An Iterative Non-overlapping Domain Decomposition Method for Optimal Boundary Control Problems Governed by Parabolic Equations

Wenyue Liu, Keying Ma

Abstract—**In this paper, we consider a numerical method for solving optimal boundary control problems governed by parabolic equations. In order to avoid large amounts of calculation produced by traditional numerical methods, we establish an iterative non-overlapping domain decomposition method. The whole domain is divided into many non-overlapping subdomains, and the optimal boundary control problem is decomposed into local problems in these subdomains. Robin conditions are used to communicate the local problems on the interfaces between subdomains. We build the iterative scheme for solving these local problems, and prove the convergence of the scheme. Finally, we present a numerical example to verify the validity of the iterative scheme.**

Index Terms—**Parabolic equations, optimal boundary control, non-overlapping domain decomposition method, iterative method, Robin conditions**

I. Introduction

IN the theory of control system, an optimal control problem is to find a control model (i.e. the control variable) admitted by the system to make the state variable tend to a target state in the process of optimizing (maximizing/ minimizing) the objective functional. If the state and control variable are subjected to partial differential equations, the optimal control problem is called the optimal control problem governed by partial differential equations (PDEs).

In the field of science and engineering, many problems, such as the Stefan-Boltzmann radiation law, the Lotka-Volterra model in population dynamics, can be described by optimal control problems governed by partial differential equations. As well known, reference [1] discussed systematically the theory and numerical methods of optimal control problems governed by PDEs. References [2]-[5] made further studies. Among the numerical methods to solve the optimal control problem governed by PDEs, an effective one is the finite element method, i.e. building the finite element space for the state variable and the control variable respectively; establishing the discretized schemes for the governing PDEs; then developing the discrete algebraic equation systems to be solved. If the domain is large, there needs a large amount of calculation, which can be settled by the method of parallel computation.

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A natural way in parallel computation is non-overlapping domain decomposition method. This method can divide the whole domain into many subdomains, and decompose the optimal control problem into many local problems, which are independent ones on subdomains and can be calculated parallel. Hence, this method can reduce much the amount of computation. Until now, there have been a lot of articles considering the application of this method to different types of partial differential equations, such as references [6]-[9]. References [10]-[13] discussed some iterative non-overlapping domain decomposition methods for optimal boundary control problems governed by PDEs. The important character of these methods is how to build internal boundary conditions of state/co-state variables to communicate the local problems on the interfaces between subdomains. Reference [14] presented an iterative another non-overlapping domain decomposition method for optimal boundary control problems governed by hyperbolic equations and proposed an internal boundary condition (called as Robin condition). The author proved the convergence of the method. But we should point out that article [14] only considered the case in which the control variable is defined in the interior of the domain, but not on the boundary.

Invoked by the work of [14], we will discuss an iterative non-overlapping domain decomposition method for optimal boundary control problems governed by parabolic equations. The structure of this article is as follows: in Section II, we give an optimal boundary control problem governed by parabolic equations, and build the co-state equations and optimal boundary conditions; In Section III, we set up the iterative non-overlapping domain decomposition scheme by using Robin conditions and prove the convergence; In Section IV, we present an numerical example, and verify the validity of the iterative scheme. We make some conclusions in Section V.

II. Model

Let $\Omega \subset \mathbf{R^2}$ be a bounded convex domain with an smooth boundary $\partial\Omega$ and $[0, T]$ be a time interval. Throughout the paper, we adopt the standard notations for Sobolev spaces on Ω. We will take the state space $L^2(0, T; V)$ with $V = H^1(\Omega)$ and the control space $L^2(0, T; U)$ with $U \subset L^2(\partial\Omega)$.

We consider the following optimal boundary control problems governed by parabolic equations:

$$\min_{u \in U} J(u, y(u)) \tag{1}$$

subject to

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} - \Delta y(x,t) = f(x,t), & (x,t) \text{ in } \Omega \times (0,T), \\ \dfrac{\partial y(x,t)}{\partial \vec{\nu}} = u(x,t) + g(x,t), & (x,t) \text{ on } \Gamma_N \times (0,T), \\ y(x,t) = y_D(x,t), & (x,t) \text{ on } \Gamma_D \times (0,T), \\ y(x,0) = y_0(x), & x \text{ in } \Omega, \end{cases} \tag{2}$$

where, the state variable $y(u) \in L^2(0,T;V)$ and the control variable $u \in L^2(0,T;U)$, Γ_N and Γ_D are Neuman and Dirichlet boundary, respectively, $\partial\Omega = \Gamma_N \cup \Gamma_D, \Gamma_N \cap \Gamma_D = \varnothing$, $\vec{\nu}$ is an unit outer normal vector, $f(x,t)$, $g(x,t)$, $y_0(x)$ and $y_D(x,t)$ are known functions.

