Fast Monte Carlo Simulation for Pricing Covariance Swap under Correlated Stochastic Volatility Models

Junmei Ma, Ping He

Abstract—**The modeling and pricing of covariance swap derivatives under correlated stochastic volatility models are studied. The pricing problem of the derivative under the case of discrete sampling covariance is mainly discussed, by efficient Control Variate Monte Carlo simulation. Based on the closed form solutions derived for approximate models with correlated deterministic volatility by partial differential equation method, two kinds of acceleration methods are therefore proposed when the volatility processes obey the Hull-White stochastic processes. Through analyzing the moments for the underlying processes, the efficient control volatility under the approximate model is constructed to make sure the high correlation between the control variate and the problem. The numerical results illustrate the high efficiency of the control variate Monte Carlo method; the results coincide with the theoretical results. The idea in the paper is also applicable for the valuation of other financial derivatives with discrete features under multi-factor models.**

Index Terms—**Covariance swap, Stochastic volatility, Monte Carlo, Control variate, Variance reduction.**

I. Introduction

MONTE Carlo method is a numerical method based on the probability theory. Monte Carlo method becomes more and more popular, as the rapid growth of the financial derivative markets and the increase of the complexity of the pricing models. The advantage of Monte Carlo method is that the efficiency of the method is independent of the number of state variables. If the number of state variables is greater than three, Monte Carlo method is suitably used to solve the high dimensional pricing problem and often becomes the only computationally feasible means of derivative pricing.

Suppose the price of the derivative $\mu = \mathrm{E}[V]$ will be estimated by mean value $\overline{V}_n = \frac{1}{n}\sum_{i=1}^{n} V_i$, where $\{V_i\}_{i=1}^{n}$ are independent and identical distribution $(i.i.d)$ samples of the random variable V. Then by the Central Limit Theorem, the price μ asymptotically falls into the interval

$$[\overline{V}_n - \frac{\sigma_n}{\sqrt{n}} Z_{\frac{\delta}{2}}, \quad \overline{V}_n + \frac{\sigma_n}{\sqrt{n}} Z_{\frac{\delta}{2}}],$$

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with probability $1 - \delta$, where σ_n is the standard deviation of estimation of V, δ is significance level and $Z_{\frac{\delta}{2}}$ is the quantile of standard normal distribution under $\frac{\delta}{2}$. The error is $\frac{\sigma_n}{\sqrt{n}} Z_{\frac{\delta}{2}}$, which just rely on the deviation σ and sample storage n.However, although Monte Carlo method is easy to implement and effective to solve the multi-dimensional problem, it's disadvantage is its slow convergence. The convergence rate of Monte Carlo method is $O(n^{-\frac{1}{2}})$. We have to spend 100 times as much as computer time in order to reduce the error by a factor of 10 [8].

Many financial practitioners and researchers concentrate on the problem of computational efficiency, several approaches to speed up Monte Carlo simulation, such as control variate, antithetic variables, importance sampling and stratification[7] have been proposed over the last few years. These techniques aim to reduce the variance per Monte Carlo simulation so that a given level of accuracy can be achieved with a smaller number of simulations. Control variate is one of the most widely used variance reduction techniques, mainly because of the simplicity of its implementations, and the fact that it can be accommodated in an existing Monte Carlo calculator with a small effort. This paper mainly studies the pricing of covariance derivatives under correlated stochastic volatility models through fast Monte Carlo simulation.

Different control variates are proposed to accelerate the convergence rate of the simulation errors, based on the closed form solutions of simplified models. The numerical experiments show the high efficiency of our acceleration methods, which can also be extended to the pricing of other path-dependent derivatives in an extended Black-Scholes framework, such as Asian options, Lookback options and other variance derivatives.

The covariance swap is a covariance forward contact of the underlying assets S_1 and S_2, and its payoff at expiration is equal to

$$P = M \times (Cov_R(S_1, S_2) - K_{cov}^2),$$

where K_{cov} is a strike price, M is the notional amount, and $Cov_R(S_1, S_2)$ is the realized covariance between two assets S_1 and S_2. Therefore, in the risk-neutral world, the fair strike covariance can be attained by $\mathrm{E}[Cov_R(S_1, S_2)]$. The procedure for calculating the realized covariance is usually clearly specified in the contract and includes details about the source and observation frequency of the price of the underlying assets, and the method to calculate the covariance.

The covariance swap is the main representation of the new generation of variance and volatility swap derivatives, and becoming very popular and actively traded in financial practice in recent years. Variance and volatility derivatives

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are based on a single underlying asset. Their payoffs just depend on the realized variance or volatility of the underlying asset or an index. The market for variance and covariance swaps has been growing, which are now actively quoted and used to speculate on future (co)variance levels, to trade the spread between implied and realized (co)variance levels, or to hedge the (co)variance exposures of other positions by many financial institutions. The objective of this paper is to demonstrate the modeling and the pricing of a covariance swap. This contract pays the excess of the realized covariance between two currencies over a constant specified at the outset of the contract. Such a contract is growing as a useful complement for the variance contracts that trade OTC on several currencies. By combining variance and covariance swaps, the realized variance of return on a portfolio of currencies can be locked in.

Logically, a correlation swap is a correlation forward contract of two underlying rates S_1 and S_2, whose payoff at expiration is equal to:

$$P = M \times (Corr_R(S_1, S_2) - K_{corr}),$$

where K_{corr} is a strike correlation, M is the notional amount, and $Corr_R(S_1, S_2)$ is the realized correlation of two underlying assets S_1 and S_2. Therefore the fair strike correlation is $E[Corr_R(S_1, S_2)]$. The correlation $Corr_R(S_1, S_2)$ can be presented by the covariance,

$$Corr_R(S_1, S_2) = \frac{Cov_R(S_1, S_2)}{\sigma_R(S_1)\sigma_R(S_2)}.$$

Therefore, the modeling and pricing of correlation swap will be attained with the research results of covariance swap.

The research on the pricing of variance swap and volatility swap is very massive and we do not intend to give a list here. We will give a brief introduction of the pricing of covariance swap and correlation swap. As these are two-dimensional path-dependent derivatives. Their pricing problems are more difficult than general variance swap or volatility swap which is only based on single underlying asset. Andrei Badescu etal [1] discussed the analytical pricing of pseudo-variance, pseudo-volatility, pseudo-covariance and pseudo-correlation swaps. Sebastien Bossu [2]and[3] from JPMorgan proposed a "toy model" for modeling and pricing correlation swaps on the components of an equity index and found that the fair strike of a correlation swap is approximately equal to a particular measure of implied correlation. Giovanni Salvi and Anatoliy Swishchuk[4] researches the modeling and pricing of covariance and correlation swaps with semi-Markov volatilities assumption. Da Foneseca et al.[5] solved a portfolio optimization problem in a market with risky assets and volatility derivatives to discuss the influence of variance and covariance swap in a market. J. Drissien etal [6] discussed the price of correlation risk for equity options. The research on the covariance swap and correlation swap is developed extensively due to its importance and broad applicability in risk-hedging, arbitrage and manage the risk of the fund in the financial transactions.

In this paper, by PDE method and Monte Carlo variance reduction techniques, the modeling and pricing of covariance swap derivatives under the correlated stochastic volatility structure are researched. Modeling and pricing of the derivatives are separately discussed under the cases of continuously and discretely sampled covariance. For the continuously sampled covariance derivatives, a closed form solution is derived directly by the PDE method. For the discretely sampled covariance derivatives, two kinds of fast Monte Carlo simulation methods are developed and studied. The discretely sampled covariance derivatives are more applicable in financial practice as the assets "covariance" cannot be observed directly in the market. The choice of efficient control variates is also discussed in the paper.

The rest of the paper is arranged as follows. We begin by modeling covariance swaps with the PDE method and Δ-hedging principle under the stochastic volatility assumption in Section 2. In Section 3, the first control variate method of Monte Carlo simulation for the valuation of the covariance swap is proposed. The analytical solution for the control variate is attained. Then, with the moment analysis for the underlying and auxiliary stochastic processes, an approximate "optimal volatility" is provided for developing the highly efficient control variate in this framework. In Section 4, the second improved control variate method is constructed based on the algorithm in Section 3 and the choice of the optimal control volatility constants is also considered. In Section 5, the computational and analytical results are illustrated and coincide with the theoretical analysis well. Concluding remarks are given in Section 6.

II. Modeling and Pricing of Covariance Swap

In this section, the partial differential equation pricing model for the covariance swap under the correlated stochastic volatility assumption is obtained. The variance products are all derivatives based on stochastic volatility model. The research on the stochastic volatility starts from the early 1980's. In 1973, Black and Scholes made a major breakthrough by deriving pricing formulas for vanilla options written on the stock. The Black-Scholes model assumes that the volatility term is a constant. This assumption is not always satisfied by real-life options as the probability distribution of an equity has a fatter left tail and thinner right tail than the lognormal distribution as in Hull [14], and the assumption of constant volatility in financial model is incompatible with derivatives prices observed in the market, verified by volatility smile. The concept of stochastic volatility was introduced by Hull and White (1987) [15], and subsequent developments include the work of Scott (1987) [19], Stein and Stein (1987)[16], Ball and Roma (1994) [17], and Heston (1993) [18]. They proposed and improved different stochastic volatility models to satisfy the various needs in financial practice.

