Matrix Measure Strategies for Stability of Cellular Neural Networks with Proportional Delays

Changjin Xu, and Peiluan Li

Abstract—**This paper is concerned with cellular neural networks with proportional delays. The proportional delay is a time-varying unbounded delay which is different from the constant delay, bounded time-varying delay and distributed delay. Using matrix measure and generalized Halanay inequality, some sufficient conditions are obtained to ensure the pth exponential stability of cellular neural networks with proportional delays. The obtained results are simple and easy to be verified. An example is given to illustrate the effectiveness of the obtained results. This paper ends with a brief conclusion.**

Index Terms—**Cellular neural networks, exponential stability, proportional delays, matrix measure.**

I. Introduction

IT is well known that considerable attention has been paid to cellular neural networks as well as various generalizations for their potential applications in many fields such as associative memories, pattern recognition, optimization and image processing and so on [1-13]. On the one hand, the existence and stability of the equilibrium point of cellular neural networks plays an important role in practical application. On the other hand, time delay is inevitable due to the finite switching speed of information processing and the inherent communication time of neurons, moreover, its existence may cause the instability of networks [14]. Thus many interesting stability results on cellular neural networks with delays have been available [14-19]. At present, the time delays considered for cellular neural networks can be classified as constant delays [4,16,21], time-varying delays [3,15,19,21], and distributed delays [22-24]. Here we would like to point out that the proportional delay which is a special delay type exists in some fields such as physics, biology systems , control theory and Web quality of service (QoS) routing decision. Since the presence of an amount of parallel pathways of a variety of axon sizes and lengths, a neural network usually has a spatial structure, it is reasonable to introduce the proportional delays into the neural networks. In an amount of parallel pathways, affected by different materials and topology, there may be some unbounded delays that is proportional to the time, thus we should choose suitable proportional delays factors in view of different cases and adopt proportional delays to characterize these unbounded delays [25]. Recently, there are only very few papers that focus on this aspect. For example, Zhou et al. [14] considered the asymptotic stability of cellular neural networks with multiple proportional delays, Zheng et al. [26] established the stability criteria for high-order networks with proportional delay. Zhou [27] adressed the delay-dependent exponential stability of cellular neural networks with multi-proportional delays, Zhou [28] discussed the delay-dependent exponential synchronization of recurrent neural works with multiple proportional delays, Zhou [29] analyzed the global asymptotic stability of cellular neural networks with proportional delays, Zhou [30] investigated the dissipativity of a class of cellular neural networks with proportional delays. In details, one can see [34,35]. We must point out that cellular neural networks with proportional delays have been widely applied in many fields such as light absorption in the star substance and nonlinear dynamic systems. Therefore the study on the cellular neural networks with proportional delays has important theoretical and practical value.

Manuscript received December 21, 2015; revised March 2, 2016. This work was supported in part by the This work is supported by National Natural Science Foundation of China(No.11261010 and No.11101126), Natural Science and Technology Foundation of Guizhou Province(J[2015]2025 and J[2015]2026), 125 Special Major Science and Technology of Department of Education of Guizhou Province ([2012]011) and Natural Science Foundation of the Education Department of Guizhou Province(KY[2015]482). E-mail: xcj403@126.com

C. Xu is with the Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550004, PR China e-mail: xcj403@126.com.

P. Li is with School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, PR China e-mail: lpllpl_lpl@163.com

In 2015, Zhou and Zhang [31] investigated the global exponential stability of the following cellular neural networks with multi-proportional delays

$$\begin{cases} \dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n}[a_{ij}f_j(x_j(t)) + b_{ij}g_j(x_j(q_1 t)) \\ \qquad\qquad + c_{ij}h_j(x_j(q_2 t))] + I_i, \\ x_i(s) = x_{i0}, s \in [q,1], \end{cases} \tag{1}$$

