsilon_2)\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))x^T(t-\tau)x(t-\tau)$$
$$+\varepsilon_1^{-1}(1+\varepsilon_2^{-1})\left\|\frac{U_b-L_b}{2}\right\|^2 x^T(t-\tau)x(t-\tau) \quad (3.7)$$

By assumptions (A2), we have

$$Trace[\sigma^T(x(t),x(t-\tau),t)\sigma(x(t),x(t-\tau),t)]$$
$$\le \alpha x^T(t)x(t)+\beta x^T(t-\tau)x(t-\tau) \quad (3.8)$$

Substituting of (3.6) - (3.8) into (3.5), we can get

$$LV(x(t),t)$$
$$\le [1+r+\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2})$$
$$+2\left\|\frac{U_a-L_a}{2}\right\|+\varepsilon_1+\alpha]e^{rt}x^T(t)x(t)$$
$$+[\varepsilon_1^{-1}(1+\varepsilon_2)\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))$$
$$+\varepsilon_1^{-1}(1+\varepsilon_2^{-1})\left\|\frac{U_b-L_b}{2}\right\|^2+\beta]e^{rt}x^T(t-\tau)x(t-\tau) \quad (3.9)$$

Using Itô's differential formula and integrating both sides from 0 to t> 0 and then taking expectation, we have

$$EV(x(t),t)$$
$$=EV(x(0),0)+\int_0^t ELV(x(s),s)ds$$
$$\le \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2+[1+r+\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2})$$
$$+2\left\|\frac{U_a-L_a}{2}\right\|+\varepsilon_1+\alpha]\int_0^t e^{rs}E|x(s)|^2 ds$$
$$+[\varepsilon_1^{-1}(1+\varepsilon_2)\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))$$
$$+\varepsilon_1^{-1}(1+\varepsilon_2^{-1})\left\|\frac{U_b-L_b}{2}\right\|^2+\beta]\int_0^t e^{rs}E|x(s-\tau)|^2 ds \quad (3.10)$$

and

$$\int_0^t e^{rs}E|x(s-\tau)|^2 ds$$
$$\le e^{r\tau}\int_0^t e^{ru}E|x(u)|^2 du+\tau e^{r\tau}\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2 \quad (3.11)$$

From (3.10) and (3.11), noting the definitions of k_1, k_2, we can obtain that

$$EV(x(t),t)$$
$$\le (1+k_2\tau e^{r\tau})\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2$$
$$+[r+\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2})$$
$$+k_1+k_2e^{r\tau}]\int_0^t e^{rs}E|x(s)|^2 ds \quad (3.12)$$

Let

$$f(r)=r+\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2})+k_1+k_2e^{r\tau}.$$

Then, $f'(r)=1+k_2\tau e^{r\tau}$. Since $f'(r)>0$,

$$f(0)=\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2})+k_1+k_2,$$

and $f(+\infty)=+\infty$, when (3.1) holds, (3.3) must have a uniquely positive solution r.

Therefore, we have

$$EV(x(t),t)\le(1+k_2\tau e^{r\tau})\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2$$

That is,

$$E|x(t,\xi)|^2\le(1+k_2\tau e^{r\tau})e^{-rt}\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2,\ t\ge 0$$

which implies that system (2.1) is exponentially stable in mean square. This completes the proof of Theorem 3.1.

Of cause, the stability criterion (3.1) has some shortages. When $\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2})$ is more than or equal to the negative, we will not confirm whether system (2.1) is the mean-square exponential stability. By the similar method, we can solve this problem, see [22].

