Some Results of Resistance Distance and Kirchhoff Index Based on R-Graph

Qun Liu

***Abstract*—The resistance distance between any two vertices of a connected graph is defined as the effective resistance between them in the electrical network constructed from the graph by replacing each edge with a (unit) resistor. The Kirchhoff index $Kf(G)$ is the sum of resistance distances between all pairs of vertices in G. For a graph G, let $R(G)$ be the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the corresponding edge. Let $G_1 \odot G_2$, $G_1 \ominus G_2$ be the R-vertex corona and R-edge corona of G_1 and G_2. In this paper, formulate for the resistance distance and the Kirchhoff index in $G_1 \odot G_2$ and $G_1 \ominus G_2$ whenever G_1 and G_2 are arbitrary graphs are obtained. This improves and extends some earlier results.**

***Index Terms*—Kirchhoff index, Resistance distance, R-vertex corona, R-edge corona, Generalized inverse**

I. Introduction

IN1993, Klein and Randić [1] introduced a distance function named resistance distance on the basis of electrical network theory. The resistance distance $r_{ij}(G)$ between any two vertices i and j in G is defined to be the effective resistance between them when unit resistors are placed on every edge of G. The Kirchhoff index $Kf(G)$ is the sum of resistance distances between all pairs of vertices of G. The computation of two-point resistances in networks and the Kirchhoff index are classical problem in electric theory and graph theory. The resistance distance and the Kirchhoff index attracted extensive attention due to its wide applications in physics, chemistry and others. For more information on resistance distance and Kirchhoff index of graphs, the readers are referred to Refs. [2]-[12] and the references therein.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let d_i be the degree of vertex i in G and $D_G = diag(d_1, d_2, \cdots d_{|V(G)|})$ the diagonal matrix with all vertex degrees of G as its diagonal entries. For a graph G, let A_G and B_G denote the adjacency matrix and vertex-edge incidence matrix of G, respectively. The matrix $L_G = D_G - A_G$ is called the Laplacian matrix of G, where D_G is the diagonal matrix of vertex degrees of G. We use $\mu_1(G) \geq u_2(G) \geq \cdots \geq \mu_n(G) = 0$ to denote the spectrum of L_G. If G is connected, then any principal submatrix of L_G is nonsingular.

In [13], new graph operations based on $R(G)$ graphs: R-vertex corona and R-edge corona, are introduced, and their A-spectrum(resp., L-spectrum) are investigated. For a graph G, let $R(G)$ be the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the corresponding edge. Let $I(G)$ be the set of newly added vertices, i.e. $I(G) = V(R(G)) \setminus V(G)$. Let G_1 and G_2 be two vertex-disjoint graphs.

Definition 1. ([13]) The R-vertex corona of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the ith vertex of $V(G_1)$ to every vertex in the ith copy of G_2.

Definition 2. ([13]) The R-edge corona of G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the ith vertex of $|I(G_1)|$ to every vertex in the ith copy of G_2.

Note that if G_i has n_i vertices and m_i edges for $i = 1, 2$, then $G_1 \odot G_2$ has $n_1 + m_1 + n_1 n_2$ vertices and $3m_1 + n_1 m_2 + n_1 n_2$ edges, $G_1 \ominus G_2$ has $n_1 + m_1 + m_1 n_2$ vertices and $3m_1 + m_1 m_2 + m_1 n_2$ edges.

This paper is organized as follows. In Section 2, some auxiliary lemma are given. In Section 3, we obtain formulas for resistance distances of R-vertex corona and R-edge corona of two arbitrary graphs. In Section 4, we obtain formulas for Kirchhoff index of R-vertex corona and R-edge corona of two arbitrary graphs.

II. Preliminaries

Let M be a square matrix M. The $\{1\}$-inverse of M is a matrix X such that $MXM = M$. If M is singular, then M has infinitely many $\{1\}$-inverse [14]. The group inverse of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. It is known [9, 11] that $M^{\#}$ exists if and only if $rank(M) = rank(M^2)$. If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric $\{1\}$- inverse of M. Actually, $M^{\#}$ is equal to the Moore-Penrose inverse of M since M is symmetric [15].

We use $M^{(1)}$ to denote any $\{1\}$- inverse of a matrix M. Let $(M)_{uv}$-denote the (u, v)-entry of M.

Lemma 1. ([16]) Let G be a connected graph. Then

$$\begin{aligned} r_{uv}(G) &= (L_G^{(1)})_{uu} + (L_G^{(1)})_{vv} - (L_G^{(1)})_{uv} - (L_G^{(1)})_{vu} \\ &= (L_G^{\#})_{uu} + (L_G^{\#})_{vv} - 2(L_G^{\#})_{uv}. \end{aligned}$$

Let 1_n denotes the column vector of dimension n with all the entries equal one. We will often use 1 to denote an all-ones column vector if the dimension can be read from the context.

Lemma 2. ([17]) For any graph, we have $L_G^{\#} 1 = 0$.

Lemma 3. ([18]) Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

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be a nonsingular matrix. If A and D are nonsingular, then

$$\begin{aligned} M^{-1} &= \begin{pmatrix} A^{-1}+A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}, \end{aligned}$$

where $S = D - CA^{-1}B$.

For a vertex i of a graph G, let $T(i)$ denote the set of all neighbors of i in G.

