The New Estimates of Diagonally Dominant Degree for the Perron Complement of Three Known Subclasses of H-matrices

Jingjing Cui, Guohua Peng, Quan Lu and Zhong Xu

Abstract—**The theory of Perron complement is very important in many fields such as control theory and computational mathematics. In this paper, some new estimates of diagonally dominant degree for the Perron complement of matrices are derived by using the entries and spectral radius of the original matrix, which are better than the corresponding ones obtained by Wang and Liu(J. Inequal. Appl. 2015:9, 2015). Finally, a numerical example is provided to confirm the theoretical results obtained in this paper.**

Index Terms—**Perron complement, diagonally dominant degree, H-matrix, nonnegative irreducible matrix, spectral radius**

I. INTRODUCTION

FOR a positive integer n, N denotes the set $\{1, 2, \cdots, n\}$, and $\mathbb{R}^{n\times n}(\mathbb{C}^{n\times n})$ denotes the set of all real (complex) matrices throughout. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}(n \geq 2)$. Denote $|A| = (|a_{ij}|)$ and

$$R_i(A) = \sum_{j\neq i} |a_{ij}|, \quad S_i(A) = \sum_{j\neq i} |a_{ji}|, \quad i \in N,$$
$$N_r(A) = \{i : |a_{ii}| > R_i(A), i \in N\},$$
$$N_c(A) = \{i : |a_{ii}| > S_i(A), i \in N\}.$$

The comparison matrix of A, denoted by $\mu(A) = (\mu_{ij})_{n\times n}$, is defined to be

$$\mu_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j. \end{cases}$$

A matrix A is called a nonsingular M-matrix if there exist a nonnegative matrix B and a real number $s > \rho(B)$ such that $A = sI - B$, where $\rho(B)$ is the spectral radius of B. It is well known that A is a nonsingular H-matrix if and only if $\mu(A)$ is a nonsingular M-matrix. We denote by $\mathbb{H}_n$ and $\mathbb{M}_n$ the sets of all $n \times n$ H-matrix and M-matrix, respectively.

Recall that A is a (row) diagonally dominant matrix (D_n) if for all $i = 1, 2, \cdots, n$,

$$|a_{ii}| \geq R_i(A). \tag{1}$$

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A is a doubly diagonally dominant matrix (DD_n) if for all $i, j \in N, i \neq j$,

$$|a_{ii}||a_{jj}| \geq R_i(A)R_j(A). \tag{2}$$

A is a γ-diagonally dominant matrix (D_n^γ) if there exists $\gamma \in [0, 1]$ such that

$$|a_{ii}| \geq \gamma R_i(A) + (1 - \gamma)C_i(A), \quad \forall i \in N. \tag{3}$$

A is a product γ-diagonally dominant matrix (PD_n^γ) if there exists $\gamma \in [0, 1]$ such that

$$|a_{ii}| \geq [R_i(A)]^\gamma [C_i(A)]^{1-\gamma}, \quad \forall i \in N. \tag{4}$$

If all inequalities in (1)-(4) hold, then A is said to be a strictly (row) diagonally dominant matrix (SD_n), a strictly doubly diagonally dominant matrix (SDD_n), a strictly γ-diagonally dominant matrix (SD_n^γ) and a strictly product γ-diagonally dominant matrix (SPD_n^γ), respectively. As in [1] and [2], for $1 \leq i \leq n$ and $\gamma \in [0, 1]$, we call $|a_{ii}| - R_i(A)$, $|a_{ii}| - \gamma R_i(A) - (1 - \gamma)C_i(A)$ and $|a_{ii}| - [R_i(A)]^\gamma [C_i(A)]^{1-\gamma}$ the i-th (row) dominant degree, γ-dominant degree and product γ-dominant degree of A, respectively.

For $A \in \mathbb{C}^{n\times n}$, nonempty index sets $\alpha, \beta \subseteq N$, we denote by $|\alpha|$ the cardinality of α and $\alpha' = N - \alpha$ the complement α of in N . Let $A(\alpha, \beta)$ denote the sub-matrix of A lying in the rows indexed by α and the columns indexed by β. $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. If $A(\alpha)$ is nonsingular, then the Schur complement of $A(\alpha)$ in A is given by

$$A/\alpha = A/A(\alpha) = A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha').$$

Meyer [3] introduced, for an $n \times n$ non-negative and irreducible matrix A, the notion of the Perron complement. Again, let $\emptyset \neq \alpha$, and $\alpha' = N - \alpha$, then the Perron complement of A with respect to $A(\alpha)$, which is denoted by $P(A/A(\alpha))$ or simply $P(A/\alpha)$, is defined as

$$A(\alpha') + A(\alpha', \alpha)[\rho(A)I - A(\alpha)]^{-1}A(\alpha, \alpha'),$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix. Recall that as A is irreducible, $\rho(A) > \rho(A(\alpha))$, so the expression of the above definition is well defined.

The Perron complement has been proved to be a useful tool in many fields such as control theory, statistics and computational mathematics, such as solving the constrained optimization problems [4], realized range-based threshold estimation for Jump-diffusion models [5] and so forth. Meyer [3], [6] has derived several interesting and useful properties of $P(A/A(\alpha))$, obtained that the Perron complement of a nonnegative irreducible matrix is nonnegative irreducible, and first used the closure property of a nonnegative irreducible matrix to construct a divided and conquer algorithm to compute the Perron vector for a Markov chain. Moreover, many works have been done on it (see [7], [8], [9]). So far as we know, if a given matrix has a sharper diagonally dominant degree, then the designed iterative algorithms have faster convergent rate than the ordinary ones [10]. At the same time, if a given matrix has a sharper diagonally dominant degree, then we may discuss more properties about generalized nonlinear diagonal dominance in [11]. Motivated by the useful applications, we will study the diagonal dominant degree for the perron complement of several cases based on the nonnegative and irreducible nature. In this paper, we exhibit some new estimates of diagonally dominant, γ-diagonally and product γ-diagonally dominant degree for the Perron complement of matrices. These bounds improve the related results.