The stochastic volatility model used in this paper is the Geometric Brownian Motion proposed by Hull and White in 1987. Under the martingale measure, the underlying assets and volatilities are assumed obeying the stochastic differential equations:

$$\frac{dS^{(1)}(t)}{S^{(1)}(t)} = rdt + \sigma_t^{(1)}dW_t^{(1)}, \sigma_t^{(1)} = \sqrt{Y^{(1)}(t)},$$

$$\frac{dY^{(1)}(t)}{Y^{(1)}(t)} = \mu^{(1)}dt + \hat{\sigma}^{(1)}dW_t^{(3)}. \quad (1)$$

$$\frac{dS^{(2)}(t)}{S^{(2)}(t)} = rdt + \sigma_t^{(2)}dW_t^{(2)}, \sigma_t^{(2)} = \sqrt{Y^{(2)}(t)},$$

$$\frac{dY^{(2)}(t)}{Y^{(2)}(t)} = \mu^{(2)}dt + \hat{\sigma}^{(2)}dW_t^{(4)}. \quad (2)$$

Where r is deterministic interest rate; $\mu^{(i)} > 0, (i = 1, 2)$ is the drift of the volatility; $\hat{\sigma}^{(i)} > 0, (i = 1, 2)$ is the volatility of volatility; $W_t^{(i)}$, $(i = 1, 2, 3, 4)$ are Wiener processes, and $Cov(dW_t^{(i)}, dW_t^{(j)}) = \rho_{ij}dt$. We further suppose that the market is no-arbitrage and no transaction costs. There are N observation dates $0 = T_0 < T_1 < T_2 <, \cdots, < T_N = T$. $S_i^{(1)} = S^{(1)}(T_i)$ and $S_i^{(2)} = S^{(2)}(T_i)$ are the asset price on the ith date, and $Y_i^{(1)} = Y^{(1)}(T_i)$ and $Y_i^{(2)} = Y^{(2)}(T_i)$ are the instantaneous variance at T_i. The payoff function for the covariance swap at the maturity T is

$$V|_{t=T} = M \times [\sum_{i=1}^{N} \frac{1}{T}(\ln(\frac{S_i^{(1)}}{S_{i-1}^{(1)}}))(\ln(\frac{S_i^{(2)}}{S_{i-1}^{(2)}})) - K_{cov}^2]$$
$$= h(S_0^{(1)}, S_0^{(2)}, S_1^{(1)}, S_1^{(2)}, \cdots, S_N^{(1)}, S_N^{(2)}). \quad (3)$$

For the case of continuously sampled covariance, the payoff function denoted by $W(.)$ for the covariance swap at the maturity T is

$$W|_{t=T} = M \times (\frac{1}{T}\int_0^T \rho_{12}\sqrt{Y_t^{(1)}}\sqrt{Y_t^{(2)}}dt - K_{cov}^2), \quad (4)$$

as the realized covariance of the two assets $S^{(1)}$ and $S^{(2)}$ is given by Giovanni S.etal. [4]

$$Cov_R(S^{(1)}, S^{(2)}) = \frac{1}{T}[\ln S_T^{(1)}, \ln S_T^{(2)}] = \frac{1}{T}\int_0^T \rho_{12}\sigma_t^{(1)}\sigma_t^{(2)}dt.$$

Then the pricing model for the price of the covariance swap with continuously sampled covariance will be established, and the closed form solution will be deduced.

Define a auxiliary state variable I_t to measure the accumulated covariance of $S^{(1)}$ and $S^{(2)}$,

$$I_t = \frac{\int_0^t \rho_{12}\sqrt{Y^{(1)}(s)}\sqrt{Y^{(1)}(s)}ds}{t}.$$

This state variable is known at time t and satisfies the ordinary differential equation,

$$\frac{dI_t}{dt} = \frac{(\rho_{12}\sqrt{Y^{(1)}(t)}\sqrt{Y^{(2)}(t)} - I_t)}{t}.$$

The price process of the derivative is denoted by $W = W(Y_t^{(1)}, Y_t^{(2)}, I_t, t)$. During the small time interval $(t, t + dt)$, by $It\hat{o}$ lemma, the change of the price of the covariance swap will be

$$dW = \frac{\partial W}{\partial t}dt + \frac{\partial W}{\partial Y^{(1)}}dY^{(1)} + \frac{\partial W}{\partial I}dI + \frac{1}{2}\frac{\partial^2 W}{\partial {Y^{(1)}}^2}d{Y^{(1)}}^2 +$$
$$\frac{1}{2}\frac{\partial^2 W}{\partial {Y^{(2)}}^2}d{Y^{(2)}}^2 + \rho_{34}\frac{\partial^2 W}{\partial Y^{(1)}\partial Y^{(2)}}dY^{(1)}dY^{(2)} + \frac{\partial W}{\partial Y^{(2)}}dY^{(2)}$$

As there is no default risk and based on risk neutral conditions, $E[dW] = rWdt$ will be satisfied. Then the partial differential equation of the price process $W(Y_t^{(1)}, Y_t^{(2)}, I_t, t)$ of the covariance swap with payoff (4) is attained as follows,

$$\begin{cases} \frac{\partial W}{\partial t} + \mu^{(1)}Y^{(1)}\frac{\partial W}{\partial Y^{(1)}} + \frac{1}{2}(\hat{\sigma}^{(1)})^2(Y^{(1)})^2\frac{\partial^2 W}{\partial {Y^{(1)}}^2} + \\ \quad \mu^{(2)}Y^{(2)}\frac{\partial W}{\partial Y^{(2)}} + \frac{1}{2}(\hat{\sigma}^{(2)})^2(Y^{(2)})^2\frac{\partial^2 W}{\partial {Y^{(2)}}^2} + \\ \rho_{34}\hat{\sigma}^{(1)}\hat{\sigma}^{(2)}Y^{(1)}Y^{(2)}\frac{\partial^2 W}{\partial Y^{(1)}\partial Y^{(2)}} + \frac{\partial W}{\partial I}\frac{\rho_{12}\sqrt{Y^{(1)}Y^{(2)}} - I}{t} \\ \quad -rW = 0, 0 < I, Y^{(1)}, Y^{(2)} < \infty, 0 \le t \le T, \\ W|_{t=T} = M(I_T - K_{cov}^2). \end{cases} \quad (5)$$

The solution of the above equation (5) has the following semi-linear expression,

$$W(Y^{(1)}, Y^{(2)}, I, t) = A(t;T)I + B(t;T)\sqrt{Y^{(1)}Y^{(2)}} + C(t;T),$$

where

$$A(t;T) = \frac{Mte^{-r(T-t)}}{T}, C(t;T) = -MK_{cov}^2 e^{-r(T-t)},$$
$$B(t;T) = \frac{M\rho_{12}[e^{-a(T-t)} - e^{-r(T-t)}]}{T(r-a)},$$

$$a = \frac{1}{8}(\hat{\sigma}^{(1)})^2 + \frac{1}{8}(\hat{\sigma}^{(2)})^2 - \frac{1}{4}\rho_{34}\hat{\sigma}^{(1)}\hat{\sigma}^{(2)} - \frac{1}{2}\mu^{(1)} - \frac{1}{2}\mu^{(2)} + r. \quad (6)$$

Therefore the price formula of this kind of covariance swap is obtained from the above price expression.

Theorem 2.1 The price of the covariance swap for the case of continuously sampled covariance with payoff (4) is

$$W|_{t=0} = \frac{M\rho_{12}\sqrt{Y_0^{(1)}Y_0^{(2)}}[e^{-aT} - e^{-rT}]}{T(r-a)} - K_{cov}^2 Me^{-rT},$$

where $K_{cov}, Y_0^{(1)}, Y_0^{(2)}, M, \mu, T, r$ and a are given as in equations (1), (2), (3)and (6).

For the case of discretely sampled covariance, the payoff function is expressed by equation (3). Then we will discuss the pricing model for the covariance derivative with PDE method. There are four sources of randomness about this problem. When constructing a risk-free portfolio, the derivatives cannot be perfectly hedged with just the underlying assets. Instead we need another two covariance swaps called V_t^{1*} and V_t^{2*} on the same underlying assets. Then during an arbitrary observation interval (T_{i-1}, T_i), we construct a riskless portfolio Π, containing the product V, the quantity Δ_1 of the underlying asset S_1, the quantity Δ_2 of the underlying asset S_2, the quantity Δ_3 of another traded covariance swap V_t^{1*} and the quantity Δ_4 of another traded covariance swap V_t^{2*}. By $It\hat{o}$ lemma and $\Delta-$hedging principle, choosing $\Delta_1, \cdots, \Delta_4$ to make Π risk-less during $[t, t + dt]$. Then the equation governing V can be written as in Jiang [20]

$$\mathcal{L}V = \frac{\partial V}{\partial t} + rS^{(1)}\frac{\partial V}{\partial S^{(1)}} + \mu^{(1)}Y^{(1)}\frac{\partial V}{\partial Y^{(1)}} + rS^{(2)}\frac{\partial V}{\partial S^{(2)}} + \mu^{(2)}Y^{(2)}\frac{\partial V}{\partial Y^{(2)}} - rV + \frac{1}{2}\sum_{i,j=1}^{4}\rho_{ij}a_i a_j Z_i Z_j\frac{\partial^2 V}{\partial Z_i \partial Z_j} = 0. \quad (7)$$

Where, for convenience with $a_1 = \sqrt{Y^{(1)}}, a_2 = \sqrt{Y^{(2)}}, a_3 = \hat{\sigma}^{(1)}, a_4 = \hat{\sigma}^{(2)}$ and $Z_1 = S_1, Z_2 = S_2, Z_3 = Y^{(1)}, Z_4 = Y^{(2)}$ in the last part of the above expression.