Lemma 4. ([17]) Let G be a connected graph. For any $i, j \in V(G)$,

$$r_{ij}(G) = d_i^{-1}\left(1 + \sum_{k\in T(i)} r_{kj}(G) - d_i^{-1}\sum_{k,l\in T(i)} r_{kl}(G)\right).$$

Lemma 5. ([16]) Let G be a connected graph on n vertices. Then

$$Kf(G) = ntr(L_G^{(1)}) - 1^T L_G^{(1)} 1 = ntr(L_G^{\#}).$$

Lemma 6. ([4]) Let G be a connected graph of order n with edge set E. Then

$$\sum_{u<v, uv\in E} r_{uv}(G) = n-1.$$

Lemma 7. Let

$$L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

be the Laplacian matrix of a connected graph. If D is nonsingular, then

$$X = \begin{pmatrix} H^{\#} & -H^{\#}BD^{-1} \\ -D^{-1}B^TH^{\#} & D^{-1}+D^{-1}B^TH^{\#}BD^{-1} \end{pmatrix}$$

is a symmetric $\{1\}$-inverse of L, where $H = A - BD^{-1}B^T$.

Proof Let $H = A - BD^{-1}B^T$, and

$$\begin{aligned} &X \\ &= \begin{pmatrix} H^{\#} & -H^{\#}BD^{-1} \\ -D^{-1}B^TH^{\#} & D^{-1}+D^{-1}B^TH^{\#}BD^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ -D^{-1}B^T & I \end{pmatrix}\begin{pmatrix} H^{\#} & 0 \\ 0 & D^{-1} \end{pmatrix} \\ &\quad \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}. \end{aligned}$$

Since D is nonsingular, then

$$\begin{aligned} L &= \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \\ &= \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} H & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & 0 \\ D^{-1}B^T & I \end{pmatrix}. \end{aligned}$$

By computation, we have

$$\begin{aligned} &LX \\ &= \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} H & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & 0 \\ D^{-1}B^T & I \end{pmatrix} \\ &\quad \begin{pmatrix} I & 0 \\ -D^{-1}B^T & I \end{pmatrix}\begin{pmatrix} H^{\#} & 0 \\ 0 & D^{-1} \end{pmatrix} \\ &\quad \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} HH^{\#} & 0 \\ 0 & I \end{pmatrix} \\ &\quad \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} HH^{\#} & -HH^{\#}BD^{-1}+BD^{-1} \\ 0 & I \end{pmatrix} \end{aligned}$$

$$\begin{aligned} LXL &= \begin{pmatrix} HH^{\#} & BD^{-1}-HH^{\#}BD^{-1} \\ 0 & I \end{pmatrix} \\ &\quad \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \\ &= \begin{pmatrix} HH^{\#}(A-BD^{-1}B^T)+BD^{-1}B^T & B \\ B^T & D \end{pmatrix} \\ &= L. \end{aligned}$$

Hence X is a symmetric $\{1\}$-inverse of L, where $H = A - BD^{-1}B^T$.

Remarks: The above result is similar to Lemma 2.8 in [16], this is another form of Lemma 2.8, but in the process of computing the resistance distance and Kirchhoff index of graph $G_1 \odot G_2$ and $G_1 \ominus G_2$, we use this formula to be more superior than Lemma 2.8 in [16].

III. Resistance Distance in R-vertex Corona and R-edge Corona of Two Graphs

We first give formulae for resistance distance between two arbitrary vertexes in $G_1 \odot G_2$.

Theorem 1. Let G_1 be a graph with n_1 vertices and m_1 edges and G_2 be a graph with n_2 vertices and m_2 edges. Then the following holds:

(a) For any $i, j \in V(G_1)$, we have

$$r_{ij}(G_1\odot G_2) = \frac{2}{3}(L_{G_1}^{\#})_{ii} + \frac{2}{3}(L_{G_1}^{\#})_{jj} - \frac{4}{3}(L_{G_1}^{\#})_{ij}.$$

(b) For any $i, j \in V(G_2)$, we have

$$\begin{aligned} r_{ij}(G_1\odot G_2) &= (I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1})_{ii} + (I_{n_1}\otimes(L_{G_2} \\ &\quad +I_{n_2})^{-1})_{jj} - 2(I_{n_1}\otimes(L_{G_2}+I_{n_2}))^{-1}. \end{aligned}$$

(c) For any $i \in V(G_1)$, $j \in V(G_2)$, we have

$$\begin{aligned} r_{ij}(G_1\odot G_2) &= \frac{2}{3}(L_{G_1}^{\#})_{ii} + (I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1})_{jj} \\ &\quad -\frac{2}{3}(L_{G_1}^{\#})_{ij}. \end{aligned}$$

(d) For any $i \in I(G_1)$, $j \in V(G_1)\cup V(G_2)$, let $u_iv_i \in E(G_1)$ denote the edge corresponding to i, we have

$$\begin{aligned} &r_{ij}(G_1\odot G_2) \\ &= \frac{1}{2} + \frac{1}{2}r_{u_ij}(G_1\odot G_2) + \frac{1}{2}r_{v_ij}(G_1\odot G_2) \\ &\quad -\frac{1}{4}r_{u_iv_i}(G_1\odot G_2). \end{aligned}$$

(e) For any $i, j \in I(G_1)$, let $u_iv_i, u_jv_j \in E(G_1)$ denote the edges corresponding to i, j, we have

$$
\begin{aligned}
r_{ij}(G_1 \odot G_2)& \\
&= 1+\frac{1}{4}(r_{u_iu_j}(G_1 \odot G_2)+r_{u_iv_j}(G_1 \odot G_2) \\
&\quad +r_{v_iu_j}(G_1 \odot G_2)+r_{v_iv_j}(G_1 \odot G_2) \\
&\quad -r_{u_iv_i}(G_1 \odot G_2)-r_{u_jv_j}(G_1 \odot G_2).
\end{aligned}
$$

Proof Let R be the incidence matrix of G_1. Then with a proper labeling of vertices, the Laplacian matrix of $G = G_1 \odot G_2$ can be written as

$$
L(G) = \begin{pmatrix} L_{G_1}+D_{G_1}+n_2I_{n_1} & -R & -I_{n_1}\otimes 1_{n_2}^T \\ -R^T & 2I_{m_1} & 0_{m_1\times n_1n_2} \\ -I_{n_1}\otimes 1_{n_2} & 0_{n_1n_2\times m_1} & I_{n_1}\otimes(L_{G_2}+I_{n_2}) \end{pmatrix},
$$

where $0_{s,t}$ denotes the $s\times t$ matrix with all entries equal to zero.