Let the objective functional be

$$\begin{aligned} &J(u, y(u)) \\ &= \frac{1}{2}\{\int_{\Omega\times(0,T)} \gamma|y - z_d|^2 dxdt + \int_{\Gamma_N\times(0,T)} \alpha|u|^2 dsdt\}. \end{aligned} \tag{3}$$

Here, $z_d(x,t)$ is the desired state variable, the constants $\alpha > 0$ and $\gamma > 0$ play the roles of balancing the contributions of the state variable y and control variable u.

According to references [1], [3], we can derive the adjoint equation of (2)

$$\begin{cases} -\dfrac{\partial p(x,t)}{\partial t} - \Delta p(x,t) = \gamma(y(x,t) - z_d(x,t)), \\ \qquad\qquad (x,t) \text{ in } \Omega \times (0,T), \\ \dfrac{\partial p(x,t)}{\partial \vec{\nu}} = 0, \quad (x,t) \text{ on } \Gamma_N \times (0,T), \\ p(x,t) = 0, \quad (x,t) \text{ on } \Gamma_D \times (0,T), \\ p(x,T) = 0, \quad x \text{ in}\, \Omega, \end{cases} \tag{4}$$

where $p(x,t)$ is the co-state variable of $y(x,t)$.

And we know that when the objective functional J gets its optimum, the control variable $u \in L^2(0,T;U)$ should satisfy

$$J'(u)(v-u) \geq 0, \quad \forall\, v \in L^2(0,T;U). \tag{5}$$

According to the definition of the directional derivative of the objective functional and references [1], [3], we can deduce that the inequality (5) equals to

$$J'(u)(v-u) = \int_{\Gamma_N\times(0,T)} \big(\alpha u + p\big)(v-u) dsdt \geq 0, \tag{6}$$

$\forall\, v \in L^2(0,T;U)$. This inequality is called as the optimality condition.

Then, the optimal boundary control problems (1)-(2) are equivalent to an optimality system composed of the state equation (2), the co-state equation (4) and the optimality condition (6). We can get the solution of problems (1)-(2) by solving the optimality system (2), (4) and (6).

III. Iterative non-overlapping domain decomposition

In this section, we will build an iterative non-overlapping domain decomposition scheme for the system (2), (4) and (6), and prove the convergence.

A. Iterative Domain Decomposition

First, we divide Ω into several non-overlapping subdomains Ω_i, $i = 1, 2, \cdots, N$,

$$\Omega = \bigcup_{i}^{N} \bar{\Omega}_i, \qquad \Omega_i \bigcap \Omega_j = \emptyset, \ \forall \ i \neq j.$$

Let $\Gamma_{D,i} = \Gamma_D \bigcap \partial\Omega_i$, $\Gamma_{N,i} = \Gamma_N \bigcap \partial\Omega_i$, $\Gamma_{D,i} \neq \varnothing$, $\Gamma_{N,i} \neq \varnothing$, $\Sigma_{ij} = \partial\Omega_i \bigcap \partial\Omega_j$ be the internal boundary between Ω_i and Ω_j, and $\Sigma_{ij} = \Sigma_{ji}$. Let $\vec{\nu}_i$ is the unit outer normal vector on $\partial\Omega_i$. We suppose that this decomposition holds the regularity to guarantee the global and local equations with good properties.

Then, we decompose the system of (2),(4) and (6) into several local problems on subdomains and use the iterative method to solve them. Take the local problem on the subdomain Ω_i for an example, i.e., the domain Ω and boundaries Γ_N, Γ_D are replaced by Ω_i, $\Gamma_{N,i}$, $\Gamma_{D,i}$, respectively. We define the local solution at step $k+1$ on subdomain Ω_i is $(y_i^{k+1}, p_i^{k+1}, u_i^{k+1})$. Hence, the local problem is

$$\begin{cases} \dfrac{\partial y_i^{k+1}}{\partial t} - \Delta y_i^{k+1} = f, & \text{in } \Omega_i \times (0,T), \\ \dfrac{\partial y_i^{k+1}}{\partial \vec{\nu}_i} = u_i^{k+1} + g, & \text{on } \Gamma_{N,i} \times (0,T), \\ y_i^{k+1} = y_D, & \text{on } \Gamma_{D,i} \times (0,T), \\ y_i^{k+1}(x,0) = y_0(x), & \text{in } \Omega_i, \end{cases} \tag{7}$$

$$\begin{cases} -\dfrac{\partial p_i^{k+1}}{\partial t} - \Delta p_i^{k+1} = \gamma(y_i^{k+1} - z_d), \\ \qquad\qquad \text{in } \Omega_i \times (0,T), \\ \dfrac{\partial p_i^{k+1}}{\partial \vec{\nu}_i} = 0, & \text{on } \Gamma_{N,i} \times (0,T), \\ p_i^{k+1} = 0, & \text{on } \Gamma_{D,i} \times (0,T), \\ p_i^{k+1}(T,x) = 0, & \text{in } \Omega_i, \end{cases} \tag{8}$$

and

$$\int_{\Gamma_{N,i}\times(0,T)} (p_i^{k+1} + \alpha u_i^{k+1})(v_i - u_i^{k+1}) dsdt \geq 0, \tag{9}$$