The remainder of this paper is organized as follows. In Section II, we present several new estimates for the diagonally dominant degree of the Perron complement of matrices. In Section III, we propose some estimates for the γ-diagonally and product γ-diagonally degree for the Perron complement of matrices. In Section IV is devoted to a numerical experiment to show the advantage of our derived results. Finally, we demonstrate our conclusions in Section V.

II. THE DIAGONALLY DOMINANT DEGREE FOR PERRON COMPLEMENT

In this section, we start with some lemmas which are utilized in the next proofs. Based on these lemmas and using the entries and spectral radius of the original matrix, the estimations of the diagonally dominant degree for the Perron complement are presented.

Lemma 2.1: [12] If A is a nonsingular H-matrix, then $[\mu(A)]^{-1} \geq |A^{-1}|$.

Lemma 2.2: [12] If A is a SD_n or A is a SDD_n. Then $\mu(A)$ is a nonsingular M-matrix, i.e., A is a nonsingular H-matrix.

Lemma 2.3: [13] Let $A \in \mathbb{R}^{n\times n}$. If $A \in \mathbb{M}_n$, then $\det A > 0$.

Lemma 2.4: [14] Let $A = (a_{ij})_{n\times n}, \emptyset \neq \alpha \in N$ and assume that $A(\alpha)$ is nonsingular. Then

$$\det A = \det A(\alpha) \det A/\alpha.$$

Lemma 2.5: Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be nonnegative irreducible with spectral radius $\rho(A)$, $\alpha = \{i_1, i_2, \cdots, i_k\} \subseteq N_r(A) \neq \emptyset$, $\alpha' = \{j_1, j_2, \cdots, j_l\}$, $|\alpha| < n$, and denote

$$B_{j_t} = \begin{pmatrix} x & -|a_{j_t i_1}| & \cdots & -|a_{j_t i_k}| \\ -\sum\limits_{u=1}^{l} |a_{i_1 j_u}| & & & \\ \vdots & & \mu[\rho(A)I - A(\alpha)] & \\ -\sum\limits_{u=1}^{l} |a_{i_k j_u}| & & & \end{pmatrix},$$

then for $\rho(A) \geq 2|a_{ii}|(i \in \alpha)$ and any $j_t \in \alpha'$, B_{j_t} is an M-matrix of $k+1$ and $\det B_{j_t} > 0$, if

$$x > h \sum_{w=1}^{k} |a_{j_t i_w}| \frac{P_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} \tag{5}$$

where

$$r = \max_{1\leq w\leq k} \frac{\sum\limits_{u=1}^{l} |a_{i_w j_u}|}{\rho(A) - |a_{i_w i_w}| - \sum\limits_{u=1,u\neq w}^{k} |a_{i_w i_u}|}. \tag{6}$$

$$P_{i_w}(A) = r \sum_{u=1,u\neq w}^{k} |a_{i_w i_u}| + \sum_{u=1}^{l} |a_{i_w j_u}| \tag{7}$$

$$h = \max_{1\leq w\leq k} \frac{\sum\limits_{u=1}^{l} |a_{i_w j_u}|}{P_{i_w}(A) - \sum\limits_{u=1,u\neq w}^{k} |a_{i_w i_u}| \frac{P_{i_u}(A)}{\rho(A) - |a_{i_u i_u}|}} \tag{8}$$

Proof Denote $B_{j_t} \equiv B = (b_{pq})$. Inasmuch as $\alpha \subseteq N_r(A)$ and $\rho(A) \geq 2|a_{ii}|(i \in \alpha)$, we deduce that

$$\rho(A) - |a_{i_w i_w}| - \sum_{u=1,u\neq w}^{k} |a_{i_w i_u}| > 0 (1 \leq w \leq k),$$

for all $1 \leq w \leq k$, it follows from Equation (6) that

$$r \geq \frac{\sum\limits_{u=1}^{l} |a_{i_w j_u}|}{\rho(A) - |a_{i_w i_w}| - \sum\limits_{u=1,u\neq w}^{k} |a_{i_w i_u}|},$$

that is,

$$r(\rho(A) - |a_{i_w i_w}|) \geq \sum_{u=1}^{l} |a_{i_w j_u}| + r \sum_{u=1,u\neq w}^{k} |a_{i_w i_u}| = P_{i_w}(A).$$

From the above inequality, for all $1 \leq w \leq k$,

$$0 \leq \frac{P_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} \leq r < 1.$$

Together with Equation (7), for all $1 \leq w \leq k$, we have

$$\frac{\sum\limits_{u=1}^{l} |a_{i_w j_u}|}{P_{i_w}(A) - \sum\limits_{u=1,u\neq w}^{k} |a_{i_w i_u}| \frac{P_{i_u}(A)}{\rho(A) - |a_{i_u i_u}|}} = \frac{P_{i_w}(A) - r\sum\limits_{u=1,u\neq w}^{k} |a_{i_w i_u}|}{P_{i_w}(A) - \sum\limits_{u=1,u\neq w}^{k} |a_{i_w i_u}| \frac{P_{i_u}(A)}{\rho(A) - |a_{i_u i_u}|}} \leq 1.$$