Let $A = L_{G_1}+D_{G_1}+n_2I_{n_1}$, $B = \begin{pmatrix} -R & -I_{n_1}\otimes 1_{n_2}^T \end{pmatrix}$, $B^T = \begin{pmatrix} -R^T \\ -I_{n_1}\otimes 1_{n_2} \end{pmatrix}$ and

$$
D = \begin{pmatrix} 2I_{m_1} & 0_{m_1\times n_1n_2} \\ 0_{n_1n_2}\times m_1 & I_{n_1}\otimes(L_{G_2}+I_{n_2}) \end{pmatrix}.
$$

We begin with the calculation about H.

Let $Q = I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1}$, then

$$
\begin{aligned}
H& \\
&= L_{G_1}+D_{G_1}+n_2I_{n_1}-\begin{pmatrix} -R & -I_{n_1}\otimes 1_{n_2}^T \end{pmatrix} \\
&\quad \begin{pmatrix} 2I_{m_1} & 0_{m_1\times n_1n_2} \\ 0_{n_1n_2\times m_1} & Q \end{pmatrix}^{-1} \begin{pmatrix} -R^T \\ -I_{n_1}\otimes 1_{n_2} \end{pmatrix} \\
&= L_{G_1}+D_{G_1}+n_2I_{n_1}- \\
&\quad \begin{pmatrix} -\frac{1}{2}R & -I_{n_1}\otimes 1_{n_2}^T(L_{G_2}+I_{n_2})^{-1} \end{pmatrix} \\
&\quad \begin{pmatrix} -R^T \\ -I_{n_1}\otimes 1_{n_2} \end{pmatrix} \\
&= L_{G_1}+D_{G_1}+n_2I_{n_1}-\tfrac{1}{2}RR^T-n_2I_{n_1} \\
&= \tfrac{3}{2}L_{G_1},
\end{aligned}
$$

so, we have $H^{\#} = \frac{2}{3}L_{G_1}^{\#}$.

Now we are ready to calculate $-H^{\#}BD^{-1}$ and $-D^{-1}B^TH^{\#}$.

Let $K = I_{n_1}\otimes 1_{n_2}^T$, then

$$
\begin{aligned}
-H^{\#}BD^{-1}& \\
&= -H^{\#}\begin{pmatrix} -R & -I_{n_1}\otimes 1_{n_2}^T \end{pmatrix} \\
&\quad \begin{pmatrix} 2I_{m_1} & 0_{m_1\times n_1n_2} \\ 0_{n_1n_2}\times m_1 & I_{n_1}\otimes(L_{G_2}+I_{n_2}) \end{pmatrix}^{-1} \\
&= -H^{\#}\begin{pmatrix} -\frac{1}{2}R & -I_{n_1}\otimes 1_{n_2}^T \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}H^{\#}R & H^{\#}K \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
-D^{-1}B^TH^{\#}& \\
&= -\begin{pmatrix} 2I_{m_1} & 0_{m_1\times n_1n_2} \\ 0_{n_1n_2\times m_1} & I_{n_1}\otimes(L_{G_2}+I_{n_2}) \end{pmatrix}^{-1} \\
&\quad \begin{pmatrix} -R^T \\ -I_{n_1}\otimes 1_{n_2} \end{pmatrix} H^{\#} \\
&= \begin{pmatrix} \frac{1}{2}R^T \\ I_{n_1}\otimes 1_{n_2} \end{pmatrix} H^{\#} = \begin{pmatrix} \frac{1}{2}R^TH^{\#} \\ K^TH^{\#} \end{pmatrix}.
\end{aligned}
$$

Next we are ready to compute the $D^{-1}B^TH^{\#}BD^{-1}$.

$$
\begin{aligned}
D^{-1}B^TH^{\#}BD^{-1}& \\
&= \begin{pmatrix} -\frac{1}{2}R^TH^{\#} \\ -K^TH^{\#} \end{pmatrix}\begin{pmatrix} -R & -K \end{pmatrix} \\
&\quad \begin{pmatrix} 2I_{m_1} & 0_{m_1\times n_1n_2} \\ 0_{n_1n_2\times m_1} & I_{n_1}\otimes(L_{G_2}+I_{n_2}) \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{1}{2}R^TH^{\#}R & \frac{1}{2}R^TH^{\#}K \\ K^TH^{\#}R & K^TH^{\#}K \end{pmatrix} \\
&\quad \begin{pmatrix} \frac{1}{2}I_{m_1} & 0_{m_1\times n_1n_2} \\ 0_{n_1n_2\times m_1} & I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4}R^TH^{\#}R & \frac{1}{2}R^TH^{\#}K \\ \frac{1}{2}K^TH^{\#}R & K^TH^{\#}K \end{pmatrix},
\end{aligned}
$$

where $K^THK = \frac{2}{3}K^TL_{G_1}^{\#}K = \frac{2}{3}j_{n_2\times n_2}\otimes L_{G_1}^{\#}$.