where $i = 1,2,\cdots,n, t > 1$, $x_i(t)$ stands for the state variable of the ith-cell at time t, $d_i > 0$ is a constant, a_{ij}, b_{ij} and c_{ij} represent the connection weights between the ith-cell and the jth-cell at time $t, q_1 t, q_2 t$, respectively. q_1 and q_2 are proportional delay factors and satisfy $0 < q_1 < q_2 \le 1, q = \min\{q_1, q_2\}, q_1 t = t-(1-q_1)t, q_2 t = t-(1-q_2 t)t$ in which $(1-q_1)t$ and $(1-q_2)t$ denote the transmission delays, $(1-q_1)t \to +\infty, (1-q_2)t \to +\infty$ as $t \to +\infty$, x_{i0} is a constant which denotes the initial value of $x_i(t)$ at $t \in [q,1]$ and $x(0) = (x_{10}, x_{20}, \cdots, x_{n0})^T$, $I_i(t)$ is the external input, $f_i(.), g_i(.)$ and $h_i(.)$ are the nonlinear activation functions, and satisfy the following conditions:

$$\begin{cases} f_i(.), g_i(.), h_i(.) : \mathbb{R} \to \mathbb{R}, \\ |f_i(u) - f_i(v)| \le L_i|u-v|, |f_i(u)| \le q_i, \\ |g_i(u) - g_i(v)| \le M_i|u-v|, |g_i(u)| \le r_i, \\ |h_i(u) - h_i(v)| \le N_i|u-v|, |f_i(u)| \le s_i, \end{cases} \tag{2}$$

where $i = 1,2,\cdots,n, u, v \in \mathbb{R}$ and L_i, M_i, N_i, q_i, r_i and s_i are non-negative constants. By applying Brouwer fixed point theorem and constructing the delay differential inequality, Zhou and Zhang [31] obtained some delay-independent and delay-dependent sufficient conditions to ensure the existence, uniqueness and global exponential stability of equilibrium of

(Advance online publication: 26 August 2016)

system (1). Moreover, the exponentially convergent rate is estimated.

In this paper, we will further investigate the exponential stability of the following system

$$\begin{cases} \dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n}[a_{ij} f_j(x_j(t)) + b_{ij} g_j(x_j(\tilde{q}t)) \\ \qquad + c_{ij} h_j(x_j(\tilde{q}t))] + I_i, \\ x_i(s) = x_{i0}, s \in [\tilde{q}t_0, t_0], \end{cases} \tag{3}$$

which is a revised version of system (1). Here for simplification, we let $q_1 = q_2 = \tilde{q}$ in system (1), t_0 is a constant. If $t_0 = 0, s \in [\tilde{q}t_0, t_0]$ is equal to $s = t_0 = 0$. Different from the work of Zhou and Zhang [31], we will obtain some sufficient conditions to ensure the pth exponential stability of system (3) by applying matrix measure and generalized Halanay inequality. The results of this paper are completely new and complement those of the previous studies in [31]. The approach is new.

The organization of the rest of this paper is as follows. In Section 2, some preliminaries are presented. In Section 3, some sufficient conditions are derived for the exponential stability of (3) by matrix measure and generalized Halanay inequality. In Section 4, we present three examples to illustrate the feasibility and effectiveness of our main theoretical findings in previous sections. A brief conclusion is drawn in Section 5.

II. Preliminary results

First we give some notations. Let

$$x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T,$$
$$D = \text{diag}\{d_1, d_2, \cdots, d_n\},$$
$$A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}, C = (c_{ij})_{n\times n},$$
$$f(x(t)) = (f_1(x_1(t)), f_2(x_1(t)), \cdots, f_n(x_1(t))^T,$$
$$g(x(t)) = (g_1(x_1(t)), g_2(x_1(t)), \cdots, g_n(x_1(t))^T,$$
$$h(x(t)) = (h_1(x_1(t)), h_2(x_1(t)), \cdots, h_n(x_1(t))^T,$$

$I = (I_1, I_2, \cdots, I_n)^T$. Then system (3) can be rewritten as follows:

$$\dot{x}(t) = -Dx(t) + [Af(x(t)) + Bg(x(\tilde{q}t)) + Ch(x(\tilde{q}t))] + I. \tag{4}$$

Using the transformation

$$y_i(t) = x_i(e^t), i = 1, 2, \cdots, n, \tag{5}$$

and letting $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T, \tau = -\ln \tilde{q} > 0$, we have

$$\begin{aligned} \dot{y}(t) = e^t\{&-Dy(t) + [Af(y(t)) + Bg(y(t-\tau)) \\ &+ Ch(y(t-\tau))] + I\}, \end{aligned} \tag{6}$$

with the initial condition

$$y_i(s) = \varphi_i(s), t_0 - \tau \le s \le t_0, i = 1, 2, \cdots, n, \tag{7}$$

where $\varphi_i(s) \in C([t_0 - \tau, t_0], \mathbb{R})$ is a continuous function.