Combining Equation (8) and the above inequality results in

$$0 \leq h \leq 1.$$

Otherwise, for $1 \leq w \leq k$,

$$h \geq \frac{\sum\limits_{u=1}^{l} |a_{i_w j_u}|}{P_{i_w}(A) - \sum\limits_{u=1,u\neq w}^{k} |a_{i_w i_u}| \frac{P_{i_u}(A)}{\rho(A) - |a_{i_u i_u}|}},$$

we deduce that

$$hP_{i_w}(A) \geq \sum_{u=1}^{l} |a_{i_w j_u}| + h \sum_{u=1,u\neq w}^{k} |a_{i_w i_u}| \frac{P_{i_u}(A)}{\rho(A) - |a_{i_u i_u}|}. \tag{9}$$

Inequality (5) means that there exists $\varepsilon > 0$ such that

$$x > \sum_{w=1}^{k} |a_{j_t i_w}|(h\frac{P_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} + \varepsilon). \quad (10)$$

Choose a positive matrix $D = diag(d_1, d_2, \cdots, d_{k+1})$ and $C_t = B_{j_t} D = (c_{sv})$, where

$$d_v = \begin{cases} 1, & v = 1; \\ h\frac{P_{i_{v-1}}(A)}{\rho(A) - |a_{i_{v-1} i_{v-1}}|} + \varepsilon, & 2 \le v \le k+1. \end{cases}$$

If $s = 1$, it follows from (10) that

$$\begin{aligned} & |c_{ss}| - \sum_{w=1, w\neq s}^{k+1} |c_{sw}| \\ = & \ |c_{11}| - \sum_{w=2}^{k+1} |c_{1w}| \\ = & \ x - \sum_{w=1}^{k} |a_{j_t i_w}|[h\frac{P_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} + \varepsilon] > 0; \end{aligned}$$

If $s = 2, 3, \cdots, k+1$, it follows from (9) that

$$\begin{aligned} & |c_{ss}| - \sum_{w=1, w\neq s}^{k+1} |c_{sw}| \\ = & |b_{ss}|d_s - |b_{s1}| - \sum_{w=2, w\neq s}^{k+1} |b_{sw}|d_w \\ = & \left(\rho(A) - |a_{i_{s-1} i_{s-1}}|\right) \left(h\frac{P_{i_{s-1}}(A)}{\rho(A) - |a_{i_{s-1} i_{s-1}}|} + \varepsilon\right) \\ & -\sum_{u=1}^{l} |a_{i_{s-1} j_u}| - \sum_{w=1, w\neq s-1}^{k} |a_{i_{s-1} i_w}| \left(h\frac{P_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} + \varepsilon\right) \\ = & hP_{i_{s-1}}(A) + \left(\rho(A) - |a_{i_{s-1} i_{s-1}}| - \sum_{w=1, w\neq s-1}^{k} |a_{i_{s-1} i_w}|\right) \varepsilon \\ & -\sum_{u=1}^{l} |a_{i_{s-1} j_u}| - \sum_{w=1, w\neq s-1}^{k} |a_{i_{s-1} i_w}| \frac{hP_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} \\ \ge & \sum_{u=1}^{l} |a_{i_{s-1} j_u}| + h \sum_{u=1, u\neq s-1}^{k} |a_{i_{s-1} i_u}| \frac{P_{i_u}(A)}{\rho(A) - |a_{i_u i_u}|} \\ & + \left(\rho(A) - |a_{i_{s-1} i_{s-1}}| - \sum_{w=1, w\neq s-1}^{k} |a_{i_{s-1} i_w}|\right) \varepsilon \\ & -\sum_{u=1}^{l} |a_{i_{s-1} j_u}| - \sum_{w=1, w\neq s-1}^{k} |a_{i_{s-1} i_w}| \frac{hP_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} \\ = & \left(\rho(A) - |a_{i_{s-1} i_{s-1}}| - \sum_{w=1, w\neq s-1}^{k} |a_{i_{s-1} i_w}|\right) \varepsilon > 0. \end{aligned}$$

It follows that C_t is a SD_{k+1}. Further, $B_{j_t} \in \mathbb{H}_{k+1}$. Besides, $\mu(B_{j_t}) = B_{j_t}$ is an M-matrix of $k+1$. Therefore, from Lemma 2.3, we obtain $\det B_{j_t} > 0$.

Lemma 2.6: Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be nonnegative irreducible with spectral radius $\rho(A)$, $\alpha = \{i_1, i_2, \cdots, i_k\} \subseteq N_r(A) \neq \emptyset$, $\alpha' = \{j_1, j_2, \cdots, j_l\}$, $|\alpha| < n$, $P(A/\alpha) = (a'_{ts})$, and $\rho(A) \ge 2|a_{ii}|(i \in \alpha)$,

(i) If $\alpha \subseteq N_r(A)$, then for all $1 \le t \le l$,

$$\begin{aligned} & \left| (a_{j_t i_1}, \cdots, a_{j_t i_k})[\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\ & + \sum_{s=1, s\neq t}^{l} \left| a_{j_t j_s} + (a_{j_t i_1}, \cdots, a_{j_t i_k}) \times [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\ \le & R_{j_t}(A) - w'_{j_t}. \end{aligned} \quad (11)$$

(ii) If $\alpha \subseteq N_c(A)$, then for all $1 \le t \le l$,

$$\begin{aligned} & \left| (a_{j_t i_1}, \cdots, a_{j_t i_k})[\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\ & + \sum_{s=1, s\neq t}^{l} \left| a_{j_s j_t} + (a_{j_s i_1}, \cdots, a_{j_s i_k}) \times [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\ \le & S_{j_t}(A) - w''_{j_t}. \end{aligned} \quad (12)$$