Based on Lemma 7, the following matrix

$$
N = \begin{pmatrix} \frac{2}{3}L_{G_1}^{\#} & \frac{1}{3}L_{G_1}^{\#}R & \frac{2}{3}L_{G_1}^{\#}K \\ \frac{1}{3}R^TL_{G_1}^{\#} & \frac{1}{2}I_{m_1}+\frac{1}{6}R^TL_{G_1}^{\#}R & \frac{1}{3}R^TL_{G_1}^{\#}K \\ \frac{2}{3}K^TL_{G_1}^{\#} & \frac{1}{3}K^TL_{G_1}^{\#}R & Q+\frac{2}{3}K^TL_{G_1}^{\#}K \end{pmatrix} \tag{1}
$$

is a symmetric $\{1\}$- inverse of $L(G_1\odot G_2)$, where $Q = I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1}$, $K = I_{n_1}\otimes 1_{n_2}^T$.

Let $G = G_1\odot G_2$, then For any $i,j\in V(G_1)$, by Lemma 1 and the Equation (1), we have

$$
r_{ij}(G) = \frac{2}{3}(L_{G_1}^{\#})_{ii}+\frac{2}{3}(L_{G_1}^{\#})_{jj}-\frac{4}{3}(L_{G_1}^{\#})_{ij},
$$

as stated in (a).

For any $i,j\in V(G_2)$, by Lemma 1 and the Equation (1), we have

$$
\begin{aligned}
r_{ij}(G) &= (I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1})_{ii}+(I_{n_1}\otimes(L_{G_2} \\
&\quad +I_{n_2})^{-1})_{jj}-2(I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1})_{ij},
\end{aligned}
$$

as stated in (b).

For any $i\in V(G_1)$, $j\in V(G_2)$, by Lemma 1 and the Equation (1), we have

$$
\begin{aligned}
r_{ij}(G) &= \frac{2}{3}(L_{G_1}^{\#})_{ii}+(I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1})_{jj} \\
&\quad -\frac{2}{3}(L_{G_1}^{\#})_{ij},
\end{aligned}
$$

as stated in (c).

For any $i\in I(G_1)$, $j\in V(G_1)\cup V(G_2)$, let $u_iv_i\in E(G_1)$ denote the edge corresponding to i, by Lemma 4, we have

$$
\begin{aligned}
r_{ij}(G) &= \frac{1}{2}+\frac{1}{2}r_{u_ij}(G_1\odot G_2)+\frac{1}{2}r_{v_ij}(G_1\odot G_2) \\
&\quad -\frac{1}{4}r_{u_iv_i}(G_1\odot G_2),
\end{aligned}
$$

as stated in (d).

For any $i,j\in I(G_1)$, let $u_iv_i, u_jv_j\in E(G_1)$ denote the edges corresponding to i, j, by Lemma 4, we have

$$\begin{aligned} r_{ij}(G) &= \frac{1}{2} + \frac{1}{2} r_{u_i j}(G_1 \odot G_2) + \frac{1}{2} r_{v_i j}(G_1 \odot G_2) \\ &\quad - \frac{1}{4} r_{u_i v_i}(G_1 \odot G_2) \\ &= 1 + \frac{1}{4}\left(r_{u_i u_j}(G_1 \odot G_2) + r_{u_i v_j}(G_1 \odot G_2)\right. \\ &\quad + r_{v_i u_j}(G_1 \odot G_2) + r_{v_i v_j}(G_1 \odot G_2) \\ &\quad \left. - r_{u_i v_i}(G_1 \odot G_2) - r_{u_j v_j}(G_1 \odot G_2)\right), \end{aligned}$$

as stated in (e).

The following result is proved in a way that is certainly similar in spirit to the proof of Theorem 1, but is a little more complicated in detail. Next, we will give the formulate for the resistance distance between two arbitrary vertexes in $G_1 \ominus G_2$.

Theorem 2. Let G_1 be a graph on n_1 vertices and m_1 edges and G_2 be a graph on n_2 vertices and m_2 edges, let $G = G_1 \ominus G_2$. Then the following holds:

(a) For any $i, j \in V(G_1)$, we have

$$r_{ij}(G) = \frac{2}{3}(L_{G_1}^{\#})_{ii} + \frac{2}{3}(L_{G_1}^{\#})_{jj} - \frac{4}{3}(L_{G_1}^{\#})_{ij}.$$

(b) For any $i, j \in V(G_2)$, we have
$r_{ij}(G)$

$$\begin{aligned} &= (I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1})_{ii} \\ &\quad + (I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1})_{jj} \\ &\quad - 2(I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1})_{ij}. \end{aligned}$$

(c) For any $i \in V(G_1)$, $j \in V(G_2)$, we have

$$\begin{aligned} r_{ij}(G) &= \frac{2}{3}(L_{G_1}^{\#})_{ii} + (I_{m_1} \otimes (L_{G_2} + I_{n_2} \\ &\quad - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1})_{jj} - (I_{m_1} \otimes \\ &\quad (L_{G_2} + I_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1})_{ij}. \end{aligned}$$

(d) For any $i \in I(G_1)$, $j \in V(G_1) \cup V(G_2)$, let $u_i v_i \in E(G_1)$ denote the edge corresponding to i, we have

$$\begin{aligned} r_{ij}(G) &= \frac{1}{n_2+2}\Big(1 + r_{ju_i}(G_1 \ominus G_2) + r_{jv_i}(G_1 \ominus G_2) \\ &\quad + \sum_{k \in V(G_2)} r_{jk}(G_1 \ominus G_2)\Big) - \frac{1}{(n_2+2)^2} \\ &\quad \Big(Kf(G_2) + \sum_{k \in V(G_2)} r_{u_i k}(G_1 \ominus G_2) + \\ &\quad r_{u_i v_i}(G_1 \ominus G_2) + \sum_{k \in V(G_2)} r_{v_i k}(G_1 \ominus G_2)\Big). \end{aligned}$$