As the market is assumed no arbitrage, the price of the covariance swap should be continuous at observation date T_i. Take $V = V_i(i = 1, 2, \cdots, N), t \in (T_{i-1}, T_i)$, the partial differential equation pricing model for the covariance swap is therefore given by

$$\begin{cases} \mathcal{L}V_N = 0, 0 < S^{(1)}, S^{(2)}, Y^{(1)}, Y^{(2)} < \infty, \\ \quad T_{N-1} \le t < T_N = T, \\ V_N|_{t=T} = h(S_0^{(1)}, S_0^{(2)}, \cdots, S_{N-1}^{(1)}, S_{N-1}^{(2)}, S^{(1)}, S^{(2)}), \end{cases}$$

and

$$\begin{cases} \mathcal{L}V_i = 0, 0 < S^{(1)}, S^{(2)}, Y^{(1)}, Y^{(2)} < \infty, T_{i-1} \le t < T_i. \\ V_i|_{t=T_i} = V_{i+1}|_{t=T_i}, (i = 1, 2, \cdots, N-1). \end{cases} \quad (8)$$

If the analytical solution $V_N(S_0^{(1)}, S_0^{(2)}, S_1^{(1)}, S_1^{(2)} \cdots, S_{N-1}^{(1)}, S_{N-1}^{(2)}, S^{(1)}, S^{(2)}, t)$ is obtained from the first problem, then at $t = T_{N-1}$, $S^{(1)} = S_{N-1}^{(1)}, S^{(2)} = S_{N-1}^{(2)}$, and $V_{N-1}|_{t=T_{N-1}} = V_N(S_0^{(1)}, S_0^{(2)}, S_1^{(1)}, S_1^{(2)} \cdots, S_{N-1}^{(1)}, S_{N-1}^{(2)}, T_{N-1})$, which is the terminal value of the recursion second problem with $i = N-1$. Therefore, the following solutions $V_{N-1}, \cdots, V_1$ can be solved by induction method.

It is very difficulty to derive the analytical expression of (8). If the finite difference method is used directly to solve the above path dependent variable problem, there will need too much computational cost. So in next two sections, we will use fast Monte Carlo simulation method to discuss the above multi-dimensional pricing problem.

III. Fast Monte Carlo Simulation for Pricing the Derivative

The aim of this section is to develop fast simulation algorithms for pricing covariance swap under the correlated stochastic volatility models. The control variate acceleration techniques are proposed and used to price covariance derivatives under this structure. The method of control variate is one of the most widely used variance reduction techniques. Its popularity rests on the ease of implementation, the availability of controls, and on the straight intuition of the underlying theory. The method of control variate exploits information about the errors in estimates of known quantities to reduce the error in an estimate of an unknown quantity explained in detail in Glasserman[7] and Xu[8]. These techniques aim to reduce the variance per Monte Carlo observation so that a given level of accuracy can be obtained with a smaller number of simulations. We just give a simple introduction on this method.

The main idea of control variate is through exploiting information about the errors in estimates of known quantities to reduce the errors in an estimate of an unknown quantity, as explained in detail in [7]. Consider the problem of estimating the expectation $\mathrm{E}[P]$, where the random variable P is the discounted payoff of a derivative. Denote $P_1, \cdots, P_m$ be outputs from m replications of simulations and suppose that $P_i, (i = 1, \cdots, m)$ are independent and identical distributed (i.i.d.). The Monte Carlo estimator of the price is the average

$$\overline{P} = \frac{P_1 + \cdots + P_m}{m}.$$

Suppose that on each simulation there is another output X_i along with P_i and the pairs (X_i, P_i) are i.i.d. The expectation $\mathrm{E}[X] = \mathrm{E}[X_i]$ is assumed known. Then for any fixed coefficient b we can calculate $P_i(b) = P_i - b(X_i - \mathrm{E}[X])$ from the ith simulation. So the control variate estimator of the derivative price is given by

$$\overline{P}(b) = \frac{1}{m}\sum_{i=1}^{m}(P_i - b(X_i - \mathrm{E}[X])) = \overline{P} - b(\overline{X} - \mathrm{E}[X]),$$

here the observed error $(\overline{X} - \mathrm{E}[X])$ serves as a control.

The control variate estimate $P(b) = P - b(X - \mathrm{E}[X])$ is an unbiased estimator of $\mathrm{E}(P)$, and its variance is

$$\mathrm{Var}(P(b)) = \mathrm{Var}(P) + b^2\mathrm{Var}(X) - 2b\rho_{XP}\sqrt{\mathrm{Var}(P)\mathrm{Var}(X)}.$$

The optimal coefficient $b^* = \frac{\mathrm{Cov}[X,P]}{\mathrm{Var}(X)}$ minimizes the variance, which is given by

$$\mathrm{Var}(P(b^*)) = (1 - \rho_{XP}^2)\mathrm{Var}(P).$$

This indicates that a rather high degree of correlation ρ_{XV} is needed for the control variate to yield better effects of variance reduction. And the error reduction ratio is denoted by $\frac{1}{\sqrt{1-\rho_{XP}^2}}$. The popularity of control variate rests on the ease of implementations, the availability of controls, and on the straight intuition of the underlying theory. According to the variety of problems, different types of instruments can be chosen as controls, including underlying assets, tractable options, bond prices, tractable dynamics, hedges and so on.

Control variate technique is widely used mainly because of the simplicity of its implementations, and the fact that they can be accommodated in an existing Monte Carlo calculator with a small effort. The skill of the control variate method lies in how to choose suitable and efficient control variate. Examples of successful implementations of control variate for pricing the derivatives include Hull and White(1987)[9], Kemna and Vorst(1990)[10], Turnbull and Wakeman(1991)[11], Fu, Madan and Wang (1999)[12] and Ma and Xu (2010)[13].

A. Analytical Solution for the Covariance Swap with Deterministic Volatility

In this subsection, we consider another covariance swap under the assumption of correlated deterministic volatility instead of correlated stochastic volatility models (1). The assets are assumed to follow,

$$\frac{dS^{(1)}(t)}{S^{(1)}(t)} = rdt + \sigma^{(1)}dW_t^{(1)}, \tag{9}$$

$$\frac{dS^{(2)}(t)}{S^{(2)}(t)} = rdt + \sigma^{(2)}dW_t^{(2)}, \tag{10}$$

where r is interest rate; $\sigma^{(1)}$ and $\sigma^{(2)}$ are the deterministic volatility constants which will be chosen in next subsection; $W_t^{(1)}$ and $W_t^{(2)}$ are the same Wiener processes as in (1) and (2), and $E(dW_t^{(1)}dW_t^{(2)}) = \rho_{12}dt$. Let $W = W(S,t)$ denote the price of this auxiliary derivative and the payoff function at maturity is also $h(S_0^{(1)}, S_0^{(2)}, S_1^{(1)}, S_1^{(2)}, \cdots, S_N^{(1)}, S_N^{(2)})$ as in (3). Then similar to the derivation in Section 2, the PDE pricing model of the product is given by

$$\begin{cases} \mathcal{L}_1 W := \frac{\partial W}{\partial t} + rS^{(1)}\frac{\partial W}{\partial S^{(1)}} + \frac{1}{2}{\sigma^{(1)}}^2 {S^{(1)}}^2 \frac{\partial^2 W}{\partial {S^{(1)}}^2} + \\ \quad rS^{(2)}\frac{\partial W}{\partial S^{(2)}} + \frac{1}{2}{\sigma^{(2)}}^2 {S^{(2)}}^2 \frac{\partial^2 W}{\partial {S^{(2)}}^2} \quad + \\ \quad \rho_{12}\sigma^{(1)}\sigma^{(2)}S^{(1)}S^{(2)}\frac{\partial^2 W}{\partial S^{(1)}\partial S^{(2)}} - rW = 0, \\ \quad 0 < S^{(1)}, S^{(2)} < \infty, T_{i-1} < t \le T_i, \\ W|_{t=T} = h(S_0^{(1)}, S_0^{(2)}, S_1^{(1)}, S_1^{(2)}, \cdots, S_N^{(1)}, S_N^{(2)}), \\ W|_{t=T_{i-1}^+} = W|_{t=T_{i-1}^-}, \qquad (i = 1, 2, \cdots, N). \end{cases} \tag{11}$$

The closed form solution to the above problem (11) will be derived by partial differential equation method.

