Almost Periodic Solutions for Lasota-Wazewska Model with Multiple Delays

Liyan Pang, Yongzhi Liao and Tianwei Zhang

***Abstract*—By means of Mawhin's continuation theorem of coincidence degree theory, some new and simple sufficient conditions are obtained for the existence of at least one positive almost periodic solution for a class of delayed Lasota-Wazewska model with nonnegative coefficients. Further, by some important inequalities and Lyapunov functional, the permanence and global asymptotical stability of the model have been studied. The main result of this paper improves some conditions of the result in [Z.D. Huang, S.H. Gong, L.J. Wang, Positive almost periodic solution for a class of Lasota-Wazewska model with multiple timing-varing delays, Comput. Math. Appl. 61 (2011) 755-760]. Two examples and numerical simulations are employed to illustrate the main result in this paper.**

***Index Terms*—Positive almost periodic solution; Coincidence degree; Lasota-Wazewska model; Multiple delays.**

I. Introduction

In 1999, Gopalsamy and Trofimchuk [1] studied the existence of an almost periodic solution of the Lasota-Wazewska-type delay differential equation:

$$\dot{x}(t) = -a(t)x(t) + b(t)e^{-rx(t-\tau)}, \tag{1.1}$$

which was used by Wazewska-Czyzewska and Lasota [2] as a model for the survival of red blood cells in an animal. In [1], the authors proved that Eq. (1.1) has a globally attractive almost periodic solution.

In recent years, Huang et al. [3] considered the following Lasota-Wazewska model with multiple time-varying delays:

$$\dot{x}(t) = -a(t)x(t) + \sum_{i=1}^{m} b_i(t)e^{-r_i(t)x(t-\tau_i(t))}. \tag{1.2}$$

The authors employed the contraction mapping principle to obtain a positive almost periodic solution of Eq. (1.2) with $a^- > 0$ and $b_i^- > 0$, $i = 1, 2, \ldots, m$.

In real life, periodic oscillation is more common [4-6]. In applications, if the various constituent components of the temporally nonuniform environment is with incommensurable periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity. In recent years, the almost periodic solution of the continuous models in biological populations has been studied extensively (see [7-10] and the references cited therein). To the best of the author's knowledge, so far, there are scarcely any papers concerning with the existence of positive almost periodic solutions of Eq. (1.2) by using Mawhin's continuation theorem.

Motivated by the above reason, the main purpose of this paper is to establish some new sufficient conditions on the existence of positive almost periodic solutions of Eq. (1.2) by using Mawhin's continuous theorem of coincidence degree theory. The main result of this paper improves some conditions of the result in [3].

Related to a continuous bounded function f, we use the following notations: $f^- = \inf_{s\in\mathbb{R}} f(s)$, $f^+ = \sup_{s\in\mathbb{R}} f(s)$, $\bar{f} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s$. Throughout this paper, we always make the following assumption for Eq. (1.2):

(H_1) All the coefficients of Eq. (1.2) are nonnegative almost periodic functions with $\bar{a} > 0$ and $\bar{b}_i > 0$, $i = 1, 2, \ldots, m$.

II. Preliminaries

Definition 1. ([11, 12]) $x \in C(\mathbb{R})$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $|x(t+\tau) - x(t)| < \epsilon$, $\forall t \in \mathbb{R}$. The collection of those functions is denoted by $AP(\mathbb{R})$.

Lemma 1. ([8]) *Assume that $x \in AP(\mathbb{R}) \cap C^1(\mathbb{R})$ with $\dot{x} \in C(\mathbb{R})$. For arbitrary interval $[a, b]$ with $b - a = \omega > 0$, let $\xi, \eta \in [a, b]$ and*

$$I = \big\{s \in [\xi, b] : \dot{x}(s) \ge 0\big\}, \quad J = \big\{s \in [\eta, b] : \dot{x}(s) \le 0\big\},$$

then ones have

$$x(t) \le x(\xi) + \int_I \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\xi, b],$$

$$x(t) \ge x(\eta) + \int_J \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\eta, b].$$

Lemma 2. ([8]) *If $x \in AP(\mathbb{R})$, then for arbitrary interval $I = [a, b]$ with $b - a = \omega > 0$, there exist $\xi \in [a, b]$, $\underline{\xi} \in (-\infty, a]$ and $\bar{\xi} \in [b, +\infty)$ such that*

$$x(\underline{\xi}) = x(\bar{\xi}) \quad \textit{and} \quad x(\xi) \le x(s), \ \forall s \in [\underline{\xi}, \bar{\xi}].$$

Lemma 3. ([8]) *If $x \in AP(\mathbb{R})$, then for arbitrary interval $[a, b]$ with $I = b - a = \omega > 0$, there exist $\eta \in [a, b]$, $\underline{\eta} \in (-\infty, a]$ and $\bar{\eta} \in [b, +\infty)$ such that*

$$x(\underline{\eta}) = x(\bar{\eta}) \quad \textit{and} \quad x(\eta) \ge x(s), \ \forall s \in [\underline{\eta}, \bar{\eta}].$$

Lemma 4. ([8]) *If $x \in AP(\mathbb{R})$, then for $\forall n \in \mathbb{N}^+$, there exists $\alpha_n \in \mathbb{R}$ such that $x(\alpha_n) \in [x^* - \frac{1}{n}, x^*]$, where $x^* = \sup_{s\in\mathbb{R}} x(s)$.*

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Lemma 5. ([8]) *Assume that $x \in AP(\mathbb{R})$ and $\bar{x} > 0$, then for $\forall t_0 \in \mathbb{R}$, there exists a positive constant T_0 independent of t_0 such that*

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(s)\,\mathrm{d}s \in \left[\frac{\bar{x}}{2}, \frac{3\bar{x}}{2}\right], \quad \forall T \geq T_0.$$

III. MAIN RESULTS

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin [13].

Lemma 6. ([13]) *Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a Fredholm mapping of index zero and $N : \mathbb{X} \to \mathbb{Y}$ be L-compact on $\bar{\Omega}$. If all the following conditions hold*:

(a) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0, 1)$;
(b) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;
(c) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, *where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is an isomorphism.*

Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

Theorem 1. *Assume that (H_1) holds, then* Eq. (1.2) *admits at least one positive almost periodic solution.*

Proof: Under the invariant transformation $x = e^u$, Eq. (1.2) reduces to

$$\dot{u}(t) = -a(t) + \sum_{i=1}^{m} \frac{b_i(t)}{e^{r_i(t)e^{u(t-\tau_i(t))}} e^{u(t)}}. \tag{3.0}$$

It is easy to see that if Eq. (3.0) has one almost periodic solution u, then $x = e^u$ is a positive almost periodic solution of Eq. (1.2). Therefore, to completes the proof it suffices to show that Eq. (3.0) has one almost periodic solution.

Take $\mathbb{X} = \mathbb{Y} = \mathbb{V}_1 \bigoplus \mathbb{V}_2$, where

$$\mathbb{V}_1 = \left\{u \in AP(\mathbb{R}) : \forall \varpi \in \Lambda(u), |\varpi| \geq \gamma\right\}, \quad \mathbb{V}_2 = \mathbb{R},$$

where γ is a given positive constant. Define the norm

$$\|u\|_{\mathbb{X}} = \sup_{s\in\mathbb{R}} |u(s)|, \quad \forall u \in \mathbb{X} = \mathbb{Y},$$

then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces with the norm $\|\cdot\|_{\mathbb{X}}$. Set

$$L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}, \quad Lu = \dot{u},$$

where $\mathrm{Dom}L = \{u \in \mathbb{X} : u \in C^1(\mathbb{R}), \dot{u} \in C(\mathbb{R})\}$ and

$$N : \mathbb{X} \to \mathbb{Y}, \quad Nu = \left[-a(t) + \sum_{i=1}^{m} \frac{b_i(t)}{e^{r_i(t)e^{u(t-\tau_i(t))}} e^{u(t)}}\right].$$

With these notations Eq. (3.0) can be written in the form

$$Lu = Nu, \quad \forall u \in \mathbb{X}.$$

It is not difficult to verify that $\mathrm{Ker}L = \mathbb{V}_2$, $\mathrm{Im}L = \mathbb{V}_1$ is closed in $\mathbb{Y}$ and $\dim \mathrm{Ker}L = 1 = \mathrm{codim}\,\mathrm{Im}L$. Similar to the proof of Lemma 2.12 in [5], L is a Fredholm mapping of index zero. Similar to the proof of Lemma 2.13 in [5], N is L-compact on $\bar{\Omega}$.

In order to apply Lemma 6, we need to search for an appropriate open-bounded subset Ω.

Corresponding to the operator equation $Lu = \lambda u$, $\lambda \in (0, 1)$, we have

$$\dot{u}(t) = \lambda\left[-a(t) + \sum_{i=1}^{m} \frac{b_i(t)}{e^{r_i(t)e^{u(t-\tau_i(t))}} e^{u(t)}}\right] \tag{3.1}$$

is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{u(t)} = \lambda\left[-a(t)e^{u(t)} + \sum_{i=1}^{m} \frac{b_i(t)}{e^{r_i(t)e^{u(t-\tau_i(t))}}}\right]. \tag{3.2}$$

Suppose that $u \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of Eqs. (3.1)-(3.2) for some $\lambda \in (0, 1)$.

By Lemma 4, there exists a sequence $\{\alpha_n : n \in \mathbb{N}^+\}$ such that

$$u(\alpha_n) \in \left[u^* - \frac{1}{n}, u^*\right], \quad u^* = \sup_{s\in\mathbb{R}} u(s), \quad n \in \mathbb{N}^+. \tag{3.3}$$

By (H_1) and Lemma 5, for $\forall t_0 \in \mathbb{R}$, there exists a constant $\omega \in [0, +\infty)$ independent of t_0 such that

$$\frac{1}{T}\int_{t_0}^{t_0+T} a(s)\,\mathrm{d}s \in \left[\frac{\bar{a}}{2}, \frac{3\bar{a}}{2}\right],$$

$$\frac{1}{T}\int_{t_0}^{t_0+T} b_i(s)\,\mathrm{d}s \in \left[\frac{\bar{b}_i}{2}, \frac{3\bar{b}_i}{2}\right], \quad \forall T \geq \omega, \tag{3.4}$$

where $i = 1, 2\ldots, m$.

For $\forall n_0 \in \mathbb{N}^+$, we consider $[\alpha_{n_0} - \omega, \alpha_{n_0}]$, where ω is defined as that in (3.4). By Lemma 2, there exist $\xi \in [\alpha_{n_0} - \omega, \alpha_{n_0}]$, $\underline{\xi} \in (-\infty, \alpha_{n_0} - \omega]$ and $\bar{\xi} \in [\alpha_{n_0}, +\infty)$ such that

$$u(\underline{\xi}) = u(\bar{\xi}), \quad u(\xi) \leq u(s), \quad \forall s \in \left[\underline{\xi}, \bar{\xi}\right]. \tag{3.5}$$

By (3.5), we obtain from Eq. (3.2) that

$$0 = \int_{\underline{\xi}}^{\bar{\xi}} \left[-a(s)e^{u(s)} + \sum_{i=1}^{m} \frac{b_i(s)}{e^{r_i(s)e^{u(s-\tau_i(s))}}}\right] \mathrm{d}s. \tag{3.6}$$

From (3.6), it follows from (3.4)-(3.5) that

$$\begin{aligned}\frac{\bar{a}}{2}e^{u(\xi)} &\leq \frac{1}{\bar{\xi} - \underline{\xi}}\int_{\underline{\xi}}^{\bar{\xi}} a(s)e^{u(s)}\,\mathrm{d}s \\ &= \frac{1}{\bar{\xi} - \underline{\xi}}\int_{\underline{\xi}}^{\bar{\xi}} \sum_{i=1}^{m} \frac{b_i(s)}{e^{r_i(s)e^{u(s-\tau_i(s))}}}\,\mathrm{d}s \\ &\leq \sum_{i=1}^{m} b_i^+,\end{aligned}$$

which implies that

$$e^{u(\xi)} \leq \sum_{i=1}^{m} \frac{2b_i^+}{\bar{a}}. \tag{3.7}$$

Let $I = \left\{s \in [\xi, \alpha_{n_0}] : \frac{\mathrm{d}}{\mathrm{d}s}e^{u(s)} \geq 0\right\}$. It follows from Eq. (3.2) that

$$\begin{aligned}\int_I \frac{\mathrm{d}}{\mathrm{d}s}e^{u(s)}\,\mathrm{d}s &\leq \int_I \sum_{i=1}^{m} b_i(s)\,\mathrm{d}s \leq \int_{\alpha_{n_0}-\omega}^{\alpha_{n_0}} \sum_{i=1}^{m} b_i(s)\,\mathrm{d}s \\ &\leq \sum_{i=1}^{m} b_i^+\omega. \end{aligned} \tag{3.8}$$

By Lemma 1, it follows from (3.7)-(3.8) that

$$e^{u(t)} \le e^{u(\xi)} + \int_I \frac{\mathrm{d}}{\mathrm{d}s} e^{u(s)}\,\mathrm{d}s \le \sum_{i=1}^m \frac{2b_i^+}{\bar{a}} + \sum_{i=1}^m b_i^+ \omega$$

for $\forall t \in [\xi, \alpha_{n_0}]$, which implies that

$$u(\alpha_{n_0}) \le \ln\left[\sum_{i=1}^m \frac{2b_i^+}{\bar{a}} + \sum_{i=1}^m b_i^+\omega\right] := f^+.$$

In view of (3.3), letting $n_0 \to +\infty$ in the above inequality leads to

$$u^* = \lim_{n_0 \to +\infty} u(\alpha_{n_0}) \le f^+.$$

On the other hand, by Lemma 4, there exists a sequence $\{\beta_n : n \in \mathbb{N}^+\}$ such that

$$u(\beta_n) \in \left[u_*, u_* + \frac{1}{n}\right], \quad u_* = \inf_{s\in\mathbb{R}} u(s), \quad n \in \mathbb{N}^+. \quad (3.9)$$

For $\forall n_0 \in \mathbb{N}^+$, we consider $[\beta_{n_0} - \omega, \beta_{n_0}]$, where ω is defined as that in (3.4). By Lemma 3, there exist $\eta \in [\beta_{n_0} - \omega, \beta_{n_0}]$, $\underline{\eta} \in (-\infty, \beta_{n_0} - \omega]$ and $\bar{\eta} \in [\beta_{n_0}, +\infty)$ such that

$$u(\underline{\eta}) = u(\bar{\eta}), \quad u(\eta) \ge u(s), \quad \forall s \in [\underline{\eta}, \bar{\eta}]. \quad (3.10)$$

By (3.10), we obtain from Eq. (3.1) that

$$0 = \int_{\underline{\eta}}^{\bar{\eta}} \left[-a(s) + \sum_{i=1}^m \frac{b_i(s)}{e^{r_i(s)e^{u(s-\tau_i(s))}} e^{u(s)}}\right] \mathrm{d}s. \quad (3.11)$$

From (3.11), it follows from (3.4) and (3.10) that

$$\begin{aligned} a^+ &\ge \frac{1}{\bar{\eta} - \underline{\eta}} \int_{\underline{\eta}}^{\bar{\eta}} a(s)\,\mathrm{d}s \\ &= \frac{1}{\bar{\eta} - \underline{\eta}} \int_{\underline{\eta}}^{\bar{\eta}} \sum_{i=1}^m \frac{b_i(s)}{e^{r_i(s)e^{u(s-\tau_i(s))}} e^{u(s)}}\,\mathrm{d}s \\ &\ge \sum_{i=1}^m \frac{\bar{b}_i}{2e^{r_i^+ e^{f^+}} e^{u(\eta)}}, \end{aligned}$$

which implies that

$$u(\eta) \ge \ln\left[\sum_{i=1}^m \frac{\bar{b}_i}{2a^+ e^{r_i^+ e^{f^+}}}\right]. \quad (3.12)$$

Let $J = \{s \in [\eta, \beta_{n_0}] : \dot{u}(s) \le 0\}$. It follows from Eq. (3.1) that

$$\begin{aligned} \int_J \dot{u}(s)\,\mathrm{d}s &\ge -\int_J \lambda a(s)\,\mathrm{d}s \ge -\int_{\beta_{n_0}-\omega}^{\beta_{n_0}} a(s)\,\mathrm{d}s \\ &\ge -a^+\omega. \end{aligned} \quad (3.13)$$

By Lemma 1, it follows from (3.12)-(3.13) that

$$\begin{aligned} u(t) &\ge u(\eta) + \int_J \dot{u}(s)\,\mathrm{d}s \\ &\ge \ln\left[\sum_{i=1}^m \frac{\bar{b}_i}{2a^+ e^{r_i^+ e^{f^+}}}\right] - a^+\omega := f^-, \quad \forall t \in [\eta, \beta_{n_0}], \end{aligned}$$

which implies that

$$u(\beta_{n_0}) \ge f^-.$$

In view of (3.9), letting $n_0 \to +\infty$ in the above inequality leads to

$$u_* = \lim_{n_0 \to +\infty} u(\beta_{n_0}) \ge f^-.$$

Set $C = |f^-| + |f^+| + 1$. Clearly, C is independent of $\lambda \in (0, 1)$. Let $\Omega = \{u \in \mathbb{X} : \|u\|_{\mathbb{X}} < C\}$. Therefore, Ω satisfies condition (a) of Lemma 6.

Now we show that condition (b) of Lemma 6 holds, i.e., we prove that $QNu \ne 0$ for all $u \in \partial\Omega \cap \mathrm{Ker}L = \partial\Omega \cap \mathbb{R}$. If it is not true, then there exists at least one constant vector $u \in \partial\Omega$ such that

$$0 = m\left[-a(t)u_0 + \sum_{i=1}^m b_i(t)e^{-r_i(t)u_0}\right].$$

Similar to the above argument it follows that

$$f^- \le u_0 \le f^+.$$

Then $u_0 \in \Omega \cap \mathbb{R}$. This contradicts the fact that $u_0 \in \partial\Omega$. This proves that condition (b) of Lemma 6 holds.

Finally, we will show that condition (c) of Lemma 6 is satisfied. Let us consider the homotopy

$$H(\iota, u) = \iota QNu + (1-\iota)\Phi u, \ \ (\iota, u) \in [0,1] \times \mathbb{R},$$

where

$$\Phi u = -\bar{a}u + \sum_{i=1}^m \bar{b}_i e^{-r_i^+ f^+}.$$

From the above discussion it is easy to verify that $H(\iota, u) \ne 0$ on $\partial\Omega \cap \mathrm{Ker}L$, $\forall \iota \in [0,1]$. Further, $\Phi u = 0$ has a solution:

$$u^* = \sum_{i=1}^m \frac{\bar{b}_i}{\bar{a}e^{r_i^+ f^+}} \in \Omega.$$

A direct computation yields

$$\deg\big(\Phi, \Omega \cap \mathrm{Ker}L, 0\big) = \mathrm{sign}\,|-\bar{a}| = -1.$$

By the invariance property of homotopy, we have

$$\deg\big(JQN, \Omega \cap \mathrm{Ker}L, 0\big) = \deg\big(\Phi, \Omega \cap \mathrm{Ker}L, 0\big) \ne 0,$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$. Obviously, all the conditions of Lemma 6 are satisfied. Therefore, Eq. (3.0) has at least one almost periodic solution, that is, Eq. (1.2) has at least one positive almost periodic solution. This completes the proof. ■

Corollary 1. *Assume that (H_1) holds. Suppose further that a and b_i of* Eq. (1.2) *are continuous nonnegative periodic functions with periods α and β_i, respectively, $i = 1, 2, \ldots, m$, then* Eq. (1.2) *has at least one positive almost periodic solution.*

Corollary 2. *Assume that (H_1) holds and all the coefficients of* Eq. (1.2) *are continuous nonnegative ω-periodic functions, then* Eq. (1.2) *has at least one positive ω-periodic solution.*

IV. Permanence

In this section, we establish a permanence result for system (1.2).

Let

$$x^* := \sum_{i=1}^m \frac{b_i^+}{a^-}, \quad x_* := \sum_{i=1}^m \frac{b_i^-}{a^+ e^{r_i^+ x^*}}.$$

Theorem 2. *Assume that* (H_1) *holds. Then Eq.* (1.2) *is permanent. That is, any positive solution* $x(t)$ *of Eq.* (1.2) *satisfies*

$$x_* \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le x^*.$$

Proof: Let $x(t)$ be any solution of Eq. (1.2). Then

$$\dot{x}(t) \le -a^- x(t) + \sum_{i=1}^{m} b_i^+. \tag{4.1}$$

From (4.1), it leads

$$x(t) \le x(0)e^{-a^- t} + \int_0^t e^{-a^-(t-s)} \sum_{i=1}^{m} b_i^+ \,\mathrm{d}s,$$

which implies that

$$\limsup_{t\to\infty} x(t) \le \sum_{i=1}^{m} \frac{b_i^+}{a^-} := x^*. \tag{4.2}$$

By (4.2), there exists a constant $\epsilon > 0$ small enough and a larger constant $T = T(\epsilon) > 0$ such that

$$x(t) \le x^* + \epsilon \quad \text{for } t \ge T. \tag{4.3}$$

In view of Eq. (1.2), it follows from (4.3) that

$$\dot{x}(t) \ge -a^+ x(t) + \sum_{i=1}^{m} b_i^- e^{-r_i^+(x^*+\epsilon)} \tag{4.4}$$

for $t \ge T$. By (4.4), we have

$$x(t) \ge x(T)e^{-a^+(t-T)} + \int_T^t e^{-a^+(t-s)} \sum_{i=1}^{m} b_i^- e^{-r_i^+(x^*+\epsilon)} \,\mathrm{d}s,$$

which implies that

$$\liminf_{t\to\infty} x(t) \ge \sum_{i=1}^{m} \frac{b_i^-}{a^+ e^{r_i^+(x^*+\epsilon)}}. \tag{4.5}$$

Letting $\epsilon \to 0$ in (4.5), we obtain the result of this theorem. This completes the proof. ■

V. Global asymptotical stability

The main result of this section concerns the global asymptotical stability of system (1.2).

Theorem 3. *Assume that* (H_1) *holds. Suppose further that*

(H_2) $\tau \in C^1(\mathbb{R})$, $\dot{\tau}^+ := \sup_{t\in\mathbb{R}} \dot{\tau}(t) < 1$, *and*

$$\Theta := a^- - \sum_{i=1}^{n} \frac{b_i^+ e^{-r_i^- x_*}}{1-\dot{\tau}_i{}^+} > 0.$$

Then system (1.2) *is globally asymptotically stable.*

Proof: Suppose that $x(t)$ and $y(t)$ are any two solutions of system (1.2).

Construct a Lyapunov functional as follows:

$$V(t) = V_1(t) + V_2(t), \quad \forall t \ge T_5,$$

where

$$V_1(t) = |x(t) - y(t)|,$$

$$V_2(t) = \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \frac{b_i^+ e^{-r_i^- x_*}}{1-\dot{\tau}_i{}^+} |x(s) - y(s)| \,\mathrm{d}s.$$

Calculating the upper right derivative of $V_1(t)$ along the solution of system(1.2), it follows that

$$\begin{aligned} D^+V_1(t) &= \operatorname{sgn}[x(t)-y(t)][\dot{x}(t)-\dot{y}(t)] \\ &\le -a^-|x(t)-y(t)| \\ &\quad + \sum_{i=1}^{n} b_i^+ e^{-r_i^- x_*} |x(t-\tau_i(t)) - y(t-\tau_i(t))|. \end{aligned}$$

Moreover, we obtain that

$$\begin{aligned} D^+V_2(t) &\le \sum_{i=1}^{n} \frac{b_i^+ e^{-r_i^- x_*}}{1-\dot{\tau}_i{}^+} |x(t)-y(t)| \\ &\quad - \sum_{i=1}^{n} b_i^+ e^{-r_i^- x_*} |x(t-\tau_i(t)) - y(t-\tau_i(t))|. \end{aligned}$$

From the above inequalities, one has

$$\begin{aligned} D^+V(t) &\le -\left[a^- - \sum_{i=1}^{n} \frac{b_i^+ e^{-r_i^- x_*}}{1-\dot{\tau}_i{}^+}\right] |x(t)-y(t)| \\ &= -\Theta|x(t)-y(t)|. \end{aligned} \tag{5.1}$$

Therefore, V is non-increasing. Integrating (5.1) from 0 to t leads to

$$V(t) + \Theta \int_0^t |x(t)-y(t)| \,\mathrm{d}s \le V(0) < +\infty, \quad \forall t \ge 0,$$

that is,

$$\int_0^{+\infty} |x(t)-y(t)| \,\mathrm{d}s < +\infty,$$

which implies that

$$\lim_{s\to+\infty} |x(t)-y(t)| = 0.$$

Thus, system (1.2) is globally asymptotically stable. This completes the proof. ■

Together with Theorem 1, we obtain

Theorem 4. *Assume that* (H_1)*-*(H_2) *hold, then system* (1.2) *has at least one positive almost periodic solution, which is globally asymptotically stable.*

Corollary 3. *Assume that* (H_1)*-*(H_2) *hold. Suppose further that all the coefficients in system* (1.2) *are* ω*-periodic functions, then system* (1.2) *has at least one positive* ω*-periodic solution, which is globally asymptotically stable.*

VI. Two examples and numerical simulations

Example 1. Consider the following Lasota-Wazewska model with multiple delays:

$$\begin{aligned} \dot{u}(t) &= -[2+|\sin(\sqrt{2}t)|]u(t) \\ &\quad +0.1\sin^2(\sqrt{3}t)e^{-|\cos t|u(t-|\sin t|)} \\ &\quad +0.1\cos^2(\sqrt{5}t)e^{-|\sin t|u(t-|\cos t|)}. \end{aligned} \tag{6.1}$$

Then Eq. (6.1) has at least one positive almost periodic solution, which is globally asymptotically stable.

Proof: Corresponding to Eq. (1.2), we have $a(s) = |\sin(\sqrt{2}s)|$, $b_1(s) = 10\sin^2(\sqrt{3}s)$, $b_2(s) = \cos^2(\sqrt{5}s)$, $r_1(s) = |\cos s| = \tau_2(s)$, $r_2(s) = |\sin s| = \tau_1(s)$, $\forall s \in \mathbb{R}$. It is easy to verify that (H_1) in Theorem 4 holds. By Theorem 4, Eq. (6.1) has at least one positive almost periodic solution (see Figure 1), which is globally asymptotically stable (see Figure 2). This completes the proof. ■

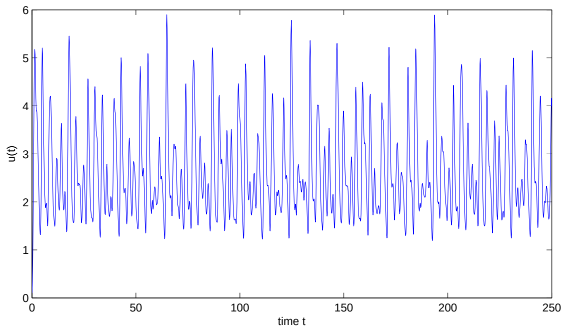

Fig. 1 State variable u of Eq. (4.1)

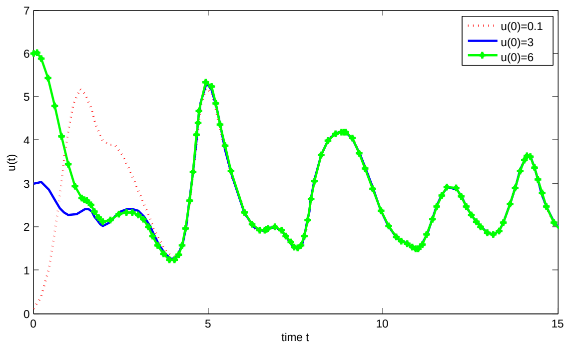

Fig. 2 Global asymptotical stability of Eq. (4.1)

Remark 1. In Eq. (6.1), $|\sin(\sqrt{2}t)|$ is $\frac{\sqrt{2}\pi}{2}$-periodic function and $\sin^2(\sqrt{3}t)$ is $\frac{\sqrt{3}\pi}{3}$-periodic function. So Eq. (6.1) is with incommensurable periods. Through all the coefficients of Eq. (6.1) are periodic functions, the positive periodic solutions of Eq. (6.1) could not possibly exist. However, by Theorem 1, the positive almost periodic solutions of Eq. (6.1) exactly exist.

Example 2. Consider the following almost periodic Lasota-Wazewska model with multiple delays:

$$\begin{aligned}\dot{u}(t) &= -(|\sin(\sqrt{2}t)| + |\sin(\sqrt{3}t)|)u(t) \\ &\quad +10\sin^2(\sqrt{3}t)e^{-|\cos t|u(t-1)} \\ &\quad +\cos^2(\sqrt{5}t)e^{-|\sin t|u(t-2)}. \qquad (6.2)\end{aligned}$$

In Eq. (6.2), $|\sin\sqrt{2}t|+|\sin\sqrt{3}t|$ is an almost periodic function, which is not periodic function. Similar to the argument as that in Example 6.1, it is easy to obtain that Eq. (6.2) admits at least one positive almost periodic solution (see Figure 3), which is globally asymptotically stable (see Figure 4).

Remark 2. Corresponding to Eq. (1.2), it is clear that all the coefficients of Eq. (6.2) is not eventually positive. Therefore, the result in [3] and the references therein cannot be applicable to prove the existence of positive almost periodic solutions of Eq. (6.2). This implies that the results of this paper are new and they complement previous results.

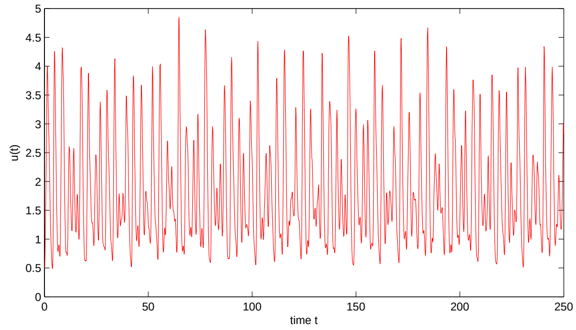

Fig. 3 State variable u of Eq. (4.2)

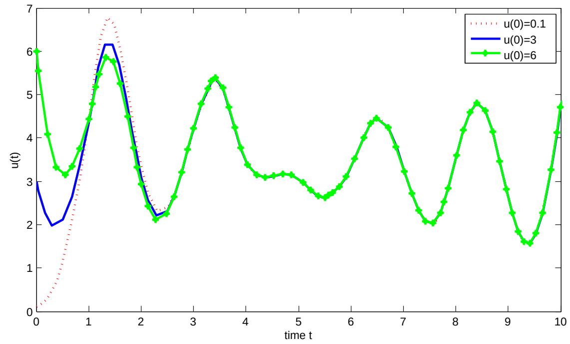

Fig. 4 Global asymptotical stability of Eq. (4.2)

VII. Conclusions

By using a fixed point theorem of coincidence degree theory, some criterions for the existence of positive almost periodic solution to a class of Lasota-Wazewska model with multiple delays are obtained. Theorems 1 gives the sufficient conditions for the existence of positive almost periodic solution of system (1.2). The results obtained in this paper improves some result in recent years. Therefore, The method used in this paper provides a possible method to study the existence of positive almost periodic solution of the models in biological populations.

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Author Contributions

L.P., Y.L. and T.Z. collectively carried out this study, collected data, and analyzed them. T.Z. and L.P. wrote the manuscript. Y.L. helped to revised the manuscript. All authors read and approved the final manuscript.

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