First, by Lemma2.1, the whitened system of system (2.1) can be written as

$$dx(t)$$
$$=[(A(\hat{\otimes})+L_b)x(t)-L_b(x(t)-x(t-\tau))$$
$$+\Delta Bx(t-\tau)]dt+\sigma(x(t),x(t-\tau),t)dw(t) \quad (3.13)$$

Since

$$x(t)-x(t-\tau)$$
$$=\int_{t+\theta}^t[A(\hat{\otimes})x(s)+B(\hat{\otimes})x(s-\tau)]ds$$

$$+\int_{t+\theta}^{t}\sigma(x(s),x(s-\tau),s)dw(s),\ \ t\ge\tau \tag{3.14}$$

So, we introduce the following functional

$$H(x,t)=\begin{cases}\int_{t+\theta}^{t}[A(\hat{\otimes})x(s)+B(\hat{\otimes})x(s-\tau)]ds\\ +\int_{t+\theta}^{t}\sigma(x(s),x(s-\tau),s)dw(s), \quad t\ge\tau\\ x(t)-x(t+\theta), \qquad 0\le t\le\tau\end{cases} \tag{3.15}$$

From (3.13) - (3.15), the whitened system of system (2.1) can be rewritten as

$$\begin{cases} dx(t)\\ =[(A(\hat{\otimes})+L_b)x(t)-L_bH(x,t)+\Delta Bx(t-\tau)]dt\\ \qquad +\sigma(x(t),x(t-\tau),t)dw(t), \quad t\ge 0\\ x_0=\xi,\ \xi\in L^2_{F_0}([-\tau,0];R^n), \quad -\tau\le t\le 0\end{cases} \tag{3.16}$$

To prove **Theorem 3.2**, the following lemma is necessary.

Lemma 3.1 For all $\xi\in L^2_{F_0}([-\tau,0];R^n)$ and $t>0$, the above functional $H(x,t)$ satisfies

$$\int_0^t e^{rs}E|H(x(s),s)|^2ds$$
$$\le(k_1\tau e^{r\tau}+k_2\tau e^{2r\tau})\int_0^t e^{rs}E|x(s)|^2ds$$
$$+k_2\tau^2e^{2r\tau}\sup_{-\tau\le\theta\le0}E|\xi(\theta)|^2+\sup_{-\tau\le\theta\le0}\psi(\theta) \tag{3.17}$$

where

$$l_1=4\tau\left(\left\|\frac{U_a+L_a}{2}\right\|+\left\|\frac{U_a-L_a}{2}\right\|\right)^2+2\alpha,$$

$$l_2=4\tau\left(\left\|\frac{U_b+L_b}{2}\right\|+\left\|\frac{U_b-L_b}{2}\right\|\right)^2+2\beta,$$

$$\psi(\theta)=\int_0^{\tau}e^{rs}E|x(s)-x(s-\tau)|^2ds$$

Proof First, noting the definition of $H(x,t)$, we have

$$E|H(x(s),s)|^2$$
$$\le 2E\left|\int_{t+\theta}^{t}[A(\hat{\otimes})x(s)+B(\hat{\otimes})x(s-\tau)]ds\right|^2$$
$$+2E\left|\int_{t+\theta}^{t}\sigma(x(s),x(s-\tau),s)dw(s)\right|^2 \tag{3.18}$$

Obviously,

$$E\left|\int_{t+\theta}^{t}[A(\hat{\otimes})x(s)+B(\hat{\otimes})x(s-\tau)]ds\right|^2$$
$$\le\tau\int_{t+\theta}^{t}E[\|A(\hat{\otimes})\|\cdot|x(s)|+\|B(\hat{\otimes})\|\cdot|x(s-\tau)|]^2ds$$
$$\le2\tau\left(\left\|\frac{U_a+L_a}{2}\right\|+\left\|\frac{U_a-L_a}{2}\right\|\right)^2\int_{t+\theta}^{t}E|x(s)|^2ds$$
$$+2\tau\left(\left\|\frac{U_b+L_b}{2}\right\|+\left\|\frac{U_b-L_b}{2}\right\|\right)^2\int_{t+\theta}^{t}E|x(s-\tau)|^2ds \tag{3.19}$$

By assumptions (A2), we see

$$E\left|\int_{t+\theta}^{t}\sigma(x(s),x(s-\tau),s)dw(s)\right|^2$$
$$\le\alpha\int_{t+\theta}^{t}E|x(s)|^2ds+\beta\int_{t+\theta}^{t}E|x(s-\tau)|^2ds \tag{3.20}$$