We use the recursive method to solve these equations. When there's a single observation point, the model is simplified as

$$\begin{cases} \mathcal{L}_1 W = 0 \qquad 0 < S^{(1)}, S^{(2)} < \infty, 0 < t \le T, \\ W|_{t=T} = h(S^{(1)}, S^{(2)}). \end{cases}$$

The analytical solution is given by

$$W(S_0^{(1)},S_0^{(2)},0) = e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(S_0^{(1)}e^{A_1-\alpha_1},S_0^{(2)}e^{A_2-\alpha_2}) f(\alpha_1,\alpha_2)d\alpha_1 d\alpha_2, \quad (12)$$

where $A_1=(r-\frac{\sigma^{(1)2}}{2})T, A_2=(r-\frac{\sigma^{(2)2}}{2})T$, and $f(\alpha_1,\alpha_2)$ is the density function of the two dimensional normal distribution $N(0,0,\sigma^{(1)2}T,\sigma^{(2)2}T,\rho_{12})$, which has the expression of

$$f(\alpha_1,\alpha_2)=\frac{1}{2\pi\sigma^{(1)}\sigma^{(2)}\sqrt{1-\rho_{12}^2}T}\times \exp(-\frac{\alpha_1^2{\sigma^{(2)}}^2-2\rho_{12}\sigma^{(1)}\sigma^{(2)}\alpha_1\alpha_2+\alpha_2^2{\sigma^{(1)}}^2}{2{\sigma^{(1)}}^2{\sigma^{(2)}}^2(1-\rho_{12}^2)T}).$$

Then substitute the payoff function into Equation (12) and yield the below Lemma 3.1.

Lemma 3.1 The price of the covariance swap with single discrete observation date under the stochastic process (9) and (10) is given by

$$W|_{t=0}=Me^{-rT}[(r-\frac{\sigma^{(1)2}}{2})(r-\frac{\sigma^{(2)2}}{2})T^2+\rho_{12}\sigma^{(1)}\sigma^{(2)}T -K_{cov}^2],$$

where $K_{cov},\sigma^{(1)},\sigma^{(2)},M,T,r,\rho_{12}$ are given as in equations (9) and (10).

Then when there are N observation points, the model is expressed as

$$\begin{cases} \mathcal{L}W_N=0, \quad 0<S^{(1)},S^{(2)}<\infty, T_{N-1}\le t<T_N=T,\\ W_N|_{t=T}=h(S_0^{(1)},S_0^{(2)},\cdots S_{N-1}^{(1)},S_{N-1}^{(2)},S^{(1)},S^{(2)}),\end{cases}$$

$$\begin{cases} \mathcal{L}W_i=0, 0<S^{(1)},S^{(2)}<\infty, T_{i-1}\le t<T_i.\\ W_i|_{t=T_i}=W_{i+1}|_{t=T_i^+}, (i=1,2,\cdots,N-1).\end{cases}$$

Where $0<T_1<\cdots<T_N=T$, $W_i(i=1,2\cdots N)$ denotes the value of W during $[T_{i-1},T_i]$. By formula (12) and the method of recursion, the pricing formula of the derivative with N observation dates can be expressed as

$$\begin{aligned} W \quad & (S_0^{(1)},S_0^{(2)},0)=e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\\ & f_1(\alpha_1,\alpha_2)f_2(\beta_1,\beta_2)\cdots f_N(\gamma_1,\gamma_2)h(S_0^{(1)}e^{A_1-\alpha_1},\\ & S_0^{(2)}e^{A_2-\alpha_2},\ldots,S_0^{(1)}e^{(A_1-\alpha_1)+(B_1-\beta_1)+\cdots+(Z_1-\gamma_1)},\\ & S_0^{(2)}e^{(A_2-\alpha_2)+(B_2-\beta_2)+\cdots+(Z_2-\gamma_2)})\\ & d\alpha_1 d\alpha_2 d\beta_1 d\beta_2\cdots d\gamma_1 d\gamma_2. \end{aligned} \quad (13)$$

Where $A_1=(r-\frac{\sigma^{(1)2}}{2})\Delta T_1$, $A_2=(r-\frac{\sigma^{(1)2}}{2})\Delta T_1$, $B_1=(r-\frac{\sigma^{(1)2}}{2})\Delta T_2, B_2=(r-\frac{\sigma^{(2)2}}{2})\Delta T_2,\cdots,Z_1=(r-\frac{\sigma^{(1)2}}{2})\Delta T_N, Z_2=(r-\frac{\sigma^{(2)2}}{2})\Delta T_N$, and $f_i(*,*),(i=1,2,\cdots,N)$ is the density function of the two dimensional correlated normal distribution $N(0,0,\sigma^{(1)2}\Delta T_i,\sigma^{(2)2}\Delta T_i,\rho_{12})$.

Substitute the payoff function h into Equation (13) and solve the $2N$ dimensional integration problem, then Theorem 3.1 is achieved, which will be an important result for fast Monte Carlo simulation for pricing covariance swap.

Theorem 3.1 The price of the covariance swap derivative with N discrete observation dates under the stochastic processes (9) and (10) is given by

$$\begin{aligned} W(S_0^{(1)},S_0^{(2)},0)= & \frac{M\exp(-rT)}{T}[\sum_{j=1}^{N}(r-\frac{\sigma^{(1)2}}{2})(r-\frac{\sigma^{(2)2}}{2})\\ & \Delta T_j^2+\rho_{12}\sigma^{(1)}\sigma^{(2)}T-K_{cov}^2T]. \end{aligned}$$

where $M,r,\sigma^{(1)},\sigma^{(2)},\Delta T_j,K_{cov},T,\rho_{12}$ are given in Section 1 and 2.

Proof Substituting $h(S_0^{(1)},S_0^{(2)},S_1^{(1)},S_1^{(2)},\cdots,S_N^{(1)},S_N^{(2)})$ into (12) yields

$$\begin{aligned} W \quad & (S_0^{(1)},S_0^{(2)},0)=\frac{Me^{-rT}}{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\\ & f_1(\alpha_1,\alpha_2)f_2(\beta_1,\beta_2)\cdots f_N(\gamma_1,\gamma_2)\\ & \quad[(A_1-\alpha_1)(A_2-\alpha_2)+(B_1-\beta_1)(B_2-\beta_2)+\cdots\\ & +(Z_1-\gamma_1)(Z_2-\gamma_2)]d\alpha_1 d\alpha_2 d\beta_1 d\beta_2\cdots d\gamma_1 d\gamma_2\\ & -K_{cov}^2Me^{-rT}=\frac{Me^{-rT}}{T}(J-K_{cov}^2T). \end{aligned}$$

Then by calculating directly,

$$\begin{aligned} J= \quad & (A_1A_2+\rho_{12}\sigma^{(1)}\sigma^{(2)}\Delta T_1)+(B_1B_2+\rho_{12}\sigma^{(1)}\sigma^{(2)}\\ & \Delta T_2)+,\cdots,+(Z_1Z_2+\rho_{12}\sigma^{(1)}\sigma^{(2)}\Delta T_N). \end{aligned}$$

Substitute these $2N$ expressions $A_1,A_2,B_1,B_2,\cdots,Z_1,Z_2$ to J,

$$J=\sum_{j=1}^{N}(r-\frac{\sigma^{(1)2}}{2})(r-\frac{\sigma^{(2)2}}{2})\Delta T_j^2+\rho_{12}\sigma^{(1)}\sigma^{(2)}T.$$

Therefore

$$\begin{aligned} W(S_0^{(1)}, \quad & S_0^{(2)},0)=\frac{M\exp(-rT)}{T}[\sum_{j=1}^{N}(r-\frac{\sigma^{(1)2}}{2})(r-\frac{\sigma^{(2)2}}{2})\\ & \Delta T_j^2+\rho_{12}\sigma^{(1)}\sigma^{(2)}T-K_{cov}^2T]. \end{aligned}$$

Remark When $N\to\infty$ and $\max_{1<i\le N}\Delta T_i\to 0$, the above expression exhibits the same as the one in the Theorem 2.1 when $\mu^{(1)}\to 0^+,\mu^{(2)}\to 0^+,\hat\sigma^{(1)}\to 0^+,\hat\sigma^{(2)}\to 0^+$.

B. Control Variate Fast Algorithm for Valuation of the Covariance Swap

The explicit algorithm for the valuation of the covariance swap by control variate Monte Carlo simulation is given as follows:

(i). Divide $[0,T]$ into n parts with mesh size $\Delta t=T/n=t_{k+1}-t_k$, and set time discrimination ponits $\{t_k\}_{k=1}^n$ cover the set of observation dates $\{T_i\}_{i=1}^N$.