$\forall\, v_i \in L^2(0,T;U_i)$, where U_i is a local control space and just the restriction of the space U on Ω_i, i.e.

$$\forall\, u \in U, \ \ u|_{\Omega_i} = u_i \ \in U_i. \tag{10}$$

For the later use, we define the following inner products and norms:

$$\begin{aligned} (y, y')_i &= \int_{\Omega_i\times(0,T)} yy' dxdt, & ||y||_i^2 &= (y,y)_i, \\ < y, y' >_{ij} &= \int_{\Sigma_{ij}\times(0,T)} yy' dxdt, & ||y||_{ij}^2 &= < y, y >_{ij}, \\ < y, y' >_{N,i} &= \int_{\Gamma_{N,i}\times(0,T)} yy' dsdt, & ||y||_{N,i}^2 &= < y, y >_{N,i}. \end{aligned} \tag{11}$$

According to these definitions, we establish the iterative scheme of (7)-(8) on Ω_i

$$\begin{cases} (\frac{\partial y_i^{k+1}}{\partial t}, v)_i + (\nabla y_i^{k+1}, \nabla v)_i = (f, v)_i \\ \quad + < u_i^{k+1} + g, v >_{\Gamma_{N,i}} + \sum_{i\neq j, \Sigma_{ij}\neq\varnothing} < \frac{\partial y_i^{k+1}}{\partial \vec{\nu}_i}, v >_{ij}, \\ -(\frac{\partial p_i^{k+1}}{\partial t}, v)_i + (\nabla p_i^{k+1}, \nabla v)_i = \gamma(y_i^{k+1} - z_d, v)_i \\ \quad + \sum_{i\neq j, \Sigma_{ij}\neq\varnothing} < \frac{\partial p_i^{k+1}}{\partial \vec{\nu}_i}, v >_{ij}, \end{cases} \tag{12}$$

where $v \in H_{0,\Gamma_{D,i}} = \{v \in H^1(\Omega_i) | v = 0, \text{ when } x \text{ on } \Gamma_{D,i}\}$.

At the same time, we should put forward the boundary conditions on the interfaces between the subdomains. These conditions are taken the form of Robin condition, where the state and co-state variables are skew-symmetrically coupled [14]:

$$\begin{cases} \frac{\partial y_i^{k+1}}{\partial \vec{\nu}_i} + \beta p_i^{k+1} = -\frac{\partial y_j^k}{\partial \vec{\nu}_j} + \beta p_j^k, & \text{on } \Sigma_{ij}, \\ \frac{\partial p_i^{k+1}}{\partial \vec{\nu}_i} - \beta y_i^{k+1} = -\frac{\partial p_j^k}{\partial \vec{\nu}_i} - \beta y_j^k, & \text{on } \Sigma_{ij}, \end{cases} \tag{13}$$

where the constant $\beta > 0$.

These conditions are called as the transmission conditions, because they can strengthen the continuity of the solutions of local problems and theirs directional derivative on Σ_{ij} at successive iteration steps. So the global problem can be composed by the local problems. The value of β will be selected in Section IV.

B. Proof of convergence

In the section, we define the local error between the global and local solution in Ω_i

$$(\bar{y}_i, \bar{p}_i, \bar{u}_i) = (y, p, u) - (y_i, p_i, u_i). \tag{14}$$

It is easy to see that this error $(\bar{y}_i, \bar{p}_i, \bar{u}_i)$ also satisfies the coupled equations (7)-(9) and (12)-(13), where $f = 0,\ g = 0,\ y_1 = 0,\ y_d = 0$.

We use the following sequence of energies on the interfaces between subdomains to prove the convergence

$$E^{k+1} = \sum_{i\neq j,\ \Sigma_{ij}\neq\varnothing} \{\|\frac{\partial \bar{y}_i^{k+1}}{\partial \vec{\nu}_i}\|_{ij}^2 + \|\beta \bar{p}_i\|_{ij}^2 + \|\frac{\partial \bar{p}_i^{k+1}}{\partial \vec{\nu}_i}\|_{ij}^2 + \|\beta \bar{y}_i^{k+1}\|_{ij}^2\}. \tag{15}$$

Now, if we take the place of $(y_i^{k+1}, p_i^{k+1}, u_i^{k+1})$ by $(\bar{y}_i^{k+1}, \bar{p}_i^{k+1}, \bar{u}_i^{k+1})$ in (12)-(13), then (15) becomes

$$\begin{aligned} E^{k+1} = E^k - 2\beta \sum_{i\neq j,\Sigma_{ij}\neq\varnothing} \{ &< \frac{\partial \bar{y}_i^{k+1}}{\partial \vec{\nu}_i},\ \bar{p}_i^{k+1} >_{ij} \\ &- < \frac{\partial \bar{p}_i^{k+1}}{\partial \vec{\nu}_i},\ \bar{y}_i^{k+1} >_{ij} + < \frac{\partial \bar{y}_i^{k}}{\partial \vec{\nu}_i},\ \bar{p}_i^{k} >_{ij} \\ &- < \frac{\partial \bar{p}_i^{k}}{\partial \vec{\nu}_i},\ \bar{y}_i^{k} >_{ij} \}. \end{aligned} \tag{16}$$