Combining (3.18) - (3.20) together and noting the definitions of k_1, k_2, then we can obtain

$$E|H(t,x(t))|^2$$
$$\le k_1\int_{t+\theta}^{t}E|x(s)|^2ds+k_2\int_{t+\theta}^{t}E|x(s-\tau)|^2ds,\ t\ge\tau \tag{3.21}$$

Using (3.21) and integrating both sides, we can get

$$\int_0^t e^{rs}E|H(x(s),s)|^2ds$$
$$=\int_0^{\tau}e^{rs}E|x(s)-x(s-\tau)|^2ds+\int_{\tau}^{t}e^{rs}E|H(x(s),s)|^2ds$$
$$\le\sup_{-\tau\le\theta\le0}\psi(\theta)+k_1\int_{\tau}^{t}e^{rs}\int_{s+\theta}^{s}E|x(u)|^2duds$$
$$+k_2\int_{\tau}^{t}e^{rs}\int_{s+\theta}^{s}E|x(u-\tau)|^2duds \tag{3.22}$$

Furthermore,

$$\int_{\tau}^{t}e^{rs}\int_{s+\theta}^{s}E|x(u-\tau)|^2duds$$
$$\le\tau e^{2r\tau}\int_0^t e^{rs}E|x(s)|^2ds+\tau^2e^{2r\tau}\sup_{-\tau\le\theta\le0}E|\xi(\theta)|^2 \tag{3.23}$$

Similarly,

$$\int_{\tau}^{t}e^{rs}\int_{s+\theta}^{s}E|x(u)|^2duds\le\tau e^{r\tau}\int_0^t e^{rs}E|x(s)|^2ds \tag{3.24}$$

Substituting of (3.23) - (3.24) into (3.22), and noting the definitions of l_1, l_2 and $\psi(\theta)$, then (3.17) holds. The proof of Lemma 3.1 is completed.

By (3.16) and Lemma 3.1, we will present another criterion of exponential stability in the mean square for system (2.1).

Theorem 3.2 If there exist scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, such that

$$\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2}+\frac{U_b+L_b}{2}+\frac{U_b^T+L_b^T}{2})$$
$$+m_1+m_2+m_3<0 \tag{3.25}$$

where

$$m_1 = 1+2\left\|\frac{U_a-L_a}{2}\right\|+\varepsilon_1+\varepsilon_2+\alpha$$

$$m_2=\varepsilon_2^{-1}\left\|\frac{U_b-L_b}{2}\right\|^2+\beta$$

$$m_3=\varepsilon_1^{-1}(l_1+l_2)\tau\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))$$

Then, for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$, we have

$$E|x(t,\xi)|^2$$
$$\le[(1+m_2\tau e^{r\tau}+\varepsilon_1^{-1}l_2\tau^2 e^{2r\tau}$$
$$\times\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2$$
$$+\varepsilon_1^{-1}\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))\sup_{-\tau\le\theta\le 0}\psi(\theta)]e^{-rt} \tag{3.26}$$

Here, r is the unique positive solution of the following equation

$$r+\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2}+\frac{U_b+L_b}{2}+\frac{U_b^T+L_b^T}{2})$$
$$+m_1+m_2e^{r\tau}+\varepsilon_1^{-1}(l_1\tau e^{r\tau}+l_2\tau e^{2r\tau})$$
$$\times\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))=0 \tag{3.27}$$

That is, system (2.1) is exponentially stable in mean square.

Proof First, following the similar line of the proof of Theorem 3.1, we have

$$LV(x(t),t)$$
$$\le[r+\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2}+\frac{U_b+L_b}{2}+\frac{U_b^T+L_b^T}{2})$$
$$+m_1]e^{rt}x^T(t)x(t)+m_2e^{rt}x^T(t-\tau)x(t-\tau)$$
$$+\varepsilon_1^{-1}\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))e^{rt}|H(x(t),t)|^2 \tag{3.28}$$