In addition, we need the following definitions and lemmas.

Definition 2.1 *An equilibrium point $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ of the system (8) is said to be pth ($p = 1, 2, \infty$) globally exponentially stable, if there exist two positive constants $M > 0$ and $\lambda > 0$ such that $||x(t, t_0, x_0) - x^*||_p \le M||x_0 - x^*||_p e^{-\lambda t}$ holds, where $x_0 = (x_{10}, x_{20}, \cdots, x_{n0})^T$ is the initial condition of the system (8), $x(t, t_0, x_0)$ is the solution of system (8).*

Definition 2.2 ([32]) *For any real matrix $A = (a_{ij})_{n\times n}$, its matrix measure is defined as*

$$\mu_p(A) = \lim_{\varepsilon\to 0^+} \frac{||E + \varepsilon A||_p - 1}{\varepsilon},$$

where $||.||_p$ denotes the matrix norm in $\mathbb{R}^{n\times n}$, E is the identity matrix, $p \in \{1, 2, \infty\}$.

Let the matrix norm be as follows:

$$||A||_1 = \max_j \left\{\sum_{i=1}^{n} |a_{ij}|\right\}, ||A||_2 = \sqrt{\lambda_{\max}(A^T A)},$$
$$||A||_\infty = \max_I \left\{\sum_{J=1}^{n} |a_{ij}|\right\}.$$

Then we get

$$\mu_1(A) = \max_j \left\{a_{jj} + \sum_{i=1, i\ne j}^{n} |a_{ij}|\right\},$$
$$\mu_2(A) = \frac{1}{2}\lambda_{\max}(A^T + A),$$
$$\mu_\infty(A) = \max_i \left\{a_{ii} + \sum_{ji=1, j\ne i}^{n} |a_{ij}|\right\}.$$

Lemma 2.1 ([32]) *For the definition of matrix measure, for any $A, B \in \mathbb{R}^{n\times n}, p = 1, 2, \infty$, we have*
(1) $-||A||_p \le \mu_p(A) \le ||A||_p$; (2) $\mu_p(\alpha A) = \alpha\mu_p(A), \forall \alpha > 0$; (3) $\mu_p(A + B) \le \mu_p(A) + \mu_p(B)$.

Lemma 2.2 ([33], Ageneralized Halanay's inequality) *Suppose*

$$\dot{x}(t) \le \gamma(t) - \alpha(t)x(t) + \beta(t) \sup_{t-\tau\le\sigma\le t} x(\sigma)$$

holds for any $t \ge t_0$. Here $\tau \ge 0$, and $\gamma(t), \alpha(t), \beta(t)$ are continuous functions such that $0 \le \gamma(t) \le \gamma^*, \alpha(t) \ge \alpha_0, 0 \le \beta(t) \le \tilde{q}\alpha(t)$ for any $t \ge t_0$ with constants $\gamma^* > 0, \alpha_0 > 0, 0 \le q < 1$. Then we have

$$x(t) \le \frac{\gamma^*}{(1-\tilde{q})\alpha_0} + Ge^{-\mu^*(t-t_0)}$$

holds for $t \ge t_0$. Here $G = \sup_{t_0-\tau\le t\le t_0} x(t)$ and $\mu^* > 0$ is defined as

$$\mu^* = \inf_{t\ge t_0}\{\mu(t) : \mu(t) - \alpha(t) + \beta(t)e^{\tau\mu(t)} = 0\}.$$

In Lemma 2.2, Letting $\gamma(t) = 0, \gamma^* \to 0$, then we can obtain the following lemma.