We take the place of $(y_i^{k+1}, p_i^{k+1}, u_i^{k+1})$ by $(\bar{y}_i^{k+1}, \bar{p}_i^{k+1}, \bar{u}_i^{k+1})$ in (7)-(8). These equations are multiplied by $\bar{p}_i^{k+1}$ and $\bar{y}_i^{k+1}$, and integrated by parts in spaces-time domain, respectively. Then, we can get the following two equations

$$\begin{aligned} &-(\bar{y}_i^{k+1}, \frac{\partial \bar{p}_i^{k+1}}{\partial t}) + (\nabla \bar{y}_i^{k+1}, \nabla \bar{p}_i^{k+1})_i \\ &=< \bar{u}_i^{k+1}, \bar{p}_i^{k+1} >_{\Gamma_{N,i}} + < \frac{\partial \bar{y}_i^{k+1}}{\partial \vec{\nu}_i}, \bar{p}_i^{k+1} >_{\partial\Omega_i/\Gamma_{N,i}}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} &-(\frac{\partial \bar{p}_i^{k+1}}{\partial t}, \bar{y}_i^{k+1}) + (\nabla \bar{p}_i^{k+1}, \nabla \bar{y}_i^{k+1})_i \\ &= \gamma(\bar{y}_i^{k+1}, \bar{y}_i^{k+1})_i + < \frac{\partial \bar{p}_i^{k+1}}{\partial \vec{\nu}_i}, \bar{y}_i^{k+1} >_{\partial\Omega_i} . \end{aligned} \tag{18}$$

Subtracting these above results (17)-(18), we obtain, for all i,

$$\begin{aligned} &- \sum_{i\neq j,\ \Sigma_{ij}\neq\varnothing} \{< \frac{\partial \bar{y}_i^{k+1}}{\partial \vec{\nu}_i}, \bar{p}_i^{k+1} >_{ij} - < \frac{\partial \bar{p}_i^{k+1}}{\partial \vec{\nu}_i}, \bar{y}_i^{k+1} >_{ij}\} \\ &= -\gamma\|\bar{y}_i^{k+1}\|_i^2 + < \bar{p}_i^{k+1}, \bar{u}_i^{k+1} >_{\Gamma_{N,i}} . \end{aligned} \tag{19}$$

Now, we use the global and local inequality (6) and (9). Under the assumptions (10), we take $v = u_i^{k+1}$ in (6) and $v_i = u$ in (9), respectively. Subtracting the two inequalities, we can get the estimate

$$< \bar{p}_i^{k+1}, \bar{u}_i^{k+1} >_{\Gamma_{N,i}} \leq -\alpha\|\bar{u}_i^{k+1}\|_{\Gamma_{N,i}}^2 \tag{20}$$

Combining (19) and (20) together, we can obtain the following decrease law for the energies:

$$\begin{aligned} E^{k+1} \leq E^k - 2\beta \sum_i \{&\gamma\|\bar{y}_i^{k+1}\|_i^2 + \alpha\|\bar{u}_i^{k+1}\|_{\Gamma_{N,i}}^2 \\ &+ \gamma\|\bar{y}_i^{k}\|_i^2 + \alpha\|\bar{u}_i^{k}\|_{\Gamma_{N,i}}^2\}. \end{aligned} \tag{21}$$

In a word, we can obtain that the sequence $\{E^k\}$ is bounded and monotone decreasing, then the limit of $\{E^k\}$ exists. The following result of convergence on each subdomain Ω_i can be derived

$$\|\bar{y}_i^{k+1}\|_i \xrightarrow{k} 0, \quad \|\bar{p}_i^{k+1}\|_i \xrightarrow{k} 0, \quad \|\bar{u}_i^{k+1}\|_{\Gamma_{N,i}} \xrightarrow{k} 0. \tag{22}$$

Hence, the convergence of the scheme (7)-(9) is proven.

IV. Numerical example

In this section, we present an example to prove the validity of the iterative non-overlapping domain decomposition method mentioned in the above section.

We consider the model (1)-(4) by choosing $\Omega = [-2, 2] \times [-1/2, 1/2]$ and $T = 1$. Let $\Gamma_N = \Gamma_l \cup \Gamma_r, \Gamma_D = \Gamma_u \cup \Gamma_d$, where Γ_l and Γ_r are the left and right edges of Ω, respectively; Γ_u and Γ_d are the upside and downside edges. To compare with the numerical solutions well, we suppose the exact solutions of the model are: $\forall\ (x_1, x_2) \in \Omega,\ t \in [0, T]$

$$\begin{cases} y = (T - t)\cos(\pi x_1)\cos(\pi x_2), \\ p = -(T - t)\cos(\pi x_1)\cos(\pi x_2), \\ u = (T - t)\cos(\pi x_2), \\ z_d = ((2\pi^2 + 1)(T - t) + 1)\cos(\pi x_1)\cos(\pi x_2), \\ f = (2\pi^2(T - t) - 1)\cos(\pi x_1)\cos(\pi x_2). \end{cases} \tag{23}$$