Using Itô's formula and integrating both sides, then taking expectation and noting that definition of $H(x,t)$, we get

$$EV(x(t),t)$$
$$=EV(x(0),0)+\int_0^t ELV(x(s),s)ds$$
$$\le[r+\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2}+\frac{U_b+L_b}{2}+\frac{U_b^T+L_b^T}{2})$$
$$+m_1+m_2e^{r\tau}+\varepsilon_1^{-1}(l_1\tau e^{r\tau}+l_2\tau e^{2r\tau})$$
$$\times\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))]\int_0^t e^{rs}E|x(s)|^2ds$$
$$+[1+m_2\tau e^{r\tau}+\varepsilon_1^{-1}l_2\tau^2e^{2r\tau}$$
$$\times\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))]\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2$$
$$+\varepsilon_1^{-1}\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))\sup_{-\tau\le\theta\le 0}\psi(\theta) \tag{3.29}$$

Let

$$g(r)$$
$$=r+\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2}+\frac{U_b+L_b}{2}+\frac{U_b^T+L_b^T}{2})$$
$$+m_1+m_2e^{r\tau}+\varepsilon_1^{-1}(l_1\tau e^{r\tau}+l_2\tau e^{2r\tau})$$
$$\times\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2})).$$

Then,

$$g^{/}(r)=1+m_2\tau e^{r\tau}+\varepsilon_2^{-1}(l_1\tau^2e^{r\tau}+2l_2\tau^2e^{2r\tau})$$
$$\times\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))$$

Since $g^{/}(r)>0$,

$$g(0)=\lambda_{\max}(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2}+\frac{U_b+L_b}{2}+\frac{U_b^T+L_b^T}{2})$$
$$+m_1+m_2+m_3$$

and $g(+\infty)=+\infty$, If (3.25) holds, (3.27) must have a uniquely positive solution r.

Therefore, we can get

$$EV(x(t),t)$$
$$\le[1+m_2\tau e^{r\tau}+\varepsilon_1^{-1}l_2\tau^2e^{2r\tau}$$
$$\times\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))]\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2$$
$$+\varepsilon_1^{-1}\lambda_{\max}((\frac{U_b+L_b}{2})^T(\frac{U_b+L_b}{2}))\sup_{-\tau\le\theta\le 0}\psi(\theta)$$

That is, (3.26) holds, the proof of Lemma 3.2 is completed.

Remark 3.1. If $A(\otimes) \equiv A$, $B(\otimes) \equiv B$, system (2.1) will become the following **deterministic** stochastic system with time delays:

$$\begin{cases} dx(t) = [Ax(t) + Bx(t-\tau)]dt + \sigma(x(t), x(t-\tau), t)dw(t) \\ x_0 = \xi,\ \xi \in L^2_{F_0}([-\tau, 0]; R^n),\ -\tau \le t \le 0 \end{cases} \tag{3.30}$$

Let $L_a = U_a = A$, $L_b = U_b = B$, and by using the similar methods of Theorem 3.1 and Theorem 3.2, we obtain the following stability criteria for system (3.30).

Corollary 3.1. If there exist scalars $\varepsilon_1 > 0, \varepsilon_2 > 0$, such that $\lambda_{\max}(A + A^T) + p_1 + p_2 < 0$ (3.31)
where

$$p_1 = 1 + \varepsilon_1 + \alpha$$

$$p_2 = \varepsilon_1^{-1}(1 + \varepsilon_2)\lambda_{\max}(B^T B) + \beta$$

Then, for all $\xi \in L^2_{F_0}([-\tau, 0]; R^n)$, we have

$$E|x(t,\xi)|^2 \le (1 + k_2 \tau e^{r\tau})e^{-rt} \sup_{-\tau \le \theta \le 0} E|\xi(\theta)|^2,\ t \ge 0 \tag{3.32}$$

Here, r is the unique positive solution of the following equation

$$r + \lambda_{\max}(A + A^T) + p_1 + p_2 e^{r\tau} = 0 \tag{3.33}$$

That is, system (3.30) is exponentially stable in mean square.