Here we denote

$$w'_{j_t} = \sum_{w=1}^{k} |a_{j_t i_w}| \frac{\rho(A) - |a_{i_w i_w}| - hP_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|},$$

$$w''_{j_t} = \sum_{w=1}^{k} |a_{i_w j_t}| \frac{\rho(A) - |a_{i_w i_w}| - \widetilde{h}Q_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|},$$

$$r = \max_{1\le w\le k} \frac{\sum_{u=1}^{l} |a_{i_w j_u}|}{\rho(A) - |a_{i_w i_w}| - \sum_{u=1, u\neq w}^{k} |a_{i_w i_u}|},$$

$$\widetilde{r} = \max_{1\le w\le k} \frac{\sum_{u=1}^{l} |a_{j_u i_w}|}{\rho(A) - |a_{i_w i_w}| - \sum_{u=1, u\neq w}^{k} |a_{i_u i_w}|},$$

$$P_{i_w}(A) = r \sum_{u=1, u\neq w}^{k} |a_{i_w i_u}| + \sum_{u=1}^{l} |a_{i_w j_u}|,$$

$$Q_{i_w}(A) = \widetilde{r} \sum_{u=1, u\neq w}^{k} |a_{i_u i_w}| + \sum_{u=1}^{l} |a_{j_u i_w}|,$$

$$h = \max_{1\le w\le k} \frac{\sum_{u=1}^{l} |a_{i_w j_u}|}{P_{i_w}(A) - \sum_{u=1, u\neq w}^{k} |a_{i_w i_u}| \frac{P_{i_u}(A)}{\rho(A) - |a_{i_u i_u}|}},$$

$$\widetilde{h} = \max_{1\le w\le k} \frac{\sum_{u=1}^{l} |a_{j_u i_w}|}{Q_{i_w}(A) - \sum_{u=1, u\neq w}^{k} |a_{i_u i_w}| \frac{Q_{i_u}(A)}{\rho(A) - |a_{i_u i_u}|}}.$$

Proof Inasmuch as $\alpha \subseteq N_r(A) \neq \emptyset$, we have $A(\alpha) \in SD_k$. Further, $\rho(A) \ge 2a_{ii}(i \in \alpha)$ yields $\rho(A)I - A(\alpha) \in SD_k$.

By Lemma 2.1 and Lemma 2.2, it follows that

$$\{\mu[\rho(A)I-A(\alpha)]\}^{-1}\geq|[\rho(A)I-A(\alpha)]^{-1}|.$$

Together with Lemma 2.4, $\forall\varepsilon>0$, we deduce that

$$\begin{aligned}
a &\triangleq \left|(a_{j_ti_1},\cdots,a_{j_ti_k})[\rho(A)I-A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right| \\
&\quad+\sum_{s=1,s\neq t}^{l}\left|a_{j_tj_s}+(a_{j_ti_1},\cdots,a_{j_ti_k})\right. \\
&\qquad\left.\times[\rho(A)I-A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_s}\\ \vdots\\ a_{i_kj_s}\end{pmatrix}\right| \\
&\leq\sum_{s=1,s\neq t}^{l}|a_{j_tj_s}|+\sum_{s=1}^{l}\left|(a_{j_ti_1},\cdots,a_{j_ti_k})\right. \\
&\qquad\left.\times[\rho(A)I-A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_s}\\ \vdots\\ a_{i_kj_s}\end{pmatrix}\right| \\
&\leq\sum_{s=1,s\neq t}^{l}|a_{j_tj_s}|+\sum_{s=1}^{l}(|a_{j_ti_1}|,\cdots,|a_{j_ti_k}|) \\
&\qquad\times\{\mu[\rho(A)I-A(\alpha)]\}^{-1}\begin{pmatrix}|a_{i_1j_s}|\\ \vdots\\ |a_{i_kj_s}|\end{pmatrix} \\
&=R_{j_t}-\sum_{u=1}^{k}|a_{j_ti_u}|+(|a_{j_ti_1}|,\cdots,|a_{j_ti_k}|) \\
&\qquad\times\{\mu[\rho(A)I-A(\alpha)]\}^{-1}\begin{pmatrix}\sum_{s=1}^{l}|a_{i_1j_s}|\\ \vdots\\ \sum_{s=1}^{l}|a_{i_kj_s}|\end{pmatrix} \\
&=R_{j_t}-w'_{j_t}+\varepsilon-\left[\sum_{u=1}^{k}|a_{j_ti_u}|-w'_{j_t}+\varepsilon-\right. \\
&\quad(|a_{j_ti_1}|,\cdots,|a_{j_ti_k}|)\{\mu[\rho(A)I-A(\alpha)]\}^{-1} \\
&\quad\left.\times\begin{pmatrix}\sum_{s=1}^{l}|a_{i_1j_s}|\\ \vdots\\ \sum_{s=1}^{l}|a_{i_kj_s}|\end{pmatrix}\right] \\
&=R_{j_t}-w'_{j_t}+\varepsilon-\frac{detB_t}{det\{\mu[\rho(A)I-A(\alpha)]\}}.
\end{aligned}$$

where

$$B_t=\begin{pmatrix}\sum_{u=1}^{k}|a_{j_ti_u}|-w'_{j_t}+\varepsilon & -|a_{j_ti_1}|\cdots & -|a_{j_ti_k}|\\ -\sum_{s=1}^{l}|a_{i_1j_s}| & & \\ \vdots & \mu[\rho(A)I-A(\alpha)] & \\ -\sum_{s=1}^{l}|a_{i_kj_s}| & & \end{pmatrix}.$$