We divide Ω into two non-overlapping subdomains: $\Omega = \Omega_1 \cup \Omega_2,\ \Omega_1 = [-2, 0] \times [-1/2, 1/2], \Omega_2 = [0, 2] \times$

$[-1/2,1/2]$, the inner boundary $\Gamma_0 = \{0\} \times [-1/2,1/2]$. In the process of calculation, the numerical solutions are computed on the triangular meshes, with different mesh size $h = 0.2, 0.1, 0.05$ sequentially. The state variable y and co-state variable p are approximated by piecewise linear finite elements, while the control variable u is approximated by piecewise constant finite elements.

Using Backward-Euler scheme to approximate the time derivative, we get the following fully discrete iterative scheme of (7)-(8)

$$\begin{cases} (\frac{y_i^{n+1,k+1}-y_i^n}{\Delta t}, v)_i + (\nabla y_i^{n+1,k+1}, \nabla v)_i \\ \quad = (f^{n+1}, v)_i + < u_i^{n+1,k+1} + g^{n+1}, v >_{\Gamma_{N,i}} \\ \quad\quad + \sum_{i\neq j, \Sigma_{ij}\neq\varnothing} < \frac{\partial y_i^{n+1,k+1}}{\partial \vec{\nu}_i}, v >_{ij}, \\ -(\frac{p_i^{n+1,k+1}-p_i^{n,k+1}}{\Delta t}, v)_i + (\nabla p_i^{n,k+1}, \nabla v)_i \\ \quad = \gamma(y_i^{n+1,k+1} - z_d^{n+1}, v)_i \\ \quad\quad + \sum_{i\neq j, \Sigma_{ij}\neq\varnothing} < \frac{\partial p_i^{n,k+1}}{\partial \vec{\nu}_i}, v >_{ij}, \end{cases} \tag{24}$$

where the first subscript $n+1$ means at time $t^{n+1} = (n+1)\Delta t$, while the second subscript $k+1$ is for the iterative step, and the time step size is $\Delta t = 0.1$.

Reference [14] showed that some eigenvalues of the discrete iteration operator are close to 1 and sometimes even exceed 1 because of numerical errors. Hence, they suggested to use an underrelaxed version of the transmission conditions instead of (13). Following their ideas to our example, we take the following form

$$\begin{cases} \frac{\partial y_i^{n+1,k+1}}{\partial \vec{\nu}_i} + \beta p_i^{n+1,k+1} \\ \quad = \rho\big(-\frac{\partial y_j^{n+1,k}}{\partial \vec{\nu}_j} + \beta p_j^{n+1,k}\big) \\ \quad\quad +(1-\rho)\big(\frac{\partial y_i^{n+1,k}}{\partial \vec{\nu}_i} + \beta p_i^{n+1,k}\big), \qquad on\ \Sigma_{ij}, \\ \frac{\partial p_i^{n+1,k+1}}{\partial \vec{\nu}_i} - \beta y_i^{n+1,k+1} \\ \quad = \rho\big(-\frac{\partial p_j^{n+1,k}}{\partial \vec{\nu}_j} - \beta y_j^{n+1,k}\big) \\ \quad\quad +(1-\rho)\big(\frac{\partial p_i^{n+1,k}}{\partial \vec{\nu}_i} - \beta y_i^{n+1,k}\big), \qquad on\ \Sigma_{ij}. \end{cases} \tag{25}$$

Here, the parameter belongs to $(0,1)$ and is always chosen as $\rho = 1/2$. It is easy to see that the similar convergence proof as Section III can also be established with (25). The parameter β has a decisive influence on the speed of convergence. We choose $\beta = 1/h$ for each case of our numerical calculations.

We present the following numerical results at $t = 0.5$ for examples. Tables I and II show L^2-norm error and convergence rate of variables y, p and u in subdomain Ω_1 and Ω_2, respectively.

We choose four points on Γ_0 as examples to show the effect of the computations. Table III considers for the state variable, where y is the exact solution, y_1 and y_2 are the approximate solutions in Ω_1 and Ω_2, respectively. And similarly, Table IV shows for the co-state variable p.

TABLE III

THE COMPARISON OF STATE VARIABLE y ON THE INTERFACE Γ_0

(x_1,x_2)	y	y_1	y_2	$\lvert y_1-y_2\rvert$
$(0, 0.40)$	0.1545	0.1543	0.1542	$1.2179e-04$
$(0, 0.15)$	0.4455	0.4428	0.4430	$1.7229e-04$
$(0, -0.10)$	0.4755	0.4720	0.4723	$2.8156e-04$
$(0, -0.35)$	0.2270	0.2265	0.2265	$1.0729e-05$

TABLE IV

THE COMPARISON OF CO-STATE VARIABLE p ON THE INTERFACE Γ_0

(x_1,x_2)	y	y_1	y_2	$\lvert y_1-y_2\rvert$
$(0, 0.40)$	-0.1545	-0.1545	-0.1544	$8.7926e-05$
$(0, 0.15)$	-0.4455	-0.4438	-0.4440	$2.6234e-04$
$(0, -0.10)$	-0.4755	-0.4737	-0.4741	$3.7825e-04$
$(0, -0.35)$	-0.2270	-0.2270	-0.2271	$5.8858e-05$

Taking the case of $h = 0.05$ for an example, Figures 1-6 below present the figure of the exact and approximate solution for variables y, p and u, respectively.