Corollary 3.2. If there exist scalars $\varepsilon_1 > 0, \varepsilon_2 > 0$, such that

$$\lambda_{\max}(A + A^T + B + B^T) + q_1 + q_2 < 0 \tag{3.34}$$

where

$$q_1 = 1 + \varepsilon_1 + \varepsilon_2 + \alpha + \beta$$

$$q_2 = \varepsilon_1^{-1}(l_1 + l_2)\tau \lambda_{\max}(B^T B)$$

Then, for all $\xi \in L^2_{F_0}([-\tau, 0]; R^n)$, we have

$$E|x(t,\xi)|^2 \le [(1 + \varepsilon_1^{-1} l_2 \tau^2 e^{2r\tau} \times \lambda_{\max}(B^T B) \sup_{-\tau \le \theta \le 0} E|\xi(\theta)|^2 + \varepsilon_1^{-1}\lambda_{\max}(B^T B) \sup_{-\tau \le \theta \le 0} \psi(\theta)]e^{-rt} \tag{3.35}$$

Here. r is the unique positive solution of the following equation

$$r + \lambda_{\max}(A + A^T + B + B^T) + q_1 + \varepsilon_1^{-1}(l_1 \tau e^{r\tau} + l_2 \tau e^{2r\tau})\ \lambda_{\max}(B^T B) = 0 \tag{3.36}$$

That is, system (3.30) is exponentially stable in mean square.

IV. Examples

Consider a grey stochastic time-delay system:

$$\begin{cases} dx(t) = [A(\otimes)x(t) + B(\otimes)x(t-0.5)]dt \\ \qquad\qquad + \sigma(x(t), x(t-0.5), t)\, dw(t) \\ x_0 = \xi,\ \xi \in L^2_{F_0}([-0.5, 0]; R^2),\quad -0.5 \le t \le 0 \end{cases} \tag{4.1}$$

where

$$L_a = \begin{bmatrix} -3.32 & 0.20 \\ 0.22 & -3.31 \end{bmatrix};\ U_a = \begin{bmatrix} -3.18 & 0.31 \\ 0.30 & -3.42 \end{bmatrix}$$

$$L_b = \begin{bmatrix} -1.13 & 0.22 \\ 0.23 & -1.15 \end{bmatrix};\ U_b = \begin{bmatrix} -1.10 & 0.23 \\ 0.32 & -1.08 \end{bmatrix}$$

L_a, U_a and L_b, U_b are the lower bound and upper bound matrices of $A(\otimes)$ and $B(\otimes)$.

$$\sigma(x(t), x(t-0.5), t) = \begin{bmatrix} \frac{1}{2}x_1(t)\sin(x_2(t-0.5)) \\ \frac{1}{2}x_2(t)\sin(x_1(t-0.5)) \end{bmatrix}$$

and

$$Trace[\sigma^T(x(t), x(t-0.5), t)\sigma(x(t), x(t-0.5), t)] \le 0.25x^2(t)$$

By using the program in [19, 22], we can calculate $\varepsilon_1, \varepsilon_2$ and get $r = 1.5587$ or $r = 1.6328$. Applying Theorem 3.1 or Theorem3.2, it is found that system (4.1) is exponentially stable in mean square.

V. Conclusion

In this paper, the exponential stability problem has been investigated for a class of grey stochastic systems with time delays. By using the Lyapunov-Krasovskii method, Itô's differential formula, and decomposition technique, two stability criteria have been obtained. The obtained criteria will guarantee the grey system in mean-square exponential stability. In addition, an illustrate example has been presented to show the correctness and effectiveness of the main results. The corresponding results will appear in the near future.

Acknowledgment

The author thanks the anonymous reviewers for those helpful comments and valuable suggestions on improving this paper.

REFERENCES

[1] M. Malek-Zavarei and M. Jamshidi, *Time-Delay Systems: Analysis, Optimation and Application,* Amsterdam, Netherlands: North-Holland, 1987.

[2] E. K. Boukas, and Z. K. Liu, *Deterministic and Stochastic Time-Delay Systems*. Boston, MA: Birkhauser, 2002.

[3] S. Arik, "Global robust stability analysis of neural networks with discrete time delays", *Chaos, Solitons and Fractals*, vol. 26, no. 5, pp. 1407-1414, 2005.

[4] J. H. Park, "Robust stabilization for dynamic systems with mulitple time-varying delays and nonlinear uncertainties," *Journal of Optimization Theory and Applications*, vol. 108, no. 1, pp. 155 -174, 2001.