(e) For any $i, j \in I(G_1)$, let $u_i v_i, u_j v_j \in E(G_1)$ denote the edges corresponding to i, j, we have

$r_{ij}(G)$

$$\begin{aligned} &= \frac{1}{n_2+2}\Big(1 + r_{iu_j}(G_1 \ominus G_2) + r_{iv_j}(G_1 \ominus G_2) \\ &\quad + \sum_{k \in V(G_2)} r_{ik}(G_1 \ominus G_2)\Big) - \frac{1}{(n_2+2)^2} \\ &\quad \Big(Kf(G_2) + \sum_{k \in V(G_2)} r_{u_j k}(G_1 \ominus G_2) \\ &\quad + r_{u_j v_j}(G_1 \ominus G_2) + \sum_{k \in V(G_2)} r_{v_j k}(G_1 \ominus G_2)\Big). \end{aligned}$$

Proof Let R be the incidence matrix of G_1. Then with a proper labeling of vertices, the Laplacian matrix of $G_1 \ominus G_2$ can be written as

$$L(G) = \begin{pmatrix} L_{G_1} + D_{G_1} & -R & 0_{n_1 \times m_1 n_2} \\ -R^T & (n_2+2) I_{m_1} & -I_{m_1} \otimes 1_{n_2}^T \\ 0_{m_1 n_2 \times n_1} & -I_{m_1} \otimes 1_{n_2} & I_{m_1} \otimes (L_{G_2} + I_{n_2}) \end{pmatrix},$$

where $0_{s,t}$ denotes the $s \times t$ matrix with all entries equal to zero.

Let $A = L_{G_1} + D_{G_1}$, $B = \begin{pmatrix} -R & 0_{n_1 \times m_1 n_2} \end{pmatrix}$, $B^T = \begin{pmatrix} -R^T \\ 0_{m_1 n_2 \times n_1} \end{pmatrix}$ and

$$D = \begin{pmatrix} (n_2+2) I_{m_1} & -I_{m_1} \otimes 1_{n_2}^T \\ -I_{m_1} \otimes 1_{n_2} & I_{m_1} \otimes (L_{G_2} + I_{n_2}) \end{pmatrix}.$$

Note that $RR^T = D_{G_1} + A_{G_1}$. Let $R_1 = [(n_2+2) I_{m_1} - (-I_{m_1} \otimes 1_{n_2}^T)(I_{m_1} \otimes (L_{G_2} + I_{n_2}))^{-1}(-I_{m_1} \otimes 1_{n_2})]^{-1} = \frac{1}{2} I_{m_1}$. By Lemma 3, we have

$$D^{-1} = \begin{pmatrix} \frac{1}{2} I_{m_1} & \frac{1}{2} I_{m_1} \otimes 1_{n_2}^T \\ \frac{1}{2} I_{m_1} \otimes 1_{n_2} & F \end{pmatrix},$$

where $F = I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1}$.

We begin with the calculation about H.

Let $T = I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1}$, $K = I_{m_1} \otimes 1_{n_2}^T$, then

$$\begin{aligned} H &= L_{G_1} + D_{G_1} - \begin{pmatrix} -R & 0_{n_1 \times m_1 n_2} \end{pmatrix} \\ &\quad \begin{pmatrix} \frac{1}{2} I_{m_1} & \frac{1}{2} I_{m_1} \otimes 1_{n_2}^T \\ \frac{1}{2} I_{m_1} \otimes 1_{n_2} & T \end{pmatrix}^{-1} \begin{pmatrix} -R^T \\ 0_{m_1 n_2 \times n_1} \end{pmatrix} \\ &= L_{G_1} + D_{G_1} - \begin{pmatrix} -\frac{1}{2} R & -\frac{1}{2} RK \end{pmatrix} \begin{pmatrix} -R^T \\ 0_{m_1 n_2 \times n_1} \end{pmatrix} \\ &= L_{G_1} + D_{G_1} - \frac{1}{2} RR^T = \frac{3}{2} L_{G_1}. \end{aligned}$$

So, we have $H^{\#} = \frac{2}{3} L_{G_1}^{\#}$.

Now we are ready to calculate the $-H^{\#} B D^{-1}$ and $-D^{-1} B^T H^{\#}$.

$$\begin{aligned} -H^{\#} B D^{-1} &= -H^{\#} \begin{pmatrix} -R & 0_{n_1 \times m_1 n_2} \end{pmatrix} \\ &\begin{pmatrix} \frac{1}{2} I_{m_1} & \frac{1}{2} I_{m_1} \otimes 1_{n_2}^T \\ \frac{1}{2} I_{m_1} \otimes 1_{n_2} & I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2+2} j_{n_2 \times n_2})^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} H^{\#} R & \frac{1}{2} H^{\#} RK \end{pmatrix} \end{aligned}$$

and

$-D^{-1}B^TH^{\#} =$

$$-\begin{pmatrix} \frac{1}{2}I_{m_1} & \frac{1}{2}I_{m_1}\otimes 1_{n_2}^T \\ \frac{1}{2}I_{m_1}\otimes 1_{n_2} & I_{m_1}\otimes (L_{G_2}+I_{n_2}-\frac{1}{n_2+2}j_{n_2\times n_2})^{-1} \end{pmatrix} \begin{pmatrix} -R^T \\ 0_{m_1n_2\times n_1} \end{pmatrix} H^{\#} = \begin{pmatrix} \frac{1}{2}R^TH^{\#} \\ \frac{1}{2}K^TR^TH^{\#} \end{pmatrix}.$$