(ii). Generate correlated standard normal random numbers $Z_k^{1,j},Z_k^{2,j},Z_k^{3,j},Z_k^{4,j}$, according to Cholesky factor and correlation matrix $[\rho_{kl}],(k,l=1,2,3,4)$. Based on diffusions (1) and (2), set

$$S^{(1),j}(t_{k+1})=S^{(1),j}(t_k)e^{((r-\frac{1}{2}(\sigma_{t_k}^{(1),j})^2)\Delta t+\sigma_{t_k}^{(1),j}\sqrt{\Delta t}Z_k^{1,j})},$$

$$S_{t_0}^{(1),j}=S_0^{(1)},$$

$$S^{(2),j}(t_{k+1})=S^{(2),j}(t_k)e^{((r-\frac{1}{2}(\sigma_{t_k}^{(2),j})^2)\Delta t+\sigma_{t_k}^{(2),j}\sqrt{\Delta t}Z_k^{2,j})},$$

$$S_{t_0}^{(2),j}=S_0^{(2)},$$

where $\sigma_{t_k}^{(1),j}$ and $\sigma_{t_k}^{(2),j}$ are governed by $\sigma_{t_k}^{(1),j}=\sqrt{Y_k^{(1),j}}$ and $\sigma_{t_k}^{(2),j}=\sqrt{Y_k^{(2),j}}$, then

$$Y_{k+1}^{(1),j}=Y_k^{(1),j}\exp((\mu-\frac{1}{2}(\hat\sigma^{(1)})^2)\Delta t+\hat\sigma^{(1)}\sqrt{\Delta t}Z_k^{3,j}),$$

$$Y_{k+1}^{(2),j} = Y_k^{(2),j} \exp((\mu - \frac{1}{2}(\hat{\sigma}^{(2)})^2)\Delta t + \hat{\sigma}^{(2)}\sqrt{\Delta t} Z_k^{4,j}),$$

The replication j of the stock prices $S_t^{(1)}, S_t^{(2)}$ following processes (1) and (2) are simulated.

(iii). According to the clause of the covariance swap and replication j in (ii), set the *jth* price of the product

$$V_j = M \exp(-rT)(\sum_{i=1}^{N}(\frac{1}{T}\ln(\frac{S_i^{(1),j}}{S_{i-1}^{(1),j}})\ln(\frac{S_i^{(2),j}}{S_{i-1}^{(2),j}}) - K_{cov}^2)).$$

(iv). Based on the auxiliary processes (8) and (9), set

$$S^{(1c),j}(t_{k+1}) = S^{1c,j}(t_k)e^{((r-\frac{1}{2}(\sigma^{(1)})^2)\Delta t + \sigma^{(1)}\sqrt{\Delta t} Z_k^{1,j})},$$

$$S_{t_0}^{(1),j} = S_0^{(1)}, S_{t_0}^{(2),j} = S_0^{(2)},$$

$$S^{(2c),j}(t_{k+1}) = S^{2c,j}(t_k)e^{((r-\frac{1}{2}(\sigma^{(2)})^2)\Delta t + \sigma^{(2)}\sqrt{\Delta t} Z_k^{1,j})},$$

with the same sequences $Z_k^{1,j}$ and $Z_k^{2,j}$ as in (ii). Then the replication j of the auxiliary stock prices $S_t^{(1)}, S_t^{(2)}$ following processes (9) and (10) are simulated.

(v). According to the covariance swap contract and replication j in (iv), set the *jth* value of the control variate

$$X_j = M \exp(-rT)(\sum_{i=1}^{N}(\frac{1}{T}\ln(\frac{S_i^{(1c),j}}{S_{i-1}^{(1c),j}})\ln(\frac{S_i^{(2c),j}}{S_{i-1}^{(2c),j}}) - K_{cov}^2)).$$

(vi). Set the *jth* control variate estimate

$$V_j(b) = V_j - b(X_j - E[X]),$$

where $E[X]$ is given by $W(S_0^{(1)}, S_0^{(2)}, 0)$ in Theorem 3.1.

(vii). The control variate estimate of the price of covariance swap derivative under stochastic volatility structure (1),(2) is finally obtained from the mean of m replications,

$$\overline{V}(b) = \frac{1}{m}\sum_{j=1}^{m} V_j(b) = \overline{V} - b(\overline{X} - E[X]).$$

Then the optimal coefficient b^* which minimizes the variance $V_j(b)$ is estimated by $\hat{b}_m = \sum_{i=1}^{m}(X_i - \overline{X})(V_i - \overline{V})(\sum_{i=1}^{m}(X_i - \overline{X})^2)^{-1}$.

The important step of the control variate techniques of the above algorithms lies in how to choose suitable and favorable volatility control constants $\sigma^{(1)}$ and $\sigma^{(2)}$, to make sure high correlation between the control variate X and the problem V. We researched the first two moments of stochastic processes (1),(2) and the auxiliary processes (9),(10), and give a method to choose efficient control volatility constants. The results are shown in Theorem 3.2.

Theorem 3.2 If the control volatility constants of processes (9) and (10) satisfy

$$\sigma^{(1)} = \sqrt{Y_0^{(1)}\frac{e^{\mu^{(1)}T} - 1}{\mu^{(1)}T}}, \sigma^{(2)} = \sqrt{Y_0^{(2)}\frac{e^{\mu^{(2)}T} - 1}{\mu^{(2)}T}},$$

then the mean and the variance of the auxiliary processes (9) and (10) approximately equal to the mean and the variance of underlying processes (1) and (2) at maturity T.

Proof For the auxiliary processes (9) and (10), the mean and the variance are as follows, respectively,

$$\mathrm{E}(S_T^{(1c)}) = S_0^{(1c)}e^{rT},$$

$$Var(S_T^{(1c)}) = (S_0^{(1c)})^2 e^{2rT}(e^{(\sigma^{(1c)})^2 T} - 1).$$

$$\mathrm{E}(S_T^{(2c)}) = S_0^{(2c)}e^{rT},$$

$$Var(S_T^{(2c)}) = (S_0^{(2c)})^2 e^{2rT}(e^{(\sigma^{(2c)})^2 T} - 1).$$

For the underlying processes (1) and (2), the mean and the variance are separately

$$\mathrm{E}(S_T^{(1)}) = \mathrm{E}(S_0^{(1)}e^{(\int_0^T (r-\frac{1}{2}Y^{(1)}(t))dt + \int_0^T \sqrt{Y^{(1)}(t)}dW_{1t}))}$$

$$= S_0^{(1)}e^{rT},$$

$$Var(S_T^{(1)})$$

$$= \mathrm{E}[(S_0^{(1)})^2 e^{2rT}(e^{-\int_0^T Y^{(1)}(t)dt + 2\int_0^T \sqrt{Y^{(1)}(t)}dW_{1t}} - 1)]$$

$$= (S_0^{(1)})^2 e^{2rT}$$

$$E[e^{2\int_0^T \sqrt{Y^{(1)}(t)}dW_{1t} - 2\int_0^T Y^{(1)}(t)dt + \int_0^T Y^{(1)}(t)dt} - 1],$$

and

$$E(S_T^{(2)}) = \mathrm{E}(S_0^{(2)}e^{(\int_0^T (r-\frac{1}{2}Y^{(2)}(t))dt + \int_0^T \sqrt{Y^{(2)}(t)}dW_{2t}))}$$

$$= S_0^{(2)}e^{rT},$$

$$Var(S_T^{(2)})$$

$$= \mathrm{E}[(S_0^{(2)})^2 e^{2rT}(e^{-\int_0^T Y^{(2)}(t)dt + 2\int_0^T \sqrt{Y^{(2)}(t)}dW_{2t}} - 1)]$$

$$= (S_0^{(2)})^2 e^{2rT}$$

$$\mathrm{E}[e^{2\int_0^T \sqrt{Y^{(2)}(t)}dW_{2t} - 2\int_0^T Y^{(2)}(t)dt + \int_0^T Y^{(2)}(t)dt} - 1].$$

If the two stochastic integrals $\int_0^T Y^{(1)}(t)dt$ and $\int_0^T Y^{(2)}(t)dt$ can be approximated by the mean integral $\int_0^T \mathrm{E}(Y^{(1)}(t))dt$ and $\int_0^T \mathrm{E}(Y^{(2)}(t))dt$, then by the Exponential Martingale Theorem, the below equations can be obtained

$$Var(S_T^{(1)}) = (S_0^{(1)})^2 e^{2rT}[e^{\int_0^T Y_0^{(1)}e^{\mu^{(1)}t}dt} - 1],$$

$$Var(S_T^{(2)}) = (S_0^{(2)})^2 e^{2rT}[e^{\int_0^T Y_0^{(2)}e^{\mu^{(2)}t}dt} - 1].$$

If the initial values of stock prices (1), (2) and auxiliary processes (9), (10) are similar, i.e. $S_0^{(1)} = S_0^{(1c)}$, $S_0^{(2)} = S_0^{(2c)}$, and their control volatility expressions satisfy

$$(\sigma^{(1)})^2 = Y_0^{(1)}\frac{e^{\mu^{(1)}T} - 1}{\mu^{(1)}T}, (\sigma^{(2)})^2 = Y_0^{(2)}\frac{e^{\mu^{(2)}T} - 1}{\mu^{(2)}T},$$

then their mean and variance are equivalent at maturity T,

$$\mathrm{E}(S_T^{(1c)}) = \mathrm{E}(S_T^{(1)}), Var(S_T^{(1c)}) = Var(S_T^{(1)}).$$

$$\mathrm{E}(S_T^{(2c)}) = \mathrm{E}(S_T^{(2)}), Var(S_T^{(2c)}) = Var(S_T^{(2)}).$$

Theorem 3.2 tells us that if $\sigma^{(1)}$ and $\sigma^{(2)}$ are chosen as in above discussion, the processes (9) and (10) approximate processes (1) and (2) in the sense of ”moments” at the maturity T. The control variate and the problem are expected to have high correlation. Therefore the control variate in this case will be expected to have a well variance reduction effect, which will be testified in next numerical subsection.