Furthermore,

$$\begin{aligned}
&\sum_{u=1}^{k}|a_{j_ti_u}|-w'_{j_t}+\varepsilon \\
&=\sum_{u=1}^{k}|a_{j_ti_u}|-\sum_{w=1}^{k}|a_{j_ti_w}|\frac{\rho(A)-|a_{i_wi_w}|-hP_{i_w}(A)}{\rho(A)-|a_{i_wi_w}|}+\varepsilon \\
&=\sum_{u=1}^{k}|a_{j_ti_u}|-\sum_{w=1}^{k}|a_{j_ti_w}|\left(1-\frac{hP_{i_w}(A)}{\rho(A)-|a_{i_wi_w}|}\right)+\varepsilon \\
&=\sum_{w=1}^{k}|a_{j_ti_w}|\frac{hP_{i_w}(A)}{\rho(A)-|a_{i_wi_w}|}+\varepsilon \\
&>\sum_{w=1}^{k}|a_{j_ti_w}|\frac{hP_{i_w}(A)}{\rho(A)-|a_{i_wi_w}|}.
\end{aligned}$$

From Lemma 2.5, we have $\det B_t>0$. Having in mind that $\det\{\mu[\rho(A)I-A(\alpha)]\}>0$, we obtain

$$a\leq R_{j_t}-w'_{j_t}+\varepsilon-\frac{\det B_t}{\det\{\mu[\rho(A)I-A(\alpha)]\}}\leq R_{j_t}-w'_{j_t}+\varepsilon.$$

Let $\varepsilon\to0$, thus we easily get

$$a\leq R_{j_t}-w'_{j_t}-\frac{\det B_t}{\det\{\mu[\rho(A)I-A(\alpha)]\}}\leq R_{j_t}-w'_{j_t},$$

which implies (11). Moreover, (12) can be proved with a similar method to the above techniques.

Remark 2.1: According to $0\leq r<1, 0\leq h\leq1$, we have

$$R_{i_w}\geq P_{i_w}(A)\geq hP_{i_w}(A), 1\leq w\leq k,$$

which implies that

$$\begin{aligned}
&\sum_{w=1}^{k}|a_{j_ti_w}|\frac{R_{i_w}}{\rho(A)-|a_{i_wi_w}|} \\
&\geq\sum_{w=1}^{k}|a_{j_ti_w}|\frac{P_{i_w}(A)}{\rho(A)-|a_{i_wi_w}|} \\
&\geq h\sum_{w=1}^{k}|a_{j_ti_w}|\frac{P_{i_w}(A)}{\rho(A)-|a_{i_wi_w}|}, 1\leq t\leq l.
\end{aligned}$$

It follows from the above inequalities that, for all $1\leq t\leq l$,

$$\begin{aligned}
w'_{j_t}&=\sum_{w=1}^{k}|a_{j_ti_w}|\frac{\rho(A)-|a_{i_wi_w}|-hP_{i_w}(A)}{\rho(A)-|a_{i_wi_w}|} \\
&\geq\sum_{w=1}^{k}|a_{j_ti_w}|\frac{\rho(A)-|a_{i_wi_w}|-P_{i_w}(A)}{\rho(A)-|a_{i_wi_w}|} \\
&\geq\sum_{w=1}^{k}|a_{j_ti_w}|\frac{\rho(A)-|a_{i_wi_w}|-R_{i_w}(A)}{\rho(A)-|a_{i_wi_w}|}.
\end{aligned}$$

So it is obvious that the results of Lemma 2.5 and Lemma 2.6 are sharper than the ones of Proposition 1 and Proposition 2 in [15], respectively.

Theorem 2.1: Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be nonnegative irreducible with spectral radius $\rho(A)$, $\alpha=\{i_1,i_2,\cdots,i_k\}\subseteq N_r(A)\neq\emptyset$, $\alpha'=\{j_1,j_2,\cdots,j_l\}$, $|\alpha|<n$, $P(A/\alpha)=(a'_{ts})$, then for $\rho(A)\geq2|a_{ii}|(i\in\alpha)$ and $\forall t:1\leq t\leq l$,

$$\begin{aligned}
&|a'_{tt}|-R_t(P(A/\alpha)) \\
&\geq|a_{j_tj_t}|-R_{j_t}(A)+w'_{j_t}\geq|a_{j_tj_t}|-R_{j_t}(A), \quad (13)
\end{aligned}$$

and

$$|a'_{tt}| + R_t(P(A/\alpha)) \le |a_{j_tj_t}| + R_{j_t}(A) - w'_{j_t} \le |a_{j_tj_t}| + R_{j_t}(A). \quad (14)$$

Proof Since $\alpha \subseteq N_r(A) \neq \emptyset$, by the definition of the Perron complement and Lemma 2.6, $\forall t : 1 \le t \le l$, we have

$$\begin{aligned}
&|a'_{tt}| - R_t(P(A/\alpha)) \\
&= |a'_{tt}| - \sum_{s=1,s\neq t}^{l} |a'_{ts}| \\
&= \left| a_{j_tj_t} + (a_{j_ti_1}, \cdots, a_{j_ti_k}) \times [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix} \right| \\
&\quad - \sum_{s=1,s\neq t}^{l} \left| a_{j_tj_s} + (a_{j_ti_1}, \cdots, a_{j_ti_k}) \times [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s} \end{pmatrix} \right| \\
&\ge |a_{j_tj_t}| - \left| (a_{j_ti_1}, \cdots, a_{j_ti_k}) \times [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix} \right| \\
&\quad - \sum_{s=1,s\neq t}^{l} \left| a_{j_tj_s} + (a_{j_ti_1}, \cdots, a_{j_ti_k}) \times [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s} \end{pmatrix} \right| \\
&\ge |a_{j_tj_t}| - (R_{j_t} - w'_{j_t}).
\end{aligned}$$

Thus we get (13). With the same manner applied in the proof of (13), we can prove the result (14) of this theorem.