For the case of $h = 0.05$, Figure 7 presents the trends of objective functional $J(u)$ in Ω_1 and Ω_2, when choosing $\alpha = 0.1$ and $\alpha = 0.01$ differently. The iteration numbers of $J(u)$ change insignificantly as long as the decreasing of the value of α, since the value of $\alpha \int_{\Gamma_N \times (0,T)} u^2 ds$ has a small influence on the whole value of $J(u)$.

V. Conclusion

We have considered an iterative non-overlapping domain decomposition method for solving optimal boundary control problems governed by parabolic equations. The iterative scheme was established. The whole domain was divided into many non-overlapping subdomains, and Robin conditions were used to communicate the local problems on the interfaces between subdomains. We proved the convergence of the iterative scheme and presented a numerical example to verify the validity of the iterative scheme.

In this paper, the parabolic equations were linear, and the objective functional was defined over the whole time interval $[0,T]$. We can extend our method to the case of nonlinear parabolic equations with the objective functional at the final state. The results for this case will be presented in a forthcoming paper.

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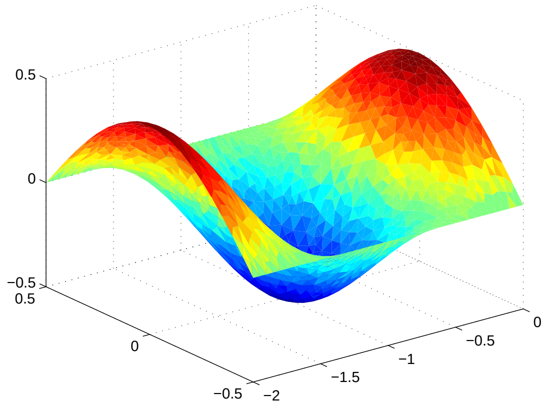
TABLE I
L^2-NORM ERROR AND CONVERGENCE RATE IN SUBDOMAIN Ω_1

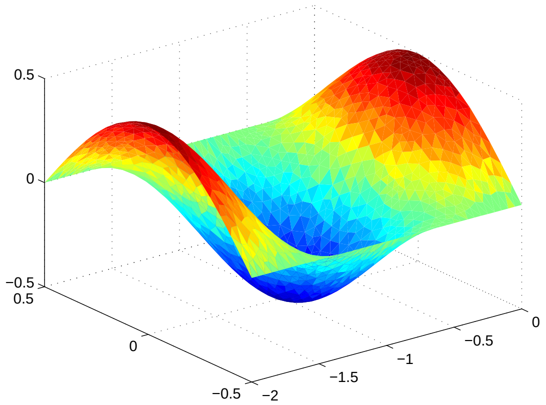
h	y		p		u	
	error	rate	error	rate	error	rate
0.2	$1.5673e-2$		$1.7312e-2$		$7.4007e-2$	
0.1	$4.3901e-3$	1.8360	$4.8983e-3$	1.8214	$3.6855e-2$	1.0058
0.05	$1.2071e-3$	1.8627	$1.2811e-3$	1.9349	$1.8421e-2$	1.0005

TABLE II
L^2-NORM ERROR AND CONVERGENCE RATE IN SUBDOMAIN Ω_2

h	y		p		u	
	error	rate	error	rate	error	rate
0.2	$1.7340e-2$		$1.4329e-2$		$1.0885e-1$	
0.1	$4.7427e-3$	1.8703	$4.0636e-3$	1.8180	$5.4242e-2$	1.0048
0.05	$1.2908e-3$	1.8774	$1.0528e-3$	1.9485	$2.7114e-2$	1.0004