Next we are ready to compute the $D^{-1}B^TH^{\#}BD^{-1}$.

$$\begin{aligned} & D^{-1}B^TH^{\#}BD^{-1} \\ &= \begin{pmatrix} -\frac{1}{2}R^TH^{\#} \\ -\frac{1}{2}K^TR^TH^{\#} \end{pmatrix} \begin{pmatrix} -R & 0_{n_1\times m_1n_2} \end{pmatrix} \begin{pmatrix} \frac{1}{2}I_{m_1} & \frac{1}{4}I_{m_1}\otimes 1_{n_2}^T \\ \frac{1}{2}I_{m_1}\otimes 1_{n_2} & T \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}R^TH^{\#}R & \frac{1}{2}R^TH^{\#}RK \\ \frac{1}{4}K^TR^TH^{\#}R & \frac{1}{4}K^TR^TH^{\#}RK \end{pmatrix}. \end{aligned}$$

Based on Lemma 3 and Lemma 7, the following matrix $N =$

$$\begin{pmatrix} \frac{2}{3}L_{G_1}^{\#} & \frac{1}{3}L_{G_1}^{\#}R & \frac{1}{3}L_{G_1}^{\#}RK \\ \frac{1}{3}R^TL_{G_1}^{\#} & \frac{1}{6}Q+\frac{1}{2}I_{m_1} & \frac{1}{6}QK+\frac{1}{2}K \\ \frac{1}{3}K^TR^TL_{G_1}^{\#} & \frac{1}{6}K^TQ+\frac{1}{2}K^T & \frac{1}{6}K^TQK+T \end{pmatrix} \tag{2}$$

is a symmetric $\{1\}$- inverse of $L(G_1 \ominus G_2)$, where $K = I_{m_1}\otimes 1_{n_2}^T$, $T = I_{m_1}\otimes(L_{G_2}+I_{n_2}-\frac{1}{n_2+2}j_{n_2\times n_2})^{-1}$, $Q = R^TL_{G_1}^{\#}R$.

For any $i,j \in V(G_1)$, by Lemma 1 and the Equation (2), we have

$$r_{ij}(G_1\ominus G_2) = \frac{2}{3}(L_{G_1}^{\#})_{ii}+\frac{2}{3}(L_{G_1}^{\#})_{jj}-\frac{4}{3}(L_{G_1}^{\#})_{ij},$$

as stated in (a).

For any $i,j \in V(G_2)$, by Lemma 1 and the Equation (2), we have $r_{ij}(G_1\ominus G_2)$

$$\begin{aligned} &= (I_{m_1}\otimes(L_{G_2}+I_{n_2}-\frac{1}{n_2+2}j_{n_2\times n_2})^{-1})_{ii} \\ &\quad +(I_{m_1}\otimes(L_{G_2}+I_{n_2}-\frac{1}{n_2+2}j_{n_2\times n_2})^{-1})_{jj} \\ &\quad -2(I_{m_1}\otimes(L_{G_2}+I_{n_2}-\frac{1}{n_2+2}j_{n_2\times n_2})^{-1})_{ij}, \end{aligned}$$

as stated in (b).

For any $i \in V(G_1)$, $j \in V(G_2)$, by Lemma 1 and the Equation (2), we have

$$\begin{aligned} r_{ij}(G_1\ominus G_2) &= \frac{2}{3}(L_{G_1}^{\#})_{ii}+(I_{m_1}\otimes(L_{G_2}+I_{n_2}- \\ &\quad \frac{1}{n_2+2}j_{n_2\times n_2})^{-1})_{jj}-(I_{m_1}\otimes(L_{G_2}+ \\ &\quad I_{n_2}-\frac{1}{n_2+2}j_{n_2\times n_2})^{-1})_{ij}, \end{aligned}$$

as stated in (c).

For any $i \in I(G_1)$, $j \in V(G_1)\cup V(G_2)$, let $u_iv_i \in E(G_1)$ denote the edge corresponding to i, by Lemma 4, we have

$$\begin{aligned} r_{ij}(G) &= \frac{1}{n_2+2}\Big(r_{ju_i}(G_1\ominus G_2)+r_{jv_i}(G_1\ominus G_2)+1 \\ &\quad +\sum_{k\in V(G_2)}r_{jk}(G_1\ominus G_2)\Big)-\frac{1}{(n_2+2)^2} \\ &\quad \Big(Kf(G_2)+\sum_{k\in V(G_2)}r_{u_ik}(G_1\ominus G_2)+ \\ &\quad r_{u_iv_i}(G_1\ominus G_2)+\sum_{k\in V(G_2)}r_{v_ik}(G_1\ominus G_2)\Big), \end{aligned}$$

as stated in (d).

For any $i,j \in I(G_1)$, let $u_iv_i, u_jv_j \in E(G_1)$ denote the edges corresponding to i,j respectively. By Lemma 4, we have

$$\begin{aligned} r_{ij}(G) &= \frac{1}{n_2+2}\Big(1+r_{iu_j}(G_1\ominus G_2)+r_{iv_j}(G_1\ominus G_2) \\ &\quad +\sum_{k\in V(G_2)}r_{ik}(G_1\ominus G_2)\Big)-\frac{1}{(n_2+2)^2} \\ &\quad \Big(Kf(G_2)+\sum_{k\in V(G_2)}r_{u_jk}(G_1\ominus G_2)+ \\ &\quad r_{u_jv_j}(G_1\ominus G_2)+\sum_{k\in V(G_2)}r_{v_jk}(G_1\ominus G_2)\Big), \end{aligned}$$

as stated in (e).