C. Numerical Results

Numerical results will be illustrated the efficiency of the acceleration simulation algorithm discussed in section 3.2. Parameters $M = 1000, r = 0.05, T = 1, \mu_1 = 0.20, \mu_2 = 0.10, S_{10} = 10, S_{20} = 12, Y_{10} = 0.15^2, Y_{20} = 0.15^2, \hat{\sigma}_1 = 0.01, \hat{\sigma}_2 = 0.02$ are chosen, $[0, T]$ is divided into 100 parts with the mesh size $\Delta t = T/100$. As strike price K_{cov} is just a constant decrement, having no effect on the numerical effects, $K_{cov} = 0$ is supposed without loss of generality.

Table 1 and Table 2 show the results of error ratios, the prices of covariance swap and estimate errors with control variate of various simulation path numbers for the case of $N = 10$ and $N = 20$ respectively. The control volatility constant $\sigma^{(1)} = 0.1578$ and $\sigma^{(2)} = 0.1538$ according to Theorem 3.2. In the tables, *Price1* and *Error1* denote the simulation results of crude Monte Carlo simulation, *Error1*$= \frac{1}{\sqrt{m}}\sqrt{\frac{1}{m-1}\sum_{j=1}^{m}(V_j - \bar{V})^2}$. *Price2* and *Error2* denote the simulation results of control variate Monte Carlo method, *Error2*$= \frac{1}{\sqrt{m}}\sqrt{\frac{1}{m-1}\sum_{j=1}^{m}(V_j(b) - \bar{V(b)})^2}$. Variance reduction ratio is denoted by *Ratio1*=*Error1*/*Error2*. The results show that the control variate algorithm has obvious variance reduction effects. All the ratios are greater than 22. As the number of simulation paths increases, all the estimated errors have a decrease tendency. The number of simulations without control variate would have to be increased by a factor of $Ratio1^2$ in order to achieve the same accuracy as a given number of simulations with control variate. Furthermore, the effects of variance reduction are stable, not influenced by the simulation paths.

Table 1: The results with $N = 10$

Paths	*Price1*	*Price2*	*Error1*	*Error2*	*Ratio1*
500	12.0457	11.6786	0.3651	0.0156	23.4738
1000	10.5649	11.6729	0.2648	0.0113	23.3957
2000	11.4888	11.6750	0.1858	0.0081	22.8342
5000	11.6548	11.6743	0.1160	0.0052	22.7432
8000	1.5963	11.6715	0.0926	0.0041	22.6306

Table 2: The results with $N = 20$

Paths	*Price1*	*Price2*	*Error1*	*Error2*	*Ratio1*
500	12.5516	11.6056	0.2557	0.0106	24.1978
1000	12.0684	11.6022	0.1799	0.0077	23.2207
2000	11.9264	11.6051	0.1281	0.0056	22.8404
5000	11.5268	11.6055	0.0813	0.0036	22.5414
8000	11.5678	11.6057	0.0649	0.0029	22.7235

The analysis of correlation between the pricing problem and the control variate are also considered. Fig. 1 and 2 shows scatter plots of simulated values with stochastic volatility against the values with deterministic volatility constant for the case of $N = 10$ and $N = 20$, respectively. They all show the strong correlations between the two cases and their resulting correlations are 0.9990.

Fig. 3 and 4 demonstrates the relationship between Error reduction ratio and control volatility constant square. An unconstrained optimization problem $max_{\sigma}\{\frac{Var[V_j]}{min_b\{Var[V_j(b)]\}}\}$ is introduced, and a direct search method is used to solve this optimization problem. From Fig. 2, we can see that Error reduction ratios are very sensitive with volatility constants square σ^2 and symmetrically distributed. The searched optimal volatility square σ^{*2} is very close to $\sigma^{(1)2} = 0.1578^2 = 0.0248$ and $\sigma^{(2)2} = 0.1538^2 = 0.0236$. These numerical results testified the high efficiency of the control variate proposed by Theorem 3.1.

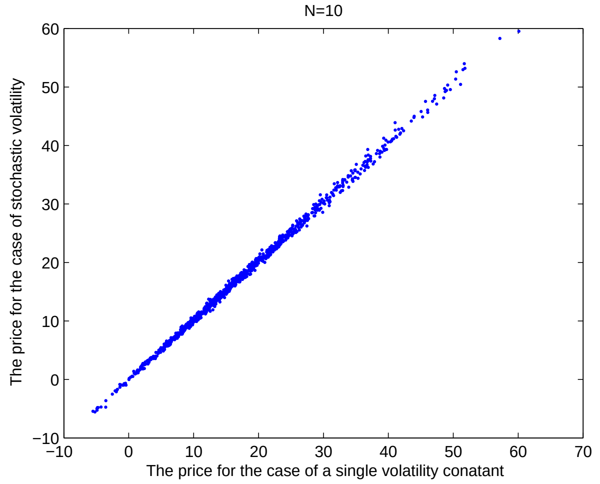

Fig. 1. Scatter plots of values of covariance swap with the stochastic volatility against the values of covariance swap with a single volatility constant for $N = 10$

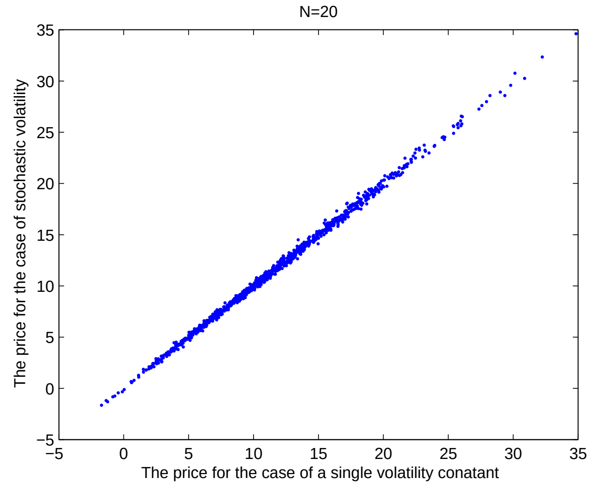

Fig. 2. Scatter plots of values of covariance swap with the stochastic volatility against the values of covariance swap with a single volatility constant for $N = 20$

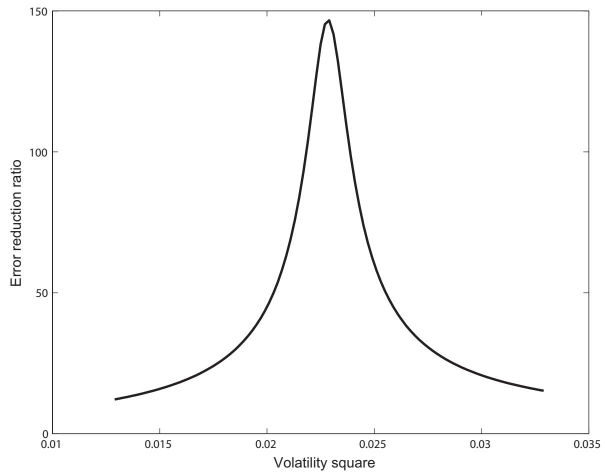

Fig. 3. The relationship between Error reduction ratio and control volatility constant square, for $N = 10$

IV. Improved Control Variate for Monte Carlo Method

A. Improved Control Variate

The control variate Monte Carlo simulation proposed in Section 3 has good variance reduction effects. The control

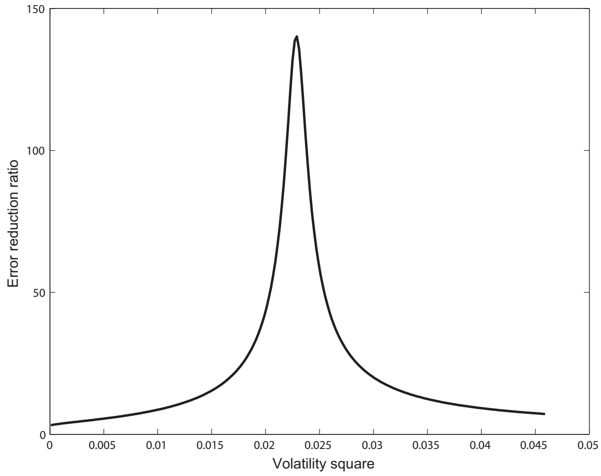

Fig. 4. The relationship between Error reduction ratio and control volatility constant square, for $N = 20$

volatility constants in processes (9) and (10) are determined by making the mean and the variance of the auxiliary processes (9) and (10) approximate those of processes (1) and (2). at the maturity T. Considering the characters of the variance swap derivative, which depends on the stock prices at observation dates T_i, $(0 = T_0 < T_1 < \cdots < T_N = T)$, we try to use piecewise constants $\sigma^1_{i,c}, \sigma^2_{i,c}, (i = 1, 2 \cdots N)$ in stead of σ^1, σ^2 of the processes (9) and (10). The improvement of the control variate algorithm is further studied in this section.