(a) Exact solution of y



(b) Approximate solution of y

Fig.1. The exact and approximate solution of y in subdomain Ω_1

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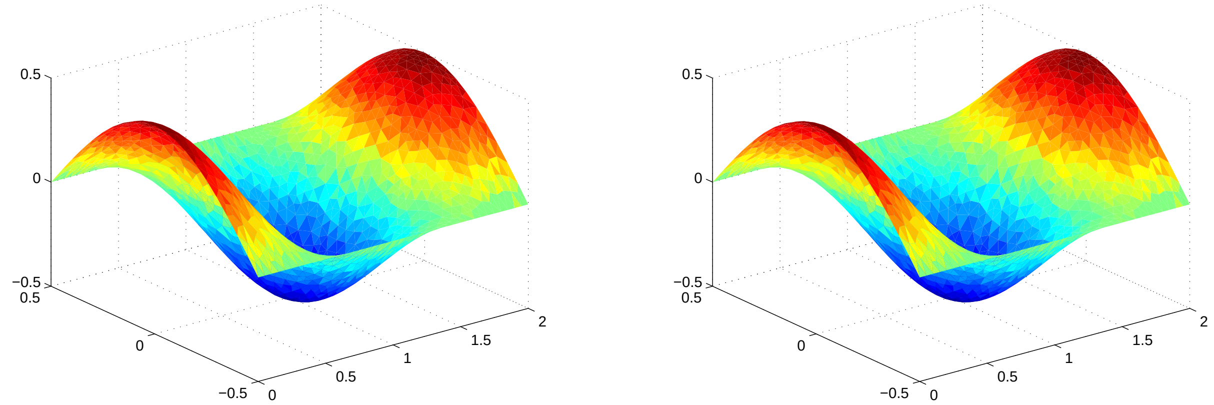
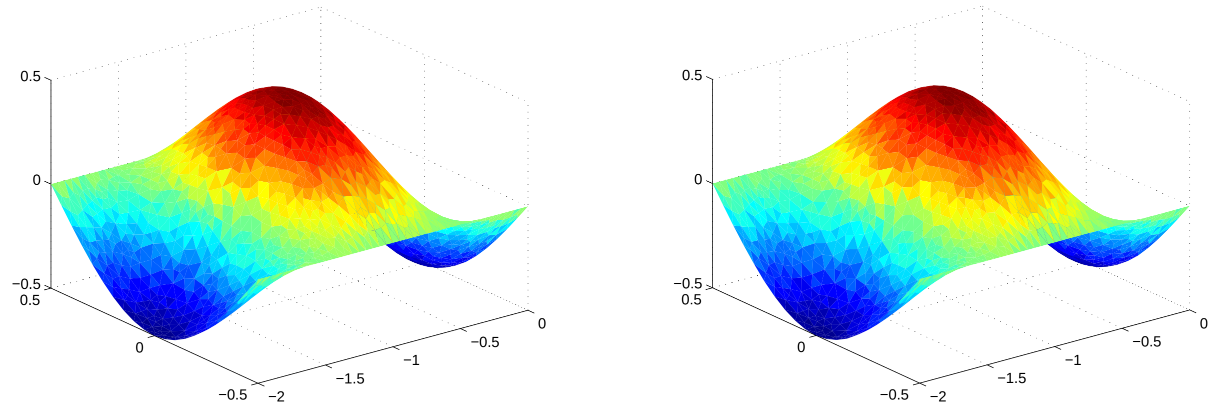


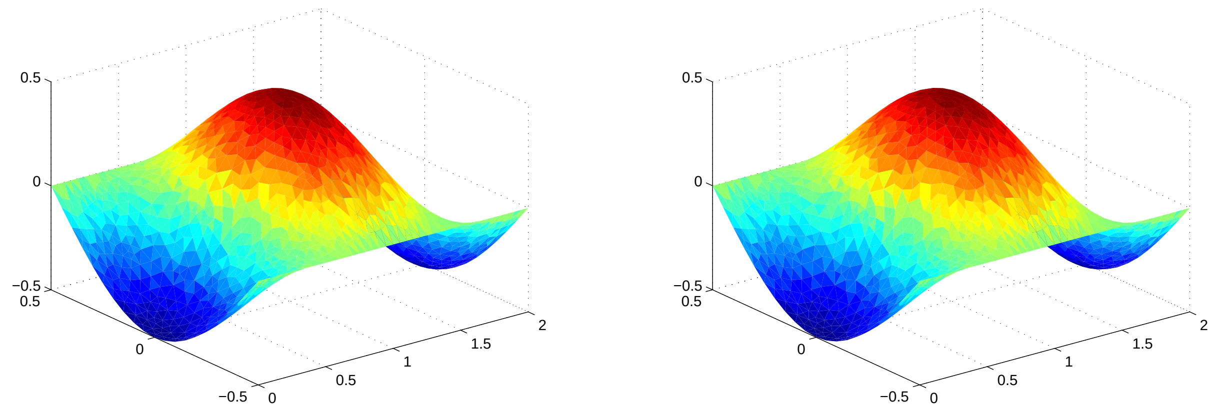
(c) Exact solution of y (d) Approximate solution of y