IV. Kirchhoff Index in R-vertex Corona and R-edge Corona of Two Graphs

In this section, we will give the formulate for the Kirchhoff index in $G_1 \odot G_2$ and $G_1 \ominus G_2$ whenever G_1 and G_2 are arbitrary graphs.

Theorem 3. Let G_1 be a graph with n_1 vertices and m_1 edges and G_2 be a graph with n_2 vertices and m_2 edges. Then

$$\begin{aligned} & Kf(G_1\odot G_2) \\ &= (n_1+m_1+n_1n_2)\Big(\frac{2+2n_2}{3n_1}Kf(G_1) \\ &\quad l+\frac{1}{3}tr(D_{G_1}L_{G_1}^{\#})+n_1\sum_{i=1}^{n_2}\frac{1}{\mu_i(G_2)+1} \\ &\quad +\frac{3m_1-n_1+1}{6}\Big)-\frac{1}{6}\pi^TL_{G_1}^{\#}\pi-\frac{1}{3} \\ &\quad 1^T(R^TL_{G_1}^{\#}K)1-\frac{1}{3}1^T(K^TL_{G_1}^{\#}R)1 \\ &\quad -n_1n_2-\frac{m_1+2n_1n_2}{2}, \end{aligned}$$

where $\pi = (d_1, d_2, \cdots, d_{n_1})^T$ and $K = I_{n_1}\otimes 1_{n_2}^T$.

Proof Let $L_{G_1\odot G_2}^{(1)}$ be the symmetric $\{1\}$-inverse of $L_{G_1\odot G_2}$. Then

$$
\begin{aligned}
&tr(L^{(1)}_{G_1\odot G_2})\\
&= \frac{2}{3}tr(L^{\#}_{G_1}) + tr(\frac{1}{2}I_{m_1} + \frac{1}{6}R^TL^{\#}_{G_1}R) + tr\\
&\quad (I_{n_1}\otimes(L_{G_2}+I_{n_2}))^{-1} + \frac{2}{3}tr(K^TL^{\#}_{G_1}K)\\
&= \frac{2}{3}tr(L^{\#}_{G_1}) + \frac{m_1}{2} + \frac{1}{6}\sum_{i<j, ij\in E(G_1)}[(L^{\#}_{G_1})_{ii}\\
&\quad +(L^{\#}_{G_1})_{jj} + 2(L^{\#}_{G_1})_{ij}] + n_1tr((L_{G_2}\\
&\quad +I_{n_2})^{-1}) + \frac{2}{3}tr(j_{n_2\times n_2}\otimes L^{\#}_{G_1}).
\end{aligned}
$$

Note that the eigenvalues of $(L_{G_2}+I_{n_2})$ are $\mu_1(G_2)+1,...,\mu_n(G_2)+1$, then $tr(L_{G_2}+I_{n_2})^{-1}=\sum_{i=1}^{n_2}\frac{1}{\mu_i(G_2)+1}$. By Lemma 1 and Lemma 5, we have

$$
\begin{aligned}
&tr(L^{(1)}_{G_1\odot G_2})\\
&= \frac{2}{3n_1}Kf(G_1) + \frac{m_1}{2} + \frac{1}{6}\sum_{i<j, ij\in E(G)}\\
&\quad [2(L^{\#}_{G_1})_{ii} + 2(L^{\#}_{G_1})_{jj} - r_{ij}(G_1)] +\\
&\quad n_1\sum_{i=1}^{n_2}\frac{1}{\mu_i(G_2)+1} + \frac{2n_2}{3n_1}Kf(G_1)\\
&= \frac{2+2n_2}{3n_1}Kf(G_1) + \frac{m_1}{2} + \frac{1}{3}tr(D_{G_1}L^{\#}_{G_1})\\
&\quad -\frac{n_1-1}{6} + n_1\sum_{i=1}^{n_2}\frac{1}{\mu_i(G_2)+1}.
\end{aligned}
$$

Next, we calculate the $1^T(L^{(1)}_{G_1\odot G_2})1$. Since $L^{\#}_G1=0$, then

$$
\begin{aligned}
1^T(L^{(1)}_{G_1\odot G_2})1 &= 1^T(\frac{1}{2}I_{m_1} + \frac{1}{6}R^TL^{\#}_{G_1}R)1 + \frac{1}{3}1^T\\
&\quad (R^TL^{\#}_{G_1}K)1 + \frac{1}{3}1^T(K^TL^{\#}_{G_1}R)1\\
&\quad +1^T(I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1})1\\
&\quad +\frac{2}{3}1^T(K^TL^{\#}_{G_1}K)1.
\end{aligned}
$$

Since $R1=\pi$, where $\pi=(d_1,d_2,\cdots,d_{n_1})^T$, then $1^TR^TL^{\#}_{G_1}R1=\pi^TL^{\#}_{G_1}\pi$.