Remark 2.2: According to Remark 2.1, we know that the results of Theorem 2.1 improve the ones of Theorem 1 in [15].

Corollary 2.1: Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and take $\alpha = \{1, 2, \cdots, n-1\} \subseteq N_r(A)$, then

$$\begin{aligned}
0 < |a_{nn}| - h \sum_{i=1}^{n-1} |a_{ni}| \frac{P_i(A)}{\rho(A) - |a_{ii}|} \le |P(A/\alpha)| \\
\le |a_{nn}| + h \sum_{i=1}^{n-1} |a_{ni}| \frac{P_i(A)}{\rho(A) - |a_{ii}|}.
\end{aligned}$$

Proof Notice that α' contains only one element $j_t = n$. Thus, $P(A/\alpha)$ is nothing but a number, and $R_t(P(A/\alpha)) = 0$. By expression of w'_{j_t}, we have

$$\begin{aligned}
w'_{j_t} = w'_n &= \sum_{i=1}^{n-1} |a_{ni}| \frac{\rho(A) - |a_{ii}| - hP_i(A)}{\rho(A) - |a_{ii}|} \\
&= \sum_{i=1}^{n-1} |a_{ni}| - h \sum_{i=1}^{n-1} |a_{ni}| \frac{P_i(A)}{\rho(A) - |a_{ii}|}. \quad (15)
\end{aligned}$$

Substituting Equation (15) into Inequalities (13) and (14) results, we can obtain the result.

III. The γ-diagonally and product γ-diagonally dominant degree for Perron complement

In this section, we obtain some estimates for the γ-diagonally and product γ-diagonally degree for Perron complement under some conditions.

Lemma 3.1: [1] Let $a > b$, $c > b$, $b > 0$ and $0 \le r \le 1$. Then

$$a^r c^{1-r} \ge (a-b)^r (c-b)^{1-r} + b.$$

Theorem 3.1: If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonnegative irreducible with spectral radius $\rho(A)$, $\alpha = \{i_1, i_2, \cdots, i_k\} \subseteq N_r(A) \bigcap N_c(A) \neq \emptyset$, $\alpha' = N - \alpha = \{j_1, j_2, \cdots, j_l\}$, $|\alpha| < n$, and $P(A/\alpha) = (a'_{ts})$. Then for $\rho(A) \ge 2|a_{ii}| (i \in \alpha), 1 \le t \le l$ and $0 \le \gamma \le 1$,

$$\begin{aligned}
&|a'_{tt}| - (R_t(P(A/\alpha)))^\gamma (S_t(P(A/\alpha)))^{1-\gamma} \\
&\ge |a_{j_tj_t}| - (R_{j_t}(A) - w'_{j_t})^\gamma (S_{j_t}(A) - w''_{j_t})^{1-\gamma} \\
&\ge |a_{j_tj_t}| - (R_{j_t}(A))^\gamma (S_{j_t}(A))^{1-\gamma}
\end{aligned}$$

and

$$\begin{aligned}
&|a'_{tt}| + (R_t(P(A/\alpha)))^\gamma (S_t(P(A/\alpha)))^{1-\gamma} \\
&\le |a_{j_tj_t}| + (R_{j_t}(A) - w'_{j_t})^\gamma (S_{j_t}(A) - w''_{j_t})^{1-\gamma} \\
&\le |a_{j_tj_t}| + (R_{j_t}(A))^\gamma (S_{j_t}(A))^{1-\gamma}.
\end{aligned}$$

Proof By the definition of the Perron complement

$$\begin{aligned}
&|a'_{tt}| - (R_t(P(A/\alpha)))^\gamma (S_t(P(A/\alpha)))^{1-\gamma} \\
&= |a'_{tt}| - \left(\sum_{s=1,s\neq t}^{l} |a'_{ts}| \right)^\gamma \left(\sum_{s=1,s\neq t}^{l} |a'_{st}| \right)^{1-\gamma} \\
&= \left| a_{j_tj_t} + (a_{j_ti_1}, \cdots, a_{j_ti_k}) [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix} \right| \\
&\quad - \left[\sum_{s=1,s\neq t}^{l} \left| a_{j_tj_s} + (a_{j_ti_1}, \cdots, a_{j_ti_k}) \times [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s} \end{pmatrix} \right| \right]^\gamma \\
&\quad \times \left[\sum_{s=1,s\neq t}^{l} \left| a_{j_sj_t} + (a_{j_si_1}, \cdots, a_{j_si_k}) \times [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix} \right| \right]^{1-\gamma} \\
&\ge |a_{j_tj_t}| - \left| (a_{j_ti_1}, \cdots, a_{j_ti_k}) \times [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix} \right| \\
&\quad - \left(\sum_{s=1,s\neq t}^{l} \left[|a_{j_tj_s}| + \left| (a_{j_ti_1}, \cdots, a_{j_ti_k}) \right. \right. \right.
\end{aligned}$$

$$\times \ [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \Bigg| \Bigg] \Bigg)^{\gamma}$$

$$\times \Bigg(\sum_{s=1, s\neq t}^{l} \Bigg[|a_{j_s j_t}| + \Bigg| (a_{j_s i_1}, \cdots, a_{j_s i_k})$$

$$\times \ [\rho(A)I - A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \Bigg| \Bigg] \Bigg)^{1-\gamma}.$$