Fig.2. The exact and approximate solution of y in subdomain Ω_2



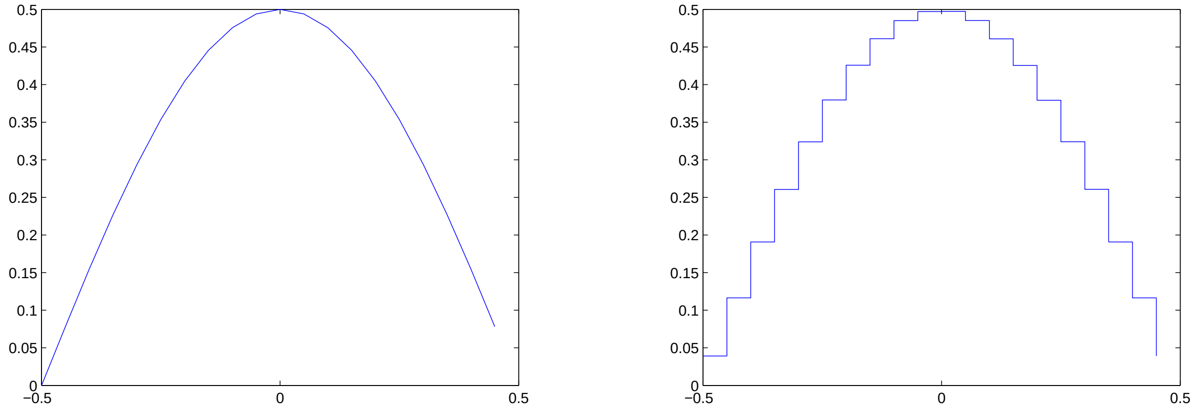
(e) Exact solution of p (f) Approximate solution of p

Fig.3. The exact and approximate solution of p in subdomain Ω_1



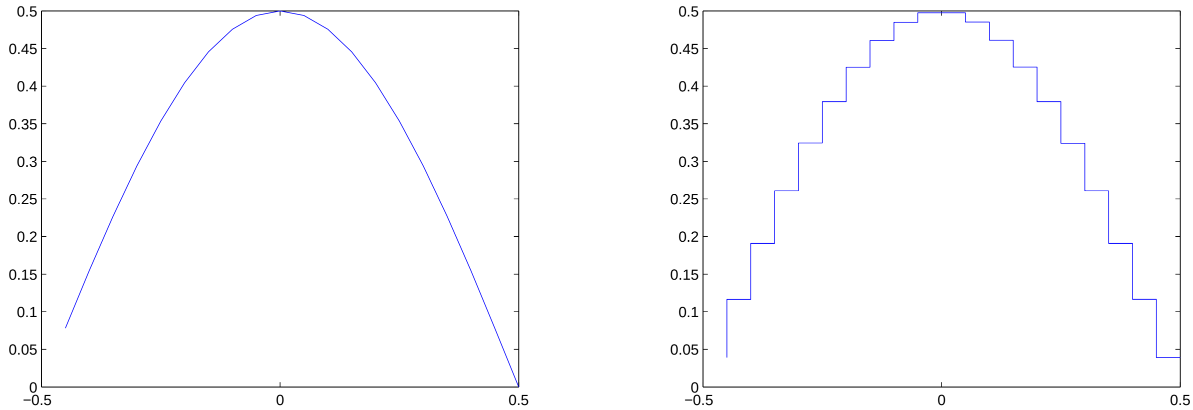
(g) Exact solution of p (h) Approximate solution of p

Fig.4. The exact and approximate solution of p in subdomain Ω_2



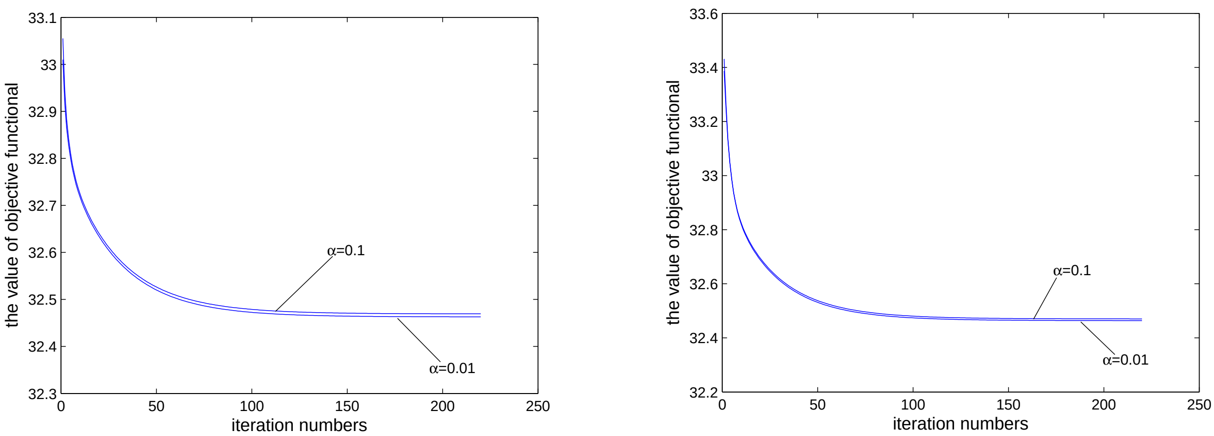
(i) Exact solution of u (j) Approximate solution of u

Fig.5. The exact and approximate solution of u in subdomain Ω_1



(k) Exact solution of u (l) Approximate solution of u

Fig.6. The exact and approximate solution of u in subdomain Ω_2



(m) in Ω_1 (n) in Ω_2

Fig.7. The functional $J(u)$ in Ω_1 and Ω_2 at $h = 0.05$