Let $T=1^T_{n_1n_2}(I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1})1_{n_1n_2}$, $M=(L_{G_2}+I_{n_2})$, $K=I_{n_1}\otimes 1^T_{n_2}$, then

$$
\begin{aligned}
T &= \begin{pmatrix} 1^T_{n_2} & 1^T_{n_2} & \cdots & 1^T_{n_2}\end{pmatrix}\\
&\quad \begin{pmatrix} M^{-1} & & & \\ & M^{-1} & & \\ & & \ddots & \\ & & & M^{-1}\end{pmatrix}\begin{pmatrix}1_{n_2}\\ 1_{n_2}\\ \cdots\\ 1_{n_2}\end{pmatrix}\\
&= n_11^T_{n_2}(L_{G_2}+I_{n_2})^{-1}1_{n_2} = n_1n_2
\end{aligned}
$$

and $1^T(K^TL^{\#}_{G_1}K)1$

$$
\begin{aligned}
&= 1^T_{n_1n_2}(I_{n_1}\otimes 1_{n_2})L^{\#}_{G_1}(I_{n_1}\otimes 1^T_{n_2})\\
&\quad 1_{n_1n_2}\\
&= \begin{pmatrix} 1^T_{n_2}1_{n_2} & \cdots & 1^T_{n_2}1_{n_2}\end{pmatrix}\\
&\quad L^{\#}_{G_1}\begin{pmatrix}1^T_{n_2}1_{n_2}\\ 1^T_{n_2}1_{n_2}\\ \cdots\\ 1^T_{n_2}1_{n_2}\end{pmatrix}\\
&= n_2^21^T_{n_1}L^{\#}_{G_1}1_{n_1} = 0,
\end{aligned}
$$

so $1^T(L^{(1)}_{G_1\odot G_2})1$

$$
\begin{aligned}
&= \frac{m_1}{2} + \frac{1}{6}\pi^TL^{\#}_{G_1}\pi + \frac{1}{3}1^T(R^TL^{\#}_{G_1}K)1\\
&\quad +\frac{1}{3}1^T(K^TL^{\#}_{G_1}R)1 + n_1n_2.
\end{aligned}
$$

Lemma 5 implies that

$$
\begin{aligned}
&Kf(L^{(1)}_{G_1\odot G_2}) =\\
&\quad (n_1+m_1+n_1n_2)tr(L^{(1)}_{G_1\odot G_2}) - 1^T(L^{(1)}_{G_1\odot G_2})1.
\end{aligned}
$$

Then plugging $tr(L^{(1)}_{G_1\odot G_2})$ and $1^T(L^{(1)}_{G_1\odot G_2})1$ into the equation above, we obtain the required result.

Corollary 1. Let G_1 be an r_1- regular graph with n_1 vertices and m_1 edges and G_2 be a graph with n_2 vertices and m_2 edges. Then

$$
\begin{aligned}
Kf(G_1\odot G_2) &= (n_1+m_1+n_1n_2)\Bigg(\frac{2+2n_2+r_1}{3n_1}\\
&\quad Kf(G_1) + n_1\sum_{i=1}^{n_2}\frac{1}{\mu_i(G_2)+1}\\
&\quad +\frac{3m_1-n_1+1}{6}\Bigg) - \frac{m_1+2n_1n_2}{2}.
\end{aligned}
$$

Proof Since $R1=r1$ and $L^{\#}_G1=0$, then $1^TK^TL^{\#}_{G_1}R1=1^T(R^TL^{\#}_{G_1}K)1=0$ and $\pi^TL^{\#}_{G_1}\pi=r_1^21^TL^{\#}_{G_1}1=0$. Then the required result is obtained by plugging $1^TK^TL^{\#}_{G_1}R1$, $1^T(R^TL^{\#}_{G_1}K)1$ and $\pi^TL^{\#}_{G_1}\pi$ into Theorem 3.

Theorem 4. Let G_1 be a graph with n_1 vertices and m_1 edges, G_2 be a graph with n_2 vertices and m_2 edges. Then

$$
\begin{aligned}
&Kf(G_1\ominus G_2)\\
&= (n_1+m_2+m_1n_2)\Bigg(\frac{2}{3n_1}Kf(G_1)\\
&\quad +\frac{1}{3}tr(D_{G_1}L^{\#}_{G_1}) + \frac{1}{6}tr(K^TR^TL^{\#}_{G_1}RK)\\
&\quad +m_1\sum_{i=1}^{n_2}\frac{1}{\mu_i(G_2)+1} + \frac{3m_1-n_1+1}{6}\Bigg)\\
&\quad -\frac{1+3n_2}{6}\pi^TL^{\#}_{G_1}\pi - \frac{m_1+4m_1n_2}{2},
\end{aligned}
$$

where $K=I_{m_1}\otimes 1^T_{n_2}$ and $\pi=(d_1,d_2,\cdots,d_{n_2})^T$.

Proof The proof is similar to that of Theorem 3 and hence we omit details.

Corollary 2. Let G_1 be an r_1-regular graph with n_1 vertices and m_1 edges and G_2 be a graph with n_2 vertices and m_2 edges. Then

$$
\begin{aligned}
Kf(G_1\ominus G_2) &= (n_1+m_2+m_1n_2)\Bigg(\frac{2+r_1}{3n_1}Kf(G_1)\\
&\quad +\frac{1}{6}tr(K^TR^TL^{\#}_{G_1}RK) + m_1\\
&\quad \sum_{i=1}^{n_2}\frac{1}{\mu_i(G_2)+1} + \frac{3m_1-n_1+1}{3}\Bigg)\\
&\quad -\frac{m_1+4m_1n_2}{2},
\end{aligned}
$$

where $K = I_{m_1} \otimes 1_{n_2}^T$.

Remark: From Theorem 3.2 and Theorem 4.2, in [13], one can immediately get the Kirchhoff index of a graph by computing the Laplacian eigenvalues of the graph. However, the expression given in Theorem 3.2 and Theorem 4.2 in [13] are somewhat complicated. In the above, by the result of Theorem 3 and Theorem 4 in a different way we obtain a much simpler expression.

V. Conclusion

We use the Laplacian generalized inverse approach to obtain the formulate for the resistance distance and the Kirchhoff index in $G_1 \odot G_2$ and $G_1 \ominus G_2$ whenever G_1 and G_2 are arbitrary graphs. This approach is more simpler than the computation of the results in [13] and improves and extends some earlier results.

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