Denote

$$f = \left| (a_{j_t i_1}, \cdots, a_{j_t i_k}) [\rho(A)I - (A(\alpha))]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right|.$$

From Lemma 2.6 and Lemma 3.1

$$\begin{aligned} & |a'_{tt}| - (R_t(P(A/\alpha)))^{\gamma} (S_t(P(A/\alpha)))^{1-\gamma} \\ \geq & |a'_{tt}| - f - [R_{j_t} - w'_{j_t} - f]^{\gamma} [S_{j_t} - w''_{j_t} - f]^{1-\gamma} \\ \geq & |a'_{tt}| - f - \left[(R_{j_t} - w'_{j_t})^{\gamma} (S_{j_t} - w''_{j_t})^{1-\gamma} - f \right] \\ = & |a'_{tt}| - [R_{j_t} - w'_{j_t}]^{\gamma} [S_{j_t} - w''_{j_t}]^{1-\gamma}. \end{aligned}$$

Thus we can get the first type of inequalities of Theorem 3.1. Similarly, we can immediately verify the other one.

Remark 3.1: According to $0 \leq r < 1, 0 \leq h \leq 1$, we have

$$R_{i_w} \geq P_{i_w}(A) \geq h P_{i_w}(A), 1 \leq w \leq k,$$

$$S_{i_w} \geq Q_{i_w}(A) \geq \widetilde{h} Q_{i_w}(A), 1 \leq w \leq k,$$

which imply that

$$\begin{aligned} & \sum_{w=1}^{k} |a_{j_t i_w}| \frac{R_{i_w}}{\rho(A) - |a_{i_w i_w}|} \\ \geq & \sum_{w=1}^{k} |a_{j_t i_w}| \frac{P_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} \\ \geq & h \sum_{w=1}^{k} |a_{j_t i_w}| \frac{P_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|}, 1 \leq t \leq l, \\ & \sum_{w=1}^{k} |a_{i_w j_t}| \frac{S_{i_w}}{\rho(A) - |a_{i_w i_w}|} \\ \geq & \sum_{w=1}^{k} |a_{i_w j_t}| \frac{Q_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} \\ \geq & \widetilde{h} \sum_{w=1}^{k} |a_{j_t i_w}| \frac{Q_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|}, 1 \leq t \leq l. \end{aligned}$$

It follows from the above inequalities that, for all $1 \leq t \leq l$,

$$\begin{aligned} w'_{j_t} &= \sum_{w=1}^{k} |a_{j_t i_w}| \frac{\rho(A) - |a_{i_w i_w}| - h P_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} \\ &\geq \sum_{w=1}^{k} |a_{j_t i_w}| \frac{\rho(A) - |a_{i_w i_w}| - P_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} \\ &\geq \sum_{w=1}^{k} |a_{j_t i_w}| \frac{\rho(A) - |a_{i_w i_w}| - R_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|}, \\ w''_{j_t} &= \sum_{w=1}^{k} |a_{i_w j_t}| \frac{\rho(A) - |a_{i_w i_w}| - \widetilde{h} Q_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} \\ &\geq \sum_{w=1}^{k} |a_{i_w j_t}| \frac{\rho(A) - |a_{i_w i_w}| - Q_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|} \\ &\geq \sum_{w=1}^{k} |a_{i_w j_t}| \frac{\rho(A) - |a_{i_w i_w}| - S_{i_w}(A)}{\rho(A) - |a_{i_w i_w}|}. \end{aligned}$$

From the above inequalities, we can conclude that the bounds of Theorem 3.1 are sharper than those of Theorem 3 in [15].

Using the same technique as the proof of Theorem 3.1, we can obtain the following Theorem.

Theorem 3.2: If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonnegative irreducible with spectral radius $\rho(A)$, $\alpha = \{i_1, i_2, \cdots, i_k\} \subseteq N_r(A) \bigcap N_c(A) \neq \emptyset$, $\alpha' = N - \alpha = \{j_1, j_2, \cdots, j_l\}$, $|\alpha| < n$, and $P(A/\alpha) = (a'_{ts})$. Then for $\rho(A) \geq 2|a_{ii}| (i \in \alpha), 1 \leq t \leq l$ and $0 \leq \gamma \leq 1$,

$$\begin{aligned} & |a'_{tt}| - \gamma R_t(P(A/\alpha)) - (1-\gamma) S_t(P(A/\alpha)) \\ \geq & |a_{j_t j_t}| - \gamma R_{j_t}(A) - (1-\gamma) S_{j_t}(A) + \gamma w'_{j_t} + (1-\gamma) w''_{j_t} \\ \geq & |a_{j_t j_t}| - \gamma R_{j_t}(A) - (1-\gamma) S_{j_t}(A) \end{aligned}$$

and

$$\begin{aligned} & |a'_{tt}| + \gamma R_t(P(A/\alpha)) + (1-\gamma) S_t(P(A/\alpha)) \\ \leq & |a_{j_t j_t}| + \gamma R_{j_t}(A) + (1-\gamma) S_{j_t}(A) - \gamma w'_{j_t} - (1-\gamma) w''_{j_t} \\ \leq & |a_{j_t j_t}| + \gamma R_{j_t}(A) + (1-\gamma) S_{j_t}(A). \end{aligned}$$

Remark 3.2: From Remark 3.1, we can conclude that the bounds of Theorem 3.2 are sharper than those of Theorem 2 in [15].

IV. NUMERICAL EXAMPLE

In this section, we present a numerical example to illustrate the theory results in this paper and show the advantages of our derived results.