The key research point of the algorithm is still on the choice of the control variate. If the approximate underlying auxiliary processes are further studied and assumed as follows, instead of process (9) and (10),

$$\frac{dS^{(1)}(t)}{S^{(1)}(t)} = rdt + \sigma^{(1)}_{i,c} dW^{(1)}_t, \qquad t \in [T_{i-1}, T_i], \quad (14)$$

$$\frac{dS^{(2)}(t)}{S^{(2)}(t)} = rdt + \sigma^{(2)}_{i,c} dW^{(2)}_t, \qquad t \in [T_{i-1}, T_i], \quad (15)$$

where r is interest rate; $\sigma^{(1)}_{i,c} > 0$ and $\sigma^{(2)}_{i,c} > 0, (i = 1 \cdots N)$ are the volatility constants during the *ith* observation period $[T_{i-1}, T_i]$. The denotation $W^{(1)}_t$ and $W^{(2)}_t$ are the same Wiener processes as in processes (1) and (2), and $E(dW^{(1)}_t dW^{(2)}_t) = \rho_{12} dt$. Then the price of this auxiliary derivative is denoted by $W_2 = W_2(S^{(1)}, S^{(2)}, t)$, and the payoff function at maturity is still denoted as

$$h(S^{(1)}_0, S^{(2)}_0, S^{(1)}_1, S^{(2)}_1, \cdots, S^{(1)}_N, S^{(2)}_N)$$

just as in Equation (3). Then similar to the derivation in Section 3.1, the PDE pricing model of the auxiliary product can be attained and the closed-form solutions of the price of the product is achieved by Theorem 4.1, with the proof omitted.

Theorem 4.1 The price of the covariance swap with N discrete observation dates under the stochastic process (13) and (14) is given by

$$\begin{aligned} W2(S^{(1)}_0, S^{(2)}_0, 0) = \; & \tfrac{M\exp(-rT)}{T}[\sum_{i=1}^{N}(r - \tfrac{\sigma^{(1)2}_{i,c}}{2})(r - \tfrac{\sigma^{(2)2}_{i,c}}{2}) \\ & \Delta T_i^2 + \rho_{12}\sigma^{(1)}_{i,c}\sigma^{(2)}_{i,c} T - K^2_{cov} T], \end{aligned}$$

where $M, r, \sigma^{(1)}_{i,c}, \sigma^{(2)}_{i,c}, \Delta T_i, K_{var}, T$ and ρ_{12} are given in above sections.

Then how to choose the list of deterministic volatility constants $\sigma_{i,c}, i = 1, \cdots, N$ in (14) and (15) to make sure the high correlation between the control variate W_2 and the derivative V. We are inspired by the idea in above section and consider the first two moments at discrete observation dates T_i, the piecewise constants $\sigma^{(1)}_{i,c}, \sigma^{(2)}_{i,c}, i = 1, \cdots, N$ are chosen to make the mean and the variance of the process (1) and (2) approximately equal to those of processes (14) and (15). Then the following Theorem 4.2 is achieved.

Theorem 4.2 If the list of piecewise volatility constants $\sigma_{i,c}, (i = 1, 2 \cdots, N)$ satisfy the equations,

$$\sigma^{(1)}_{i,c} = \sqrt{\frac{Y^{(1)}_0(\exp(\mu^{(1)} T_i) - \exp(\mu^{(1)} T_{i-1}))}{\mu^{(1)}(T_i - T_{i-1})}},$$

$$\sigma^{(2)}_{i,c} = \sqrt{\frac{Y^{(2)}_0(\exp(\mu^{(2)} T_i) - \exp(\mu^{(2)} T_{i-1}))}{\mu^{(2)}(T_i - T_{i-1})}}.$$

then the mean and the variance of the process (13) and (14) are nearly equal to those of processes (1) and (2) at the observation date T_i.

Proof We just need to prove at observations date T_i, their fist two moments are nearly equal, which is similar to the proof of Theorem 3.2 and the proof is omitted.

Then how to choose the list of deterministic volatility constants $\sigma_{i,c}, i = 1, \cdots, N$ in (13) and (14) to make sure the high correlation between the control variate W_2 and the derivative V. We are inspired by the idea in above section and consider the first two moments at discrete observation dates T_i, the piecewise constants $\sigma_{i,c}, i = 1, \cdots, N$ are chosen to make the mean and the variance of the process (1) and (2) approximately equal to those of processes (13) and (14). Then the following Theorem 4.2 is achieved.

Theorem 4.2 If the list of piecewise volatility constants $\sigma_{i,c}, (i = 1, 2 \cdots, N)$ satisfy the equations,

$$\sigma^{(1)}_{i,c} = \sqrt{\frac{Y^{(1)}_0(\exp(\mu^{(1)} T_i) - \exp(\mu^{(1)} T_{i-1}))}{\mu^{(1)}(T_i - T_{i-1})}},$$

$$\sigma^{(2)}_{i,c} = \sqrt{\frac{Y^{(2)}_0(\exp(\mu^{(2)} T_i) - \exp(\mu^{(2)} T_{i-1}))}{\mu^{(2)}(T_i - T_{i-1})}}.$$

then the mean and the variance of the process (13) and (14) are nearly equal to those of processes (1) and (2) at the observation date T_i.

Proof We just need to prove at observations date T_i, their fist two moments are nearly equal, which is similar to the proof of Theorem 3.2 and the proof is omitted.

B. Numerical Results of the Improved Acceleration Algorithm

The main difference of the improvement algorithm from the algorithm in Section 3 is that during the observation period $[T_{i-1}, T_i]$, the volatility constants decided by Theorem 4.2 which will be used to simulate the control processes. In this case, the processes (14) and (15) seem to have a better approximation to the underlying stochastic processes (1) and (2), and the better variance reduction effects will be expected.

In order to make comparison with previous algorithm. Numerical parameters are chosen as the same as in Subsection 3.3. Table 3 and Table 4 record the simulation results of the improved control variate Monte Carlo simulation. Here *Error1* denotes the simulation errors of crude Monte Carlo simulation and *Error3* denotes the simulation errors of improved control variate Monte Carlo simulation. Variance reduction ratio is denoted by *Ratio2=Error1/Error3*. The selected piecewise control volatility constants for the case of $N = 10$ by Theorem 4.2 is that $\sigma^{(1)}$ =[0.1508 0.1523 0.1538 0.1553 0.1569 0.1585 0.1601 0.1617 0.1633 0.1650] and $\sigma^{(2)}$ =[0.1504 0.1511 0.1519 0.1526 0.1534 0.1542 0.1550 0.1557 0.1565 0.1573]. Then the selected piecewise control volatility constants for the case of $N = 20$ is that $\sigma^{(1)}$ =[0.1504 0.1511 0.1519 0.1526 0.1534 0.1542 0.1550 0.1557 0.1565 0.1573 0.1581 0.1589 0.1597 0.1605 0.1613 0.1621 0.1629 0.1637 0.1645 0.1654] and $\sigma^{(2)}$ =[0.1502 0.1506 0.1509 0.1513 0.1517 0.1521 0.1525 0.1528 0.1532 0.1536 0.1540 0.1544 0.1548 0.1551 0.1555 0.1559 0.1563 0.1567 0.1571 0.1575]. Especially, when $N = 1$, the two algorithms are same.

Table 3: The results with $N = 10$ of the improved algorithm

Paths	*Price1*	*Price3*	*Error1*	*Error3*	*Ratio2*
500	13.5678	11.3672	0.3651	0.0029	124.5770
1000	12.4537	11.6705	0.2648	0.0021	124.3808
2000	11.0264	11.6736	0.1858	0.0015	121.9264
5000	11.9865	11.6729	0.1160	0.0010	120.4219

Table 4: The results with $N = 20$ of the improved algorithm

Paths	*Price1*	*Price3*	*Error1*	*Error3*	*Ratio2*
500	13.0568	11.4026	0.2557	0.0024	106.9722
1000	12.0486	11.6641	0.1799	0.0017	105.4123
2000	11.7963	11.6394	0.1281	0.0013	102.8288
5000	11.6528	11.6935	0.0813	0.0009	98.9774

From the results in Table 3 and Table 4, we can see that the improved control variate algorithm has better obvious variance reduction effects than the control variate determined by Theorem 3.2. All the ratios are nearly 100, 4 times as much as *Ratio1*. As the number of simulation paths increases, the estimated error also has a decrease tendency. The number of simulations without control variate would have to be increased by a factor of $Ratio2^2$ in order to achieve the same accuracy as a given number of simulations with improved control variate. Furthermore, the effects of variance reduction are stable, not influenced by the simulation paths. The better variance reduction effect of the improved control variate in Section 4 is because the auxiliary stochastic process with piecewise volatility constants can approximate the original process with stochastic volatility process better.

Then Fig. 5 and 6 shows scatter plots of simulated values with stochastic volatility against the values with piecewise constant volatility for the case of $N = 10$ and $N = 20$, respectively, with correlation coefficient bigger than 0.9999. They all show stronger correlations than the first cases of control variate I demonstrated by Fig. 1 and 2. The evident comparison of variance reduction ratios of the two kinds of control variate acceleration algorithms is listed in Fig. 7, where *'o'* denotes the variance reduction ratio of the improved algorithm in Section 4, and '*'denotes the variance reduction ratio of the algorithm in Section 3.