[5] W. J. Xiong, and J. L. Liang, "Novel stability criteria for neutral systems with multiple time delays," *Chaos, Solitons & Fractals*, vol. 32, no. 5, pp. 1735-1741, 2007.

[6] R. Rakkiyappan, and P. Balasubramaniam, "New global exponential stability results for neutral type neural networks with distributed time delays," *Neurocomputing*, vol. 171, no. 4, pp. 1039-1045, 2008.

[7] P. Park, "A delay-dependent stability criterion for systems with uncertain time-invariant delays". *IEEE Transactions on Automatic Control*, vol. 44, no. 4, pp. 876-877, 1999.

[8] Y. Glizer, V. Turetsky, and J. Shinar, "Terminal Cost Distribution in Discrete-Time Controlled System with Disturbance and Noise-Corrupted State Information," *IAENG International Journal of Applied Mathematics*, vol. 42, no. 1, pp. 52-59, 2012.

[9] L. Lin, "Stabilization of LTI Switched Systems with Input Time Delay," *Engineering Letters*, vol. 14, no. 2, pp. 117-123, 2007.

[10] T. C. Kuo, and Y. J. Huang, "Global Stabilization of Robot Control with Neural Network and Sliding Mode," *Engineering Letters*, vol. 16, no. 1, pp. 56-60, 2008.

[11] H. Kushner, *Stochastic Stability and Control*, Academic Press, New York, 1967.

[12] X. Mao, *Stochastic Differential Equations and Applications, Chichester*, UK: Horwood Publishing, 1997.

[13] H. Yan, Q. H. Meng, X. Huang, and H. Zhang, "Robust exponential stability of stochastic time-delay systems with uncertainties and nonlinear perturbations," *International Journal of Information Acquisition*, vol. 6, no. 1, pp. 61 - 71, 2009.

[14] X. Mao. "Robustness of exponential stability of stochastic differential delay equation," *IEEE Trans.Autom. Control*, vol. 41, no. 3, pp. 442 -447, 1996.

[15] X. Liao, and X. Mao, "Exponential stability of stochastic delay interval systems," *Systems Control Lett*, vol. 40, pp. 171–181, 2000.

[16] H. J. Chu, and L. X. Gao, "Robust exponential stability and $H\infty$ control for jumping stochastic Cohen-Grossberg neural networks with mixed delays", *Journal of Computational Information Systems*, vol. 7, no. 3, pp. 794 -806, 2011.

[17] W. Chen, and X. Lu, "Mean square exponential stability of uncertain stochastic delayed neural networks," *Physics Letter A*, vol. 372, no. 7, pp. 1061 - 1069, 2008.

[18] S. Y. Xu, J. Lam, and X. Mao, Y. Zou, "A new LMI condition for delay-dependent robust stability of stochastic time-delay systems," *Asian Journal of Control*, vol. 7, no. 4, pp. 419 – 423, 2005.

[19] C. H. Su, and S. F. Liu, "The p-moment exponential robust stability for stochastic systems with distributed delays and interval parameters," *Applied Mathematics and Mechanics*, vol. 30, no. 7, pp. 915-924, 2009.

[20] C. H. Su, and S. F. Liu, "Exponential Robust Stability of Grey Neutral Stochastic Systems with Distributed Delays," *Chinese Journal of Engineering Mathematics*, vol. 27, no. 3, pp. 403-414, 2010.

[21] C. H. Su, and S. F. Liu, "Robust Stability of Grey Stochastic Nonlinear Systems with Distributed-Delays," *Mathematics in Practice and Theory*, vol. 38, no. 22, pp. 218-223, 2008.

[22] J. J. Li, and C. H. Su, "Mean-square Exponential Robust Stability for a Class of Grey Stochastic Systems with Distributed Delays", *Chin. Quart. J. of Math.*, vol. 25, no. 3, pp. 451-458, 2010.

[23] A. Friedman, *Stochastic differential equations and their applications*, New York, Academic Press, 1976.

[24] C. H. Su, "Robust Stability of Grey Stochastic Delay Systems with impulsive effect", *Sys. Sci. & Math. Scis*, vol.32, no.5, pp.537-548, 2012.