Example 4.1: Let

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 2 & 1 & 4 & 1 \\ 1 & 2 & 1 & 4 \end{pmatrix}, \ \alpha = \{1, 2\} \subseteq N_r(A).$$

By calculation with matlab 7.1, we have $\rho(A) = 6.3028 \geq 2|a_{ii}| (i \in \alpha)$. From Theorem 1 in [15], we obtain

$$w_3 = \sum_{u=1}^{2} |a_{3u}| \frac{\rho(A) - |a_{uu}| - R_u(A)}{\rho(A) - |a_{uu}|} = 1.1883,$$

$$w_4 = \sum_{u=1}^{2} |a_{4u}| \frac{\rho(A) - |a_{uu}| - R_u(A)}{\rho(A) - |a_{uu}|} = 1.1883,$$

and

$$|a'_{11}| - R_1(P(A/\alpha)) \geq |a_{33}| - R_3(A) + w_3 = 1.1883,$$

$$|a'_{22}| - R_2(P(A/\alpha)) \geq |a_{44}| - R_4(A) + w_4 = 1.1883.$$

By Theorem 2.1 in this paper, we have

$$w'_3 = \sum_{u=1}^{2} |a_{3u}| \frac{\rho(A) - |a_{uu}| - h P_u(A)}{\rho(A) - |a_{uu}|} = 1.6972,$$

$$w'_4 = \sum_{u=1}^{2} |a_{4u}| \frac{\rho(A) - |a_{uu}| - h P_u(A)}{\rho(A) - |a_{uu}|} = 1.6972.$$

Thus

$$|a'_{11}| - R_1(P(A/\alpha)) \geq |a_{33}| - R_3(A) + w'_3 = 1.6972,$$
$$|a'_{22}| - R_2(P(A/\alpha)) \geq |a_{44}| - R_4(A) + w'_4 = 1.6972.$$

In fact, by calculation, we have

$$P(A/\alpha) = \begin{pmatrix} 4.7676 & 1.5352 \\ 1.5352 & 4.7676 \end{pmatrix},$$

then,

$$|a'_{11}| - R_1(P(A/\alpha)) = 3.2324,$$
$$|a'_{22}| - R_2(P(A/\alpha)) = 3.2324.$$

Thus, by Theorem 2.1 in this paper, we can get a better bound for the diagonally dominant degree of the Perron complement of matrices than Theorem 1 in [15].

V. Conclusions

This paper studies the diagonally dominant degree for the Perron complement of three known subclasses of H-matrices and exhibits some new estimates of diagonally dominant, γ-diagonally and product γ-diagonally dominant degree for Perron complement of matrices. Furthermore, these estimations are more accurate than the existing ones in [15]. And numerical result given in Section IV also show that the derived results improve the related results.

In this paper, we do not give the error analysis. i.e., how accurately these bounds can be computed. At present, it is very difficult for the authors to do this. Next, we will continue to study this problem in the future.

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References

[1] J. Z. Liu and Z. J. Huang, "The Schur complement of γ-diagonally and product γ-diagonally dominant matrix and their disc seperation," *Linear Algebra Appl.*, vol. 432, pp. 1090–1104, 2010.

[2] J. Z. Liu, Z. H. Huang, and J. Zhang, "The dominant degree and disc theorem for the Schur complement," *Appl. Math. Comput.*, vol. 215, pp. 4055–4066, 2010.

[3] C. D. Meyer, "Uncoupling the Perron eigenvector problem," *Linear Algebra Appl.*, vol. 114, pp. 69–94, 1989.

[4] N. T. Binh, "Smoothed lower order penalty function for constrained optimization problems," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 1, pp. 76–81, 2016.

[5] J. W. Cai, P. Chen, X. Mei, and X. Ji, "Realized range-based threshold estimation for jump-diffusion models," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 293–299, 2015.

[6] C. D. Meyer, "Stochastic complementation, uncoupling markov chains, and the theory of nearly re- ducible systems," *SIAM Rev.*, vol. 31, pp. 240–272, 1989.

[7] S. J. Kirland, M. Neumann, and J. H. Xu, "A divide and conquer approach to computing the mean firest passage matrix for Markov chains via Perron complement reductions," *Numer. Linear Algerbra Appl.*, vol. 8, pp. 287–295, 2001.

[8] L. Z. Lu and T. Markham, "Perron complement and Perron root," *Linear Algerbra Appl.*, vol. 314, pp. 239–248, 2002.

[9] Z. M. Yang, "Some closer bounds of perron root basing on generalized Perron complement," *J. Comput. Appl. Math.*, vol. 235, pp. 315–324, 2010.

[10] J. Z. Liu, Z. H. Huang, and L. Zhu, "Theorems on Schur complement of block diagonally dominant matrices and their application in reducing the order for the solution of larger scale linear systems," *Linear Algerbra Appl.*, vol. 435, pp. 3085–3100, 2011.

[11] T. B. Gan, T. Z. Huang, and J. Gao, "A note on genralized nonlinaer diagonal dominance," *J. Math. Anal. Appl.*, vol. 333, pp. 581–586, 2006.

[12] R. A. Horn and C. R. Johnsonh, *Topics in Matrix Analysis*. Cambridge University Press, 1991.

[13] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, 1979.

[14] F. Z. Zhang, *Matrix theory: Basic Results and Techniques*. Springer, 1999.

[15] L. L. Wang, J. Z. Liu, and S. Chu, "Properties for the Perron complement of three known Subclasses of H-matrices," *Journal of Inequalities and Applications*, vol. 9, p. 2015, 2015.