Lemma 2.3 *Suppose*

$$\dot{x}(t) \le -\alpha(t)x(t) + \beta(t) \sup_{t-\tau\le\sigma\le t} x(\sigma)$$

holds for any $t \ge t_0$. Here $\tau \ge 0$, and $\alpha(t), \beta(t)$ are continuous functions such that $\alpha(t) \ge \alpha_0, 0 \le \beta(t) \le q\alpha(t)$

for any $t \geq t_0$ with constants $\alpha_0 > 0, 0 \leq q < 1$. Then we have

$$x(t) \leq G e^{-\mu^*(t-t_0)}$$

for $t \geq t_0$, where $G = \sup_{t_0-\tau \leq t \leq t_0} x(t)$ and $\mu^* > 0$ is defined as

$$\mu^* = \inf_{t \geq t_0} \{\mu(t) : \mu(t) - \alpha(t) + \beta(t) e^{\tau\mu(t)} = 0\}.$$

In order to obtain the main results, we make the following assumptions.

(H1) For $i = 1, 2, \cdots, n$, there exist positive constants α_i, β_i and γ_i such that $|f_i(u) - f_i(v)| \leq \alpha_i |u - v|, |g_i(u) - g_i(v)| \leq \beta_i |u - v|, |h_i(u) - h_i(v)| \leq \gamma_i |u - v|$ for all $u, v \in \mathbb{R}$.

Denote

$$\alpha = \max_{1 \leq i \leq n} \{\alpha_i\}, \beta = \max_{1 \leq i \leq n} \{\beta_i\}, \gamma = \max_{1 \leq i \leq n} \{\gamma_i\}.$$

III. Main results

In this section, we consider the global exponential stability of system (4) by applying the matrix norm and matrix measure.

Theorem 3.1 *Under the condition (H1), let $\Theta_1 = -\mu_p(-D)$ and $\Theta_2 = \alpha ||A||_p + \beta ||B||_p + \gamma ||C||_p$. If $\Theta_1 > \Theta_2 > 0$, then the equilibrium point y^* of system (4) is pth globally exponentially stable.*

Proof Assume that $y^* = (y_1^*, y_2^*, \cdots, y_n^*)^T$ is the equilibrium point of system (6), then y^* satisfies

$$-Dy^* + [Af(y^*) + Bg(y^*) + Ch(y^*)] + I = 0.$$

Let $u(t) = y(t) - y^*$, then

$$\begin{aligned} \dot{u}(t) = e^t \{ &-Du(t) + A[f(y(t)) - f(y^*)] \\ &+B[g(y(t-\tau)) - g(y^*)] \\ &+C[h(y(t-\tau)) - h(y^*)]\}. \end{aligned} \quad (8)$$

Consider the following nonnegative function

$$V(t) = ||u(t)||_p. \quad (9)$$

Calculating the derivative of $V(t)$ along the trajectories of (9) leads to

$$\begin{aligned} D^+V(t) &= \lim_{\epsilon \to 0^+} \frac{||(u(t+\epsilon)||_p - ||u(t)||_p}{\epsilon} \\ &= \lim_{\epsilon \to 0^+} \frac{||(u(t) + \epsilon \dot{u}(t) + o(\epsilon)||_p - ||u(t)||_p}{\epsilon} \\ &\leq \lim_{\epsilon \to 0^+} \frac{E + \epsilon e^t(-D)||_p - 1}{\epsilon} ||u(t)||_p \\ &\quad + e^t \{A[f(y(t)) - f(y^*)] \\ &\quad + B[g(y(t-\tau)) - g(y^*)] \\ &\quad + C[h(y(t-\tau)) - h(y^*)]\}. \end{aligned} \quad (10)$$

In view of (H1), we get

$$\left\{ \begin{array}{l} ||f(y(t)) - f(y^*)||_p \leq \alpha ||u(t)||_p, \\ ||g(y(t-\tau)) - g(y^*)||_p \leq \beta ||u(t-\tau)||_p, \\ ||h(y(t-\tau)) - h(y^*)||_p \leq \gamma ||u(t-\tau)||_p. \end{array} \right. \quad (11)$$

It follows from (10) and (11) that

$$\begin{aligned} D^+V(t) &\leq \mu_p(-e^t D) ||u(t)||_p + e^t (\alpha ||A||_p \\ &\quad + \beta ||B||_p + \gamma ||C||_p) ||u(t-\tau)||_p \\ &\leq -\Theta_1 e^t ||u(t)||_p \\ &\quad + \Theta_2 e^t \sup_{t-\tau \leq s \leq t} ||u(s)||_p. \end{aligned} \quad (12)$$

By Lemma 2.3, we have

$$V(t) \leq \sup_{t_0-\tau \leq s \leq t_0} V(s) e^{-\mu^*(t-t_0)}, \quad (13)$$

where

$$\mu^* = \inf_{t \geq t_0} \{\mu(t) : \mu(t) - \Theta_1 e^t + \Theta_2 e^{t + \tau\mu(t)} = 0\} > 0.$$

Then the zero solution of system (8) is pth globally exponentially stable, i.e., the equilibrium point y^* of system (4) is pth globally exponentially stable. The proof of Theorem 3.1 is complete.

IV. Examples

In this section, we give three examples to illustrate our main results derived in previous sections. Consider the following three cellular neural networks with proportional delays

Example 4.1 Consider the following cellular neural networks with proportional delays

$$\left\{ \begin{aligned} \dot{x}_1(t) &= -d_1 x_i(t) + \sum_{j=1}^{3} [a_{1j} f_j(x_j(t)) + b_{1j} g_j(x_j(\tilde{q}t)) \\ &\quad + c_{1j} h_j(x_j(\tilde{q}t))] + I_1, \\ \dot{x}_2(t) &= -d_2 x_2(t) + \sum_{j=1}^{2} [a_{2j} f_j(x_j(t)) + b_{2j} g_j(x_j(\tilde{q}t)) \\ &\quad + c_{2j} h_j(x_j(\tilde{q}t))] + I_2, \\ \dot{x}_3(t) &= -d_3 x_3(t) + \sum_{j=1}^{3} [a_{3j} f_j(x_j(t)) + b_{3j} g_j(x_j(\tilde{q}t)) \\ &\quad + c_{3j} h_j(x_j(\tilde{q}t))] + I_3, \end{aligned} \right. \quad (14)$$

where

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.4 & 0.4 \\ 0.1 & 0.8 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{bmatrix},$$

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.6 & 0.1 \\ 0.8 & 0.3 & 0.5 \\ 0.5 & 0.4 & 0.2 \end{bmatrix},$$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.6 & 0.1 \\ 0.5 & 0.4 & 0.6 \\ 0.1 & 0.6 & 0.5 \end{bmatrix},$$

$$\cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.7 \end{bmatrix}, f_i(x) = \frac{1}{2}(|x+1| - |x_1|),$$

$$g_i(x) = \tanh(\frac{5}{7}x), h_i(x) = \tanh(\frac{2}{5}x)(i = 1, 2, 3).$$

Set $\bar{q} = 0.5$ Then $\alpha = \alpha_i = 1, \beta = \beta_i = \frac{5}{7}, \gamma = \gamma_i = \frac{2}{5}$. It is easy to verify that $\Theta_1 = -\mu_p(-D) = 9.0703$ and

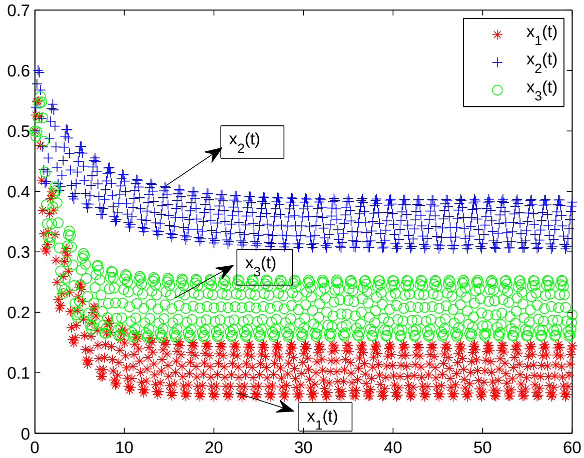

Fig. 1. Transient response of state variables $x_1(t), x_2(t)$ and $x_3(t)$.

$\Theta_2 = \alpha||A||_p + \beta||B||_p + \gamma||C||_p = 6.3423$. It follows that $\Theta_1 > \Theta_2 > 0$. Then all the conditions (H1)-(H2) of Theorem 3.1 hold. Thus system (14) has a unique equilibrium $(0.0921, 0.3532, 0.2036)$ which is globally exponentially stable. The results are illustrated in Fig. 1.

Example 4.2 Consider the following cellular neural networks with proportional delays

$$\begin{cases} \dot{x}_1(t) = -d_1 x_i(t) + \sum_{j=1}^{3}[a_{1j} f_j(x_j(t)) + b_{1j} g_j(x_j(\tilde{q}t)) \\ \qquad + c_{1j} h_j(x_j(\tilde{q}t))] + I_1, \\ \dot{x}_2(t) = -d_2 x_2(t) + \sum_{j=1}^{3}[a_{2j} f_j(x_j(t)) + b_{2j} g_j(x_j(\tilde{q}t)) \\ \qquad + c_{2j} h_j(x_j(\tilde{q}t))] + I_2, \\ \dot{x}_3(t) = -d_3 x_3(t) + \sum_{j=1}^{3}[a_{3j} f_j(x_j(t)) + b_{3j} g_j(x_j(\tilde{q}t)) \\ \qquad + c_{3j} h_j(x_j(\tilde{q}t))] + I_3, \end{cases} \tag{15}$$

where

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.7 \end{bmatrix},$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.5 & 0.7 \\ 0.2 & 0.3 & 0.5 \\ 0.7 & 0.4 & 0.8 \end{bmatrix},$$
$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.3 & 0.7 \\ 0.3 & 0.8 & 0.7 \\ 0.8 & 0.1 & 0.5 \end{bmatrix},$$
$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.9 & 0.4 \\ 0.7 & 0.3 & 0.4 \\ 0.8 & 0.9 & 0.9 \end{bmatrix},$$
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.9 \\ 0.2 \end{bmatrix}, f_i(x) = \frac{1}{2}(|x+1| - |x_1|),$$
$$g_i(x) = \tanh(\frac{7}{8}x), h_i(x) = \tanh(\frac{5}{9}x)(i = 1, 2, 3).$$

Set $\bar{q} = 0.3$ Then $\alpha = \alpha_i = 1, \beta = \beta_i = \frac{7}{8}, \gamma = \gamma_i = \frac{5}{9}$. It is easy to verify that $\Theta_1 = -\mu_p(-D) = 14.5008$ and

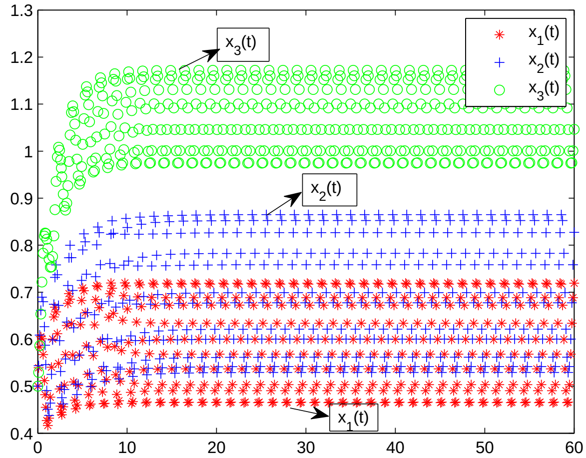

Fig. 2. Transient response of state variables $x_1(t), x_2(t)$ and $x_3(t)$.

$\Theta_2 = \alpha||A||_p + \beta||B||_p + \gamma||C||_p = 9.8402$. It follows that $\Theta_1 > \Theta_2 > 0$. Then all the conditions (H1)-(H2) of Theorem 3.1 hold. Thus system (15) has a unique equilibrium $(0.5902, 0.7023, 1.0931)$ which is globally exponentially stable. The results are illustrated in Fig. 2.

Example 4.3 Consider the following cellular neural networks with proportional delays

$$\begin{cases} \dot{x}_1(t) = -d_1 x_i(t) + \sum_{j=1}^{3}[a_{1j} f_j(x_j(t)) + b_{1j} g_j(x_j(\tilde{q}t)) \\ \qquad + c_{1j} h_j(x_j(\tilde{q}t))] + I_1, \\ \dot{x}_2(t) = -d_2 x_2(t) + \sum_{j=1}^{3}[a_{2j} f_j(x_j(t)) + b_{2j} g_j(x_j(\tilde{q}t)) \\ \qquad + c_{2j} h_j(x_j(\tilde{q}t))] + I_2, \\ \dot{x}_3(t) = -d_3 x_3(t) + \sum_{j=1}^{3}[a_{3j} f_j(x_j(t)) + b_{3j} g_j(x_j(\tilde{q}t)) \\ \qquad + c_{3j} h_j(x_j(\tilde{q}t))] + I_3, \end{cases} \tag{16}$$

where

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.9 & 0.3 \\ 0.5 & 0.8 & 0.4 \\ 0.7 & 0.4 & 0.7 \end{bmatrix},$$
$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.5 & 0.5 \\ 0.8 & 0.5 & 0.4 \\ 0.8 & 0.7 & 0.6 \end{bmatrix},$$
$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.7 & 0.7 \\ 0.3 & 0.5 & 0.8 \\ 0.7 & 0.5 & 0.6 \end{bmatrix},$$
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.3 \\ 0.8 \end{bmatrix}, f_i(x) = \frac{1}{2}(|x+1| - |x_1|),$$
$$g_i(x) = \tanh(\frac{5}{6}x), h_i(x) = \tanh(\frac{3}{4}x)(i = 1, 2, 3).$$

Set $\bar{q} = 0.2$ Then $\alpha = \alpha_i = 1, \beta = \beta_i = \frac{5}{6}, \gamma = \gamma_i = \frac{3}{4}$. It is easy to verify that $\Theta_1 = -\mu_p(-D) = 11.2232$ and

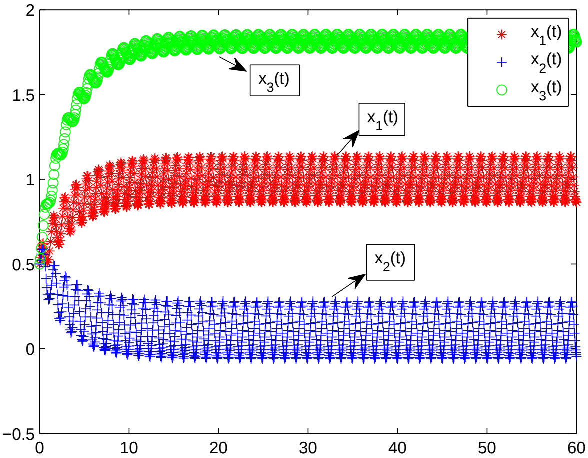

Fig. 3. Transient response of state variables $x_1(t), x_2(t)$ and $x_3(t)$.

$\Theta_2 = \alpha||A||_p + \beta||B||_p + \gamma||C||_p = 8.7865$. It follows that $\Theta_1 > \Theta_2 > 0$. Then all the conditions (H1)-(H2) of Theorem 3.1 hold. Thus system (16) has a unique equilibrium $(0.9034, 0.3215, 1.7213)$ which is globally exponentially stable. The results are illustrated in Fig. 3.

V. Conclusions

In this paper, we have investigated the global exponential stability of cellular neural networks with proportional delays. Applying matrix measure and generalized Halanay inequality, a series of new sufficient conditions to guarantee the pth exponential stability of cellular neural networks with proportional delays are established. The obtained conditions are easily to check in practice. Finally, three examples are included to illustrative the feasibility and effectiveness. To the best of our knowledge, there are no results on the anti-periodic solution and synchronization for cellular neural networks with proportional delays, which might be our future research topic.

Acknowledgment

The authors would like to thank the anonymous referees for their helpful comments and valuable suggestions, which led to the improvement of the manuscript.

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