V. Conclusion and Further Discussion

In this paper, modeling and pricing of covariance swap derivative are discussed through the combination of PDE approach and Monte Carlo simulation. Control variate Monte Carlo method is applied to the pricing of the covariance swap. Two kinds of control variate techniques are proposed, based on the closed form solutions in simplified models with constant or piecewise constant volatility. The variance reduction ratios of the improved control variate algorithm which is based on the piecewise constant volatility are smaller than that of the original control variate based on a single constant volatility.

The algorithms in the paper can also be extended to the pricing of other financial derivatives, such as Asian options, Lookback options with discrete sampling features under multi-factor stochastic volatility models. Multiple control variate techniques can be considered based on the high moments of the stochastic volatility. Furthermore, other variance reduction techniques can be applied to the research of the hedging and the risk of the derivatives next. Then the research on the optimal control variate can be considered[21].

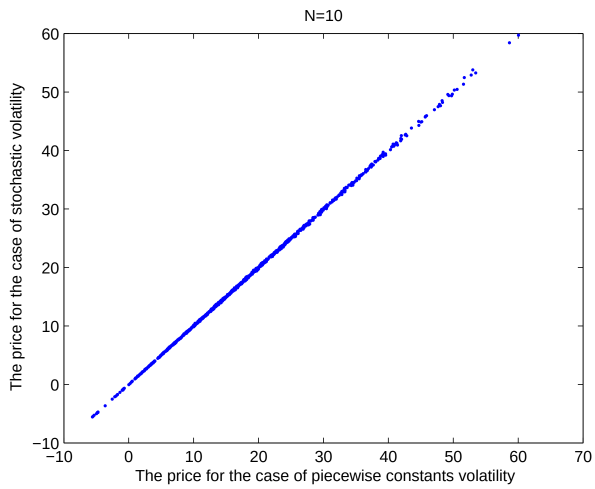

Fig. 5. Scatter plots of values of covariance swap with the stochastic volatility against the values of covariance swap with piecewise volatility constants, for $N = 10$

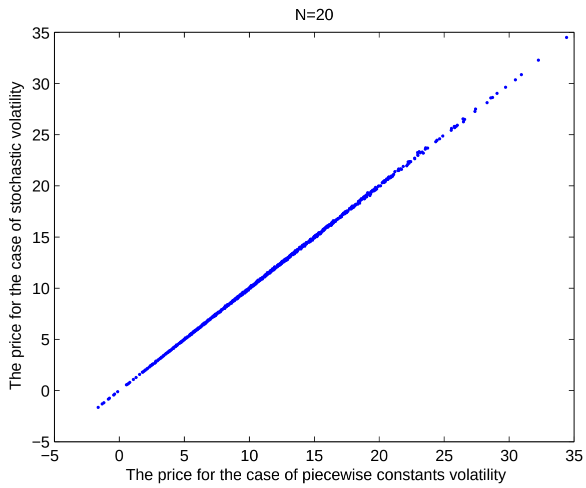

Fig. 6. Scatter plots of values of covariance swap with the stochastic volatility against the values of covariance swap with piecewise volatility constants, for $N = 20$

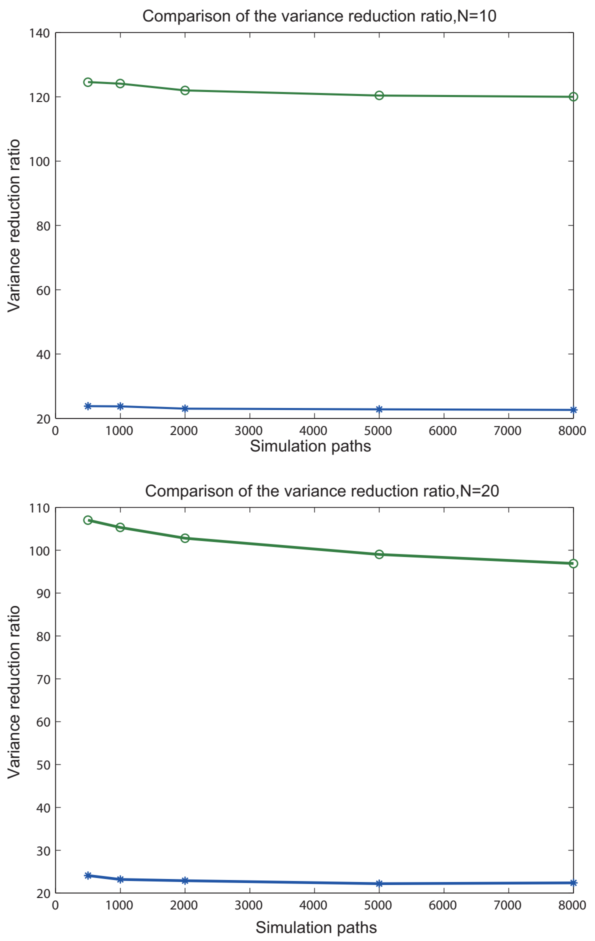

Fig. 7. Comparison of the effects of two control variate algorithms

REFERENCES

[1] A. V. Swishchuk, R. Cheng, S. Lawi, A. Badescu, H. B. Mekki, A. F. Gashaw, Y. Hua, M. Molyboga, T. Neocleous and Y. Petrachenko. Price Pseudo-Variance, Pseudo-Covariance, Pseudo-Volatility, and Pseudo-Correlation Swaps-In Analytical Closed-Forms, Proceedings of the Sixth PIMS Industrial Problems SolvingWorkshop, PIMS IPSW6, University of British Columbia, Vancouver, Canada, May 27-31, pp. 45-55 (2002).

[2] S. Bossu. Arbitrage Pricing of Equity Correlation Waps. JPMorgan Equity Deriva- tives, *Working paper*, 2005.

[3] S. Bossu. A New Approach For Modelling and Pricing Correlation Swaps. Equity Structuring - ECD London, *Working paper*, 2007.

[4] Giovanni Salvi and Anatoliy V. Swishchuk, Modeling and Pricing of Covariance and Correlation Swaps for Financial Markets with Semi-Markov Volatilities, Available at SSRN:*http://arxiv.org/pdf/1205.5565v1.pdf*.

[5] J. Da Fonseca, F. Ielpo and M. Grasselli. Hedging (Co)Variance Risk with Variance Swaps, Available at SSRN: *http://ssrn.com/abstract=1341811* (2009)

[6] J. Drissien, P. J. Maenhout and G. Vilkov. The price of Correlation Risk: Evidence from Equity Options, *The Journal of Finance*, 64(3), pp. 1377-1406,(2009).

[7] P. Glasserman, Monte Carlo Methods in Financial Engineering, Springer, New York, 2004, 185-195.

[8] ZhongJi Xu, Monte Carlo Methods, Shanghai Science and Technology Press, 1985.

[9] Hull, J., and White, A. 1987. The use of the control variate technique in option pricing. *Journal of Financial and Quantitative Analysis*, 23(3), pp. 237-251.(1987).

[10] A. G. Kemna and and A.C. Vorst, A Pricing method for options based on average asset values, *Journal of Banking and Finance*, 14(1), 113-129, 1990.

[11] Turnbull, S., andWakeman, L. 1991. A quick algorithm for pricing European average Options, *Journal of Financial and Quantitative Analysis*, 26(3), pp.377-389, 1991.

[12] Fu, M., Madan, D., andWang, T. Pricing continuous time Asian options: A comparison of Monte Carlo and Laplace transform inversion methods. *Journal of Computational Finance*, 2(2), pp 49-74, 1999.

[13] J. Ma and C. Xu. An Efficient Control Variate Method for Pricing Variance Derivatives under Stochastic Volatility and Jump Diffusion Models. *Journal of Computational and Applied Mathematics*, 235(1), pp108-119, 2010.

[14] J. Hull, Options Futures and Other Derivatives, 4th ed, Prentice Hall, New Jersey, 2000.

[15] J. Hull and A. White, The pricing of options on assets with stochastic volatilities, *Journal of Finance*,42(2), 281-300, June 1987.

[16] E. M. Stein, and J. C. Stein, Stock price distributions with stochastic volatility: an analytic approach, *Review of Financial Studies*, 29(4), 727-752, Winter 1991.

[17] C. Ball and A. Roma, Stochastic volatility option pricing, *Journal of Financial and Quantitative Analysis*, 29(4), 589-607, Dec. 1994.

[18] S. L. Heston and L. Steven, A closed-form solution for options with stochastic volatility, *Review of Financial Studies*, 6(2), 327-343, 1993.

[19] L. Scott, Pricing Stock Options in a Jump-Diffusion Model with Stochastic Volatility and Interest Rates: Applications of Fourier Inversion Methods, *Mathematical Finance*,7(4), 413-424, 1997.

[20] L. S. Jiang, C. L. Xu, X. M. Ren and S. H. Li, Mathematical Models and Cases Analysis for Financial Derivatives, in Chinese, High Eduction Press, Beijing, 2008.

[21] M. Gabli, E. M. Jaara, and E. B. Mermri, A Genetic Algorithm Approach for an Equitable Treatment of Objective Functions in Multi-objective Optimization Problems, *IAENG International Journal of Computer Science*, 41(2), 102